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DYNAMIC ANALYSIS OF BEAMS WITH ARBITRARY ELASTIC SUPPORTS AT BOTH ENDS

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1. INTRODUCTION

In a previous paper [1], the author proposed a simple and unified technique for the vibration analysis of a generally supported beam. The flexural displacement of the beam is sought as the linear combination of a Fourier series and an auxiliary polynomial function. The polynomial function is introduced to take care of all the potential discontinuities of the original displacement and its derivatives when they are periodically extended onto the entire x-axis. In other words, the Fourier expansion now only represents a residual or conditioned displacement function that has at least three continuous derivatives everywhere. As a result, not only is it always possible to expand the displacement in a Fourier series for beams with any boundary conditions, but also the solution will be drastically improved in terms of its accuracy and convergence. Another important advantage of the technique is that the modal properties of a beam can be readily determined from solving a standard matrix eigenproblem, rather than the non-linear hyperbolic equations as in many traditional techniques.

In this study, an alternative discretization scheme based on the Galerkin method, instead of the Fourier method, is used for solving the governing differential equation for beams. It will be demonstrated through numerical examples that the current method provides an improved solution with respect to both accuracy and convergence.

2. BASIC EQUATIONS

Figure 1 shows a beam elastically restrained at both ends. The differential equation for the free vibration of the beam is

$$D\,\mathrm{d}^4 w(x)/\mathrm{d}x^4 - \rho A\omega^2 w(x) = 0 \tag{1}$$

or

$$w''''(x) - \rho_D\omega^2 w(x) = 0, \tag{2}$$

where D, ρ and A are, respectively, the flexural rigidity, the mass density and the cross-sectional area of the beam, ω the frequency in radian and $\rho_D = \rho A/D$.

The boundary conditions at the ends of the beam can be expressed as

$$\hat{k}_0 w = -w''', \quad \hat{K}_0 w' = w'', \quad (\hat{k}_0 = k_0/D,\ \hat{k}_1 = k_1/D), \quad \text{at } x = 0 \tag{3, 4}$$

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Figure 1. A beam elastically restrained at both ends.

and

$$\hat{k}_1 w = w''', \quad \hat{K}_1 w' = -w'', \quad (\hat{K}_0 = K_0/D, \hat{K}_1 = K_1/D), \quad \text{at } x = L, \tag{5, 6}$$

where k_0 and k_1 are the stiffnesses of the linear springs, and K_0 and K_1 are the stiffnesses of the rotational springs at $x = 0$ and L respectively.

Many traditional boundary conditions can be simply considered as the special cases of equations (3–6). For example, the simply supported boundary condition can be essentially obtained by setting the stiffnesses of the translational and rotational springs to be extremely large and small numbers respectively.

As in reference [1], the beam displacement will be sought in the following form:

$$w(x) = \sum_{m=0}^{\infty} A_m \cos \lambda_m x + p(x), \quad 0 \leqslant x \leqslant L, \quad (\lambda_m = m\pi/L). \tag{7, 8}$$

In equation (7) the auxiliary polynomial function $p(x)$ is introduced to remove all the potential discontinuities from the displacement $w(x)$ and its relevant derivatives at the end points. As a result, the Fourier series only represents a residual or conditioned displacement that has at least three continuous derivatives everywhere. An immediate benefit of doing this is that all the required differential operations on the Fourier series can be carried out on a term-by-term basis.

Set

$$p'''(0) = w'''(0) = \alpha_0, \quad p'''(L) = w'''(L) = \alpha_1 \tag{9, 10}$$

$$p'(0) = w'(0) = \beta_0, \quad p'(L) = w'(L) = \beta_1. \tag{11, 12}$$

Then the polynomial function $p(x)$ can be readily written as

$$p = \boldsymbol{\zeta}(x)^{\mathrm{T}} \bar{\boldsymbol{\alpha}}, \tag{13}$$

where

$$\bar{\boldsymbol{\alpha}} = \{\alpha_0, \alpha_1, \beta_0, \beta_1\}^{\mathrm{T}} \tag{14}$$

and

$$\boldsymbol{\zeta}(x)^{\mathrm{T}} = \begin{Bmatrix} -(15x^4 - 60Lx^3 + 60L^2x^2 - 8L^4)/360L \\ (15x^4 - 30L^2x^2 + 7L^4)/360L \\ (6Lx - 2L^2 - 3x^2)/6L \\ (3x^2 - L^2)/6L \end{Bmatrix}. \tag{15}$$

