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VIBRATIONS OF CONTINUOUS SYSTEMS WITH A GENERAL OPERATOR NOTATION SUITABLE FOR PERTURBATIVE CALCULATIONS

M. Pakdemirli

Department of Mechanical Engineering, Celal Bayar University, 45140 Muradiye, Manisa, Turkey. E-mail: pakdemirli@hotmail.com

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The operator notation previously developed to analyze vibrations of continuous systems has been further generalized to model a system with an arbitrary number of coupled differential equations. Linear parts of the equations are expressed with an arbitrary linear differential and/or integral operators, and non-linear parts are expressed with arbitrary quadratic and cubic operators. Equations of motion are solved in their general form using the method of multiple scales, a perturbation technique. The case of primary resonances of the external excitation and one-to-one internal resonances between the natural frequencies of the equations is considered. The algorithm developed is applied to a non-linear cable vibration problem having small sag-to-span ratios.

1. INTRODUCTION

A new operator notation suitable for perturbative calculations has been developed to analyze, in a general sense, vibrations of continuous systems by Pakdemirli [1]. One-mode approximation of a continuous system with arbitrary quadratic and cubic non-linearities was considered in that analysis. Infinite mode analysis of the same system was later performed by Pakdemirli and Boyaci [2]. Using the same model of differential equation, Pakdemirli and Boyaci [3] treated the subharmonic, superharmonic and combination resonances cases in detail. A model of a coupled partial differential system with arbitrary quadratic and cubic non-linearities was solved later [4]. Two different versions of the method of multiple scales were compared using general models, one with an arbitrary cubic operator and the other with arbitrary quadratic and cubic operators [5]. Using the same concept of operator notation, arbitrary odd-non-linearity models were also considered [6, 7].

In this study, the previous work is generalized to a system of an arbitrary number of coupled partial differential equations. The dimensionless equations of motion are

$$\ddot{w}_n + \hat{\mu}_n \dot{w}_n + \mathbf{L}_n(w_n) + \mathbf{Q}_{nmp}(w_m, w_p) + \mathbf{C}_{nmpq}(w_m, w_p, w_q) = \delta_{nl}\hat{F}_n \cos \Omega t, \qquad (1)$$

where n is the free index with $n = 1, 2, \ldots, N$, N being the number of equations. Summation is carried out over all other indices m, p and q from 1 to N. δ denotes the usual Kronecker delta function. $\hat{\mu}_n$ are the viscous damping coefficients. The linear parts of the equations are expressed by operator $\mathbf{L}_n$ and are uncoupled. The quadratic and cubic non-linearities are expressed by an arbitrary spatial differential and/or integral operators by $\mathbf{Q}_{nmp}$ and

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$\mathbf{C}_{nmpq}$ respectively. In a system of N equations, there are at most N^3 quadratic and N^4 cubic operators. The external excitation is assumed to be applied to the first equation only. $\hat{F}_n$ and Ω are the external excitation amplitude and frequency respectively. The quadratic and cubic operators possess the property of being multilinear so that

$$\mathbf{Q}(c_1 w_1 + c_2 w_2, c_3 w_3 + c_4 w_4) = c_1 c_3 \mathbf{Q}(w_1, w_3) + c_1 c_4 \mathbf{Q}(w_1, w_4) + c_2 c_3 \mathbf{Q}(w_2, w_3) + c_2 c_4 \mathbf{Q}(w_2, w_4), \tag{2}$$

$$\{\text{in general } \mathbf{Q}(w_1, w_2) \neq \mathbf{Q}(w_2, w_1)\},$$

where c_i are the arbitrary constants or time-varying coefficients. This property is essential in perturbative calculations and reflects well the properties of the original quadratic and cubic non-linearities.

It is assumed that the boundary conditions for equation (1) are linear, homogenous and free from time derivatives; that is

$$\mathbf{B}_1(w_n) = 0 \quad \text{at } x = 0, \qquad \mathbf{B}_2(w_n) = 0 \quad \text{at } x = 1, \tag{3}$$

where $\mathbf{B}_1$ and $\mathbf{B}_2$ are the arbitrary spatial linear operators.