Substituting equations (7) and (9–12) into the boundary conditions (3–6), the unknown vector, $\bar{\boldsymbol{\alpha}}$, can be expressed as [1]

$$\bar{\boldsymbol{\alpha}} = \sum_{m=0}^{\infty} \mathbf{H}^{-1} \mathbf{Q}_m A_m, \tag{16}$$

where

$$\mathbf{H} = \begin{bmatrix} \frac{8\hat{k}_0 L^3}{360} + 1 & \frac{7\hat{k}_0 L^3}{360} & \frac{-\hat{k}_0 L}{3} & \frac{-\hat{k}_0 L}{6} \\ \frac{7\hat{k}_1 L^3}{360} & \frac{8\hat{k}_1 L^3}{360} + 1 & \frac{-\hat{k}_1 L}{6} & \frac{-\hat{k}_1 L}{3} \\ \frac{L}{3} & \frac{L}{6} & \hat{K}_0 + \frac{1}{L} & \frac{-1}{L} \\ \frac{L}{6} & \frac{L}{3} & \frac{-1}{L} & \hat{K}_1 + \frac{1}{L} \end{bmatrix} \tag{17}$$

and

$$\mathbf{Q}_m = \{ -\hat{k}_0 \,(-1)^m \hat{k}_1 \; -\lambda_m^2 \,(-1)^m \lambda_m^2 \}^{\mathrm{T}}. \tag{18}$$

Combining equations (7), (13) and (16) results in

$$w(x) = \sum_{m=0}^{\infty} A_m (\cos \lambda_m x + \boldsymbol{\zeta}(x)^{\mathrm{T}} \mathbf{H}^{-1} \mathbf{Q}_m). \tag{19}$$

or

$$w(x) = \sum_{m=0}^{\infty} A_m \phi_m(x), \tag{20}$$

where

$$\phi_m(x) = \cos \lambda_m x + \boldsymbol{\zeta}(x)^{\mathrm{T}} \mathbf{H}^{-1} \mathbf{Q}_m. \tag{21}$$

Equation (20) essentially defines a new set of trial or basis functions, $[\phi_m(x),\, m = 0, 1, 2, 3, \ldots\}$, that clearly satisfy all the specified boundary conditions, equations (3–6).

Substituting equation (20) into equation (2) and following the standard procedures in the Galerkin method, one is able to obtain

$$\sum_{m'=1}^{\infty} (\delta_{mm'} + S_{m'm}) \lambda_m^4 A_{m'} - \rho_D \omega^2 \left((S_{m0} + Z_{m0}) A_0 + \sum_{m'=1}^{\infty} (\delta_{mm'} + S_{mm'} + S_{m'm} + Z_{mm'}) A_{m'} \right) = 0, \tag{22}$$

$m = 1, 2, 3, \ldots,$ and

$$\mathbf{c}\mathbf{H}^{-1}\mathbf{Q}_0 A_0 + \sum_{m'=1}^{\infty} (\mathbf{c}\mathbf{H}^{-1}\mathbf{Q}_{m'} + \lambda_{m'}^4 S_{m'0}) A_{m'} - \rho_D \omega^2 \left((2 + Z_{00}) A_0 + \sum_{m'=1}^{\infty} (S_{m'0} + Z_{0m'}) A_{m'} \right) = 0, \tag{23}$$

where

$$S_{mm'} = \mathbf{P}_m \mathbf{H}^{-1} \mathbf{Q}_{m'}, \qquad \mathbf{c} = \{ -2/L \;\, 2/L \;\, 0 \;\, 0 \}, \tag{24, 25}$$

$$\mathbf{P}_m = \frac{2}{L} \left\{ \frac{1}{\lambda_m^4} \; \frac{(-1)^{m+1}}{\lambda_m^4} \; \frac{-1}{\lambda_m^2} \; \frac{(-1)^m}{\lambda_m^2} \right\}, \tag{26}$$

$$Z_{mm'} = \mathbf{Q}_m^{\mathrm{T}} \mathbf{H}^{-\mathrm{T}} \boldsymbol{\Xi} \mathbf{H}^{-1} \mathbf{Q}_{m'} \tag{27}$$

and

$$\boldsymbol{\Xi} = 2/L\int_0^L \zeta(x)^{\mathrm{T}}\zeta(x)\mathrm{d}x = \begin{bmatrix} \dfrac{2L^6}{4725} & & & \\ \dfrac{127\,L^6}{302400} & \dfrac{2L^6}{4725} & sym. & \\ -\dfrac{4L^4}{945} & -\dfrac{31L^4}{7560} & \dfrac{2L^2}{45} & \\ -\dfrac{31L^4}{7560} & -\dfrac{4L^4}{945} & \dfrac{7L^2}{180} & \dfrac{2L^2}{45} \end{bmatrix}. \tag{28}$$

In addition, it is not difficult to verify that

$$S_{m'm}\lambda_{m'}^4 = S_{mm'}\lambda_m^4, \quad Z_{m'm} = Z_{mm'}, \tag{29, 30}$$

and

$$\mathbf{cH}^{-1}\mathbf{Q}_m + \lambda_{m'}^4 S_{m'0} \equiv 0. \tag{31}$$

Making use of equation (31), equation (23) reduces to

$$\mathbf{cH}^{-1}\mathbf{Q}_0 A_0 - \rho_D\omega^2\left((2 + Z_{00})A_0 + \sum_{m'=1}^{\infty}(S_{m'0} + Z_{m'0})A_{m'}\right) = 0. \tag{32}$$

Finally, for the sake of clarity, equations (22) and (32) will be combined as

$$(\mathbf{K} - \rho_D\omega^2\mathbf{M})\mathbf{A} = 0, \tag{33}$$

where

$$\mathbf{A} = \{A_0, A_1, A_2, \dots\}^{\mathrm{T}}, \tag{34}$$

$$K_{mm'} = (1 - \delta_{0m})(1 - \delta_{0m'})(\delta_{mm'} + S_{m'm})\lambda_{m'}^4 + \delta_{0m'}\delta_{m0}\,(\mathbf{cH}^{-1}\mathbf{Q}_0) \tag{35}$$

and

$$\begin{aligned} M_{mm'} = {} & (1 - \delta_{0m})(1 - \delta_{0m'})(\delta_{mm'} + S_{mm'} + S_{m'm} + Z_{mm'}) + \delta_{0m'}\delta_{m0}(2 + Z_{00}) \\ & + \delta_{0m'}(S_{m0} + Z_{m0}) + \delta_{m0}(S_{m'0} + Z_{m'0}) \quad \text{for } m, m' = 0, 1, 2, 3, \dots . \end{aligned} \tag{36}$$

In comparison with the final equation derived in reference [1], the current PDE discretization scheme based on the Galerkin method has led to a few additional terms in the stiffness and mass matrices. Although the corresponding benefits cannot be fully realized yet at the moment, it should be pointed out that the stiffness and mass matrices have become symmetric here, which is highly desired numerically.

The modal properties of the beam can be readily determined from equation (33) by solving a standard eigenproblem. The components in each of the eigenvectors actually represent the expansion coefficients, A_m ($m = 0, 1, 2, \dots$), from which the corresponding mode can be readily obtained using equation (19). In numerical calculations equation (19) (and accordingly, equation (33)) has to be truncated to include only the first $M + 1$ terms (equations).

3. RESULTS AND DISCUSSIONS

As the first example, assume a beam that is clamped at $x = 0$, and simply supported at $x = L$. In addition, an elastic rotational spring of stiffness K_1 is applied to the right end,

TABLE 1

Frequency parameters, $\mu_i = a/\pi\ (\omega_i \sqrt{\rho A/D})^{1/2}$, for several different values of the stiffness $\hat{K}_1 L$

	$\mu_i = L/\pi\ (\omega_i \sqrt{\rho A/D})^{1/2}$				
Mode	$\hat{K}_1 L = 0$	$\hat{K}_1 L = 1$	$\hat{K}_1 L = 10$	$\hat{K}_1 L = 100$	$\hat{K}_1 L = 10^{10}$
1	1·24988	1·28656	1·4102	1·49137	1·50562
2	2·25	2·27077	2·37137	2·47681	2·49975
3	3·25	3·26479	3·3492	3·46883	3·50001
4	4·25001	4·26147	4·33354	4·46107	4·5

TABLE 2

Frequency parameters, $\mu_i = L/\pi\ (\omega_i \sqrt{\rho A/D})^{1/2}$, obtained using various numbers of terms in equation (19)