Any continuous system having viscous damping and external excitation modelled with an arbitrary number of partial differential equations having non-linearities of a quadratic and cubic type can be represented by the general format given by equations (1) and (3). The model excludes visco-elastic effects, non-linear inertial effects and gyroscopic effects. Non-linear boundary conditions and multi-frequency excitations are also excluded.

2. A GENERAL APPROXIMATE SOLUTION

Equations (1) and (3) can be solved in its general form by applying the method of multiple scales [8, 9] directly to the differential equation system. The case of primary resonances of the external excitation and one-to-one internal resonances between the natural frequencies of the equations will be considered. Other types of internal resonances can be considered in a similar way. Assume approximate expansions of the form

$$w_n(x, t; \varepsilon) = \varepsilon w_{n1}(x, T_0, T_2) + \varepsilon^2 w_{n2}(x, T_0, T_2) + \varepsilon^2 w_{n3}(x, T_0, T_2) + \cdots, \tag{4}$$

where ε is the small dimensionless measure of the deflections w_n, $T_0 = t$ the usual fast time scale and $T_2 = \varepsilon^2 t$ the slow time scale. The later analysis shows that there is no $T_1 = \varepsilon t$ dependence, hence it is omitted from the beginning. Assuming a weakly non-linear system, damping and excitation coefficients are ordered so that their effects balance the cubic non-linearities [9]

$$\hat{\mu}_n = \varepsilon^2 \mu_n, \qquad \hat{F}_n = \varepsilon^3 F_n. \tag{5}$$

In terms of the new time variables, the derivatives become

$$\mathrm{d}/\mathrm{d}t = D_0 + \varepsilon^2 D_2, \qquad \mathrm{d}^2/\mathrm{d}t^2 = D_0^2 + 2\varepsilon^2 D_0 D_2, \tag{6}$$

where $D_k = \partial/\partial T_k$. Substituting equations (4)–(6) into equations (1) and separating at each order of ε, finally gives

$$O(\varepsilon):\; D_0^2 w_{n1} + \mathbf{L}_n(w_{n1}) = 0,$$

$$\mathbf{B_1}(w_{n1}) = 0 \quad \text{at } x = 0, \qquad \mathbf{B_2}(w_{n1}) = 0 \quad \text{at } x = 1. \tag{7}$$

$$O(\varepsilon^2):\; D_0^2 w_{n2} + \mathbf{L}_n(w_{n2}) = -\,\mathbf{Q}_{nmp}(w_{m1}, w_{p1}),$$

$$\mathbf{B_1}(w_{n2}) = 0 \quad \text{at } x = 0, \qquad \mathbf{B_2}(w_{n2}) = 0 \quad \text{at } x = 1. \tag{8}$$

$$O(\varepsilon^3):\; D_0^2 w_{n3} + \mathbf{L}_n(w_{n3}) = -\,2D_0D_2 w_{n1} - \mu_n D_0 w_{n1} - \mathbf{Q}_{nmp}(w_{m1}, w_{p2}) - \mathbf{Q}_{nmp}(w_{m2}, w_{p1})$$

$$-\,\mathbf{C}_{nmpq}(w_{m1}, w_{p1}, w_{q1}) + \delta_{n1} F_n \cos \Omega T_0,$$

$$\mathbf{B_1}(w_{n3}) = 0 \quad \text{at } x = 0, \qquad \mathbf{B_2}(w_{n3}) = 0 \quad \text{at } x = 1. \tag{9}$$

At order ε, the solutions are

$$w_{n1} = (A_n(T_2)\mathrm{e}^{\mathrm{i}\omega_n T_0} + \text{c.c.})\, Y_n(x), \qquad n = 1, 2, \ldots, N, \tag{10}$$

where c.c. stands for the complex conjugate of the preceding terms and $Y_n(x)$ satisfy the following equations:

$$\mathbf{L}_n(Y_n) - \omega_n^2 Y_n = 0, \quad n = 1, 2, \ldots, N,$$

$$\mathbf{B_1}(Y_n) = 0 \quad \text{at } x = 0, \qquad \mathbf{B_2}(Y_n) = 0 \quad \text{at } x = 1, \tag{11}$$

where ω_n are the eigenvalues and Y_n are the corresponding eigenfunctions. It is well known [8, 9] that when there is damping in the system, the modes that are not excited through the external frequency or through internal resonances decay in time. In the system considered, it is assumed that one of the natural frequencies of the first equation is excited through external excitation (primary resonance case is considered) and the energy of that mode is transferred to other modes of the remaining equations through internal resonances in such a way that one mode is activated through internal resonances in each equation.

Substituting solutions (10) at order ε to the right-hand side of the equations at order ε^2, gives solutions of the form

$$w_{n2} = (A_m A_p \mathrm{e}^{\mathrm{i}(\omega_m + \omega_p)T_0} + \text{c.c.})\xi_{nmp}(x) + (A_m \bar{A}_p \mathrm{e}^{\mathrm{i}(\omega_m - \omega_p)T_0} + \text{c.c.})\eta_{nmp}(x), \tag{12}$$

where functions $\xi_{nmp}(x)$ and $\eta_{nmp}(x)$ satisfy

$$\mathbf{L}_n(\xi_{nmp}) - (\omega_m + \omega_p)^2 \xi_{nmp} = -\,\mathbf{Q}_{nmp}(Y_m, Y_p), \quad n, m, p = 1, 2, \ldots, N,$$

$$\mathbf{B_1}(\xi_{nmp}) = 0 \quad \text{at } x = 0, \qquad \mathbf{B_2}(\xi_{nmp}) = 0 \quad \text{at } x = 1, \tag{13}$$

$$\mathbf{L}_n(\eta_{nmp}) - (\omega_m - \omega_p)^2 \eta_{nmp} = -\,\mathbf{Q}_{nmp}(Y_m, Y_p), \quad n, m, p = 1, 2, \ldots, N,$$

$$\mathbf{B_1}(\eta_{nmp}) = 0 \quad \text{at } x = 0, \qquad \mathbf{B_2}(\eta_{nmp}) = 0 \quad \text{at } x = 1. \tag{14}$$

There are N^3 equations in each line above. For solutions (12), summation should be carried out over indices m and p.

At order ε^3, it is assumed that the external excitation frequency is close to the natural frequency of the first equation and all natural frequencies of other equations are excited through one-to-one internal resonances; that is

$$\Omega = \omega_1 + \varepsilon^2\rho, \qquad \omega_n = \omega_1 + \varepsilon^2\sigma_n \quad (\sigma_1 = 0), \tag{15, 16}$$

where ρ and σ_n are detuning parameters of $O(1)$. Since the homogenous parts of equations (9) have non-trivial solutions, the inhomogenous equations (9) have a solution only if a solvability condition is satisfied [8]. To find this condition, their solution is expressed in the form

$$w_{n3} = \varphi_n(x, T_2)\mathrm{e}^{\mathrm{i}\omega_n T_0} + W_n(x, T_0, T_2) + \text{c.c.}, \tag{17}$$

where φ_n represent the secular and small-divisor solutions and W_n represent non-secular solutions without small-divisor terms. Substituting the solutions at order ε and ε^2 and equations (15)–(17) into equation (9), finally gives

$$\begin{aligned}\mathbf{L}_n(\varphi_n) - \omega_n^2\varphi_n = &- \mathrm{i}\omega_n(2D_2A_n + \mu_n A_n)Y_n - \bar{A}_m A_r A_q \mathrm{e}^{\mathrm{i}(\sigma_r+\sigma_q-\sigma_m-\sigma_n)T_2}[\mathbf{Q}_{nmp}(Y_m, \xi_{prq})\\ &+ \mathbf{Q}_{npm}(\xi_{pqr}, Y_m) + \mathbf{C}_{nmrq}(Y_m, Y_r, Y_q) + \mathbf{Q}_{nqp}(Y_q, \eta_{prm}) + \mathbf{Q}_{npq}(\eta_{pmr}, Y_q)\\ &+ \mathbf{C}_{nqrm}(Y_q, Y_r, Y_m) + \mathbf{Q}_{nrp}(Y_r, \eta_{pmq}) + \mathbf{Q}_{npr}(\eta_{pqm}, Y_r)\\ &+ \mathbf{C}_{nrmq}(Y_r, Y_m, Y_q)] + (1/2)\delta_{n1}F_n\mathrm{e}^{\mathrm{i}(\rho-\sigma_n)T_2}.\end{aligned} \tag{18}$$