	$\mu_i = L/\pi\ (\omega_i \sqrt{\rho A/D})^{1/2}$			
Mode	$M = 5$	$M = 10$	$M = 15$	$M = 20$
1	1·50562 (1·50563)†	1·50562	1·50562	1·50562
2	2·49975 (2·49985)	2·49975 (2·49976)	2·49975	2·49975
3	3·50012 (3·50392)	3·50001 (3·50003)	3·50001	3·50001
4	4·50034 (4·5073)	4·5 (4·5002)	4·5 (4·50001)	4·5
5	—	5·5 (5·50044)	5·5 (5·50005)	5·5
6	—	6·5001 (6·50289)	6·5 (6·5001)	6·5 (6·50002)
7	—	7·50018 (7·50421)	7·5 (7·50045)	7·5 (7·50004)
8	—	8·5031 (8·52423)	8·50001 (8·5007)	8·5 (8·50014)
9	—	9·5042 (9·52852)	9·50008 (9·50251)	9·5 (9·50022)
10	—	—	10·5001 (10·5033)	10·5 (10·5007)

† Results in parentheses are taken from reference [1].

$x = L$. In the current study, the clamped condition can be essentially created by setting the stiffnesses, $\hat{k}_0 L^3$ and $\hat{K}_0 L$, of the constraining springs to be a very large number, say, 10^{10}. Similarly, the simply supported boundary condition at the other end can easily be created by setting $\hat{k}_1 L^3 = 10^{10}$ and $\hat{K}_1 L = 0$.

In Table 1, the first four frequency parameters, $\mu_i = a/\pi (\omega_i \sqrt{\rho A/D})^{1/2}$, are listed for several different values of the stiffness $\hat{K}_1 L$. For the two extreme values. $\hat{K}_1 L = 0$ and 10^{10}, the problem essentially turns into the classical clamped–simply supported and clamped–clamped cases. For them, the first four frequency parameters are, respectively, as follows [2]: $\mu_i = 1{\cdot}24988$, 2·25, 3·25, 4·25 and $\mu_i = 1{\cdot}50562$, 2·49975, 3·50001, 4·5. The excellent accuracy of the current solution is evident from the corresponding results in Table 1.

In the above calculations, the displacement expansion, equation (19), is truncated to $M = 10$. In order to examine the convergence of the solution. Table 2 compares, for the clamped–clamped case, the first 10 frequency parameters estimated using various numbers of terms in equation (19). The previous results (in parentheses) from reference [1] are also given in Table 2 for the purpose of comparison. The current solution has shown a meaningful improvement over the previous one.

TABLE 3

Frequency parameters, $\mu_i = (L^2\omega_i\sqrt{\rho A/D})^{1/2}$, *obtained using various numbers of terms in equation* (19)

Mode	$M = 5$	$M = 10$	$M = 20$	Reference [3]
1	1·188301	1·188301	1·188301	1·188301
2	3·144180	3·144180	3·144180	3·144179
3	6·227221 (6·22726)†	6·22722 (6·227224)	6·227220 (6·227221)	6·227220
4	9·336975 (9·33717)	9·336970 (9·337013)	9·336970 (9·336975)	9·336969
5	12·45011 (12·4514)	12·44988 (12·45001)	12·44988 (12·44990)	12·44988

† Results in parentheses are taken from reference [1].

Now, let us consider a more complicated boundary condition involving both rotational and translational restraints at each end. It is assumed that $\hat{k}_0L^3 = \hat{k}_1L^3 = 1$ and $\hat{K}_1L = \hat{K}_0L = 100$. Table 3 compares the five lowest frequency parameters, $\mu_i = (a^2\omega_i\sqrt{\rho A/D})^{1/2}$, calculated by including different numbers of terms in equation (19). It is seen that the results are identical for $M = 10$ and 20, indicating that the current solution has already converged for $M = 10$ with respect to these five frequency parameters. It should be noticed that the first four frequencies currently obtained with 6-terms are as accurate as the previous ones obtained with 21-terms. As a matter of fact, the fifth frequency will become 12·44990 if only one more term is added to the expansion (i.e., $M = 6$).

4. CONCLUSIONS

A unified technique based on the Galerkin method for PDE discretization is derived for determining the modal properties of beams arbitrarily restrained at the ends. Although the current stiffness and mass matrices appear more complicated than the original ones given in reference [1] they become numerically attractive because of their symmetric nature. As demonstrated by examples, the current solution is more accurate than the previous one based on the Fourier method. In addition, it converges at a truly remarkable speed, which is of critical importance to its future extension to the related two-dimensional problems.

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