Assuming that the linear operators $\mathbf{L}_n$ with the associated boundary conditions are self-adjoint, the solvability conditions for equations (18) are

$$\mathrm{i}\omega_n(2D_2A_n + \mu_n A_n) + \alpha_{nmrq}\bar{A}_m A_r A_q \mathrm{e}^{\mathrm{i}(\sigma_r+\sigma_q-\sigma_m-\sigma_n)T_2} - (1/2)\delta_{n1}\,\mathrm{f}_n\mathrm{e}^{\mathrm{i}(\rho-\sigma_n)T_2} = 0, \tag{19}$$

where

$$\begin{aligned}\alpha_{nmrq} = \int_0^1 & Y_n[\mathbf{Q}_{nmp}(Y_m, \xi_{prq}) + \mathbf{Q}_{npm}(\xi_{pqr}, Y_m) + \mathbf{Q}_{nqp}(Y_q, \eta_{prm}) + \mathbf{Q}_{npq}(\eta_{pmr}, Y_q)\\ &+ \mathbf{Q}_{nrp}(Y_r, \eta_{pmq}) + \mathbf{Q}_{npr}(\eta_{pqm}, Y_r) + \mathbf{C}_{nmrq}(Y_m, Y_r, Y_q) + \mathbf{C}_{nqrm}(Y_q, Y_r, Y_m)\\ &+ \mathbf{C}_{nrmq}(Y_r, Y_m, Y_q)]\,\mathrm{d}x,\end{aligned} \tag{20}$$

$$f_n = \int_0^1 Y_n F_n\,\mathrm{d}x. \tag{21}$$

Representing the solutions in this general form makes it convenient to see explicitly the contributions of each operator to the coefficients. $\int_0^1 Y_n^2\,\mathrm{d}x = 1$ normalizations are used in arriving at equation (19) which determines modulations of the complex amplitudes $A_n(T_2)$. Once the spatial solutions Y_n, ξ_{nmp} and η_{nmp} are known either analytically or numerically, the coefficients α_{nmrq} can be calculated numerically for specific operators.

Expressing the complex amplitudes in the polar form

$$A_n(T_2) = (1/2)a_n(T_2)\mathrm{e}^{\mathrm{i}\beta_n(T_2)} \tag{22}$$

and substituting into equation (19), finally gives the amplitude and phase modulation equations

$$\omega_n a_n' = -\tfrac{1}{2}\mu_n\omega_n a_n - \tfrac{1}{8}\alpha_{nmrq}a_m a_r a_q \sin(\gamma_n + \gamma_m - \gamma_r - \gamma_q) + \tfrac{1}{2}\delta_{n1} f_n \sin\gamma_n, \tag{23}$$

$$a_n\omega_n\gamma_n' = a_n\omega_n(\rho - \sigma_n) - \tfrac{1}{8}\alpha_{nmrq}a_m a_r a_q \cos(\gamma_n + \gamma_m - \gamma_r - \gamma_q) + \tfrac{1}{2}\delta_{n1} f_n \cos\gamma_n, \tag{24}$$

where

$$\gamma_n = (\rho - \sigma_n)T_2 - \beta_n. \tag{25}$$

The final approximate solutions for the problem are

$$w_n(x, t) = \varepsilon a_n \cos(\Omega t - \gamma_n) Y_n(x) + \tfrac{1}{2}\varepsilon^2 a_m a_p[\cos(2\Omega t - \gamma_m - \gamma_p)\xi_{nmp}(x)$$

$$+ \cos(\gamma_p - \gamma_m)\eta_{nmp}(x)] + O(\varepsilon^3), \tag{26}$$

where summation should be carried out over the indices m and p. The real amplitudes a_n and phases γ_n in equations (26) are governed by equations (23) and (24). This general solution algorithm will be applied to a specific problem in the next section.

3. APPLICATION TO A CABLE VIBRATION PROBLEM

In this section, the formalism derived in the previous section will be applied to a non-linear cable vibration problem with small sag-to-span ratios. Following the previous analysis, primary resonances of the external excitation and one-to-one internal resonances between the natural frequencies of the in-plane and out-of-plane vibrations will be investigated. The equations of motion first derived by Lee and Perkins [10] are

$$[v_t^2 + v_l^2 g(t)]w_1'' + \frac{v_l^2}{v_t^2} g(t) + \hat{F}(x)\cos\Omega t = \ddot{w}_1 + \hat{\mu}_1\dot{w}_1, \qquad [v_t^2 + v_l^2 g(t)]w_2'' = \ddot{w}_2 + \hat{\mu}_2\dot{w}_2, \tag{27}$$

$$g(t) = \int_0^1 \left\{-\frac{1}{v_t^2}w_1 + \frac{1}{2}[w_1'^2 + w_2'^2]\right\}\mathrm{d}x, \qquad w_{1,2}(0, t) = w_{1,2}(1, t) = 0,$$

where x is the dimensionless arclength co-ordinate and prime denotes differentiation with respect to x. The constants v_t and v_l are the dimensionless propagation speeds of transverse and longitutional waves respectively. w_1 is the in-plane and w_2 is the out-of-plane displacement.

The operators in equations (1) take the following form for this specific example:

$$\mathbf{L}_1(w_1) = -v_t^2 w_1'' + \frac{v_l^2}{v_t^4}\int_0^1 w_1\,\mathrm{d}x, \qquad \mathbf{L}_2(w_2) = -v_t^2 w_2'',$$

$$\mathbf{Q}_{111}(w_1, w_1) = \frac{v_l^2}{v_t^2}\left(w_1''\int_0^1 w_1\,\mathrm{d}x - \frac{1}{2}\int_0^1 w_1'^2\,\mathrm{d}x\right),$$

$$\mathbf{Q}_{122}(w_2, w_2) = -\frac{v_l^2}{2v_t^2}\int_0^1 w_2'^2\,\mathrm{d}x, \qquad \mathbf{Q}_{221}(w_2, w_1) = \frac{v_l^2}{v_t^2} w_1''\int_0^1 w_1\,\mathrm{d}x,$$

$$\mathbf{C}_{1111}(w_1, w_1, w_1) = -\frac{v_l^2}{2} w_1''\int_0^1 w_1'^2\,\mathrm{d}x, \tag{28}$$

$$\mathbf{C}_{1122}(w_1, w_2, w_2) = -\frac{v_l^2}{2} w_1''\int_0^1 w_2'^2\,\mathrm{d}x,$$

$$\mathbf{C}_{2211}(w_2, w_1, w_1) = -\frac{v_l^2}{2} w_2''\int_0^1 w_1'^2\,\mathrm{d}x,$$

$$\mathbf{C}_{2222}(w_2, w_2, w_2) = -\frac{v_l^2}{2} w_2''\int_0^1 w_2'^2\,\mathrm{d}x,$$

$$\mathbf{Q}_{112}(w_1, w_2) = \mathbf{Q}_{121}(w_2, w_1) = \mathbf{Q}_{211}(w_1, w_1) = \mathbf{Q}_{212}(w_1, w_2) = \mathbf{Q}_{222}(w_2, w_2) = 0,$$

$$\mathbf{C}_{1112}(w_1, w_1, w_2) = \mathbf{C}_{1121}(w_1, w_2, w_1) = \mathbf{C}_{1211}(w_2, w_1, w_1) = \mathbf{C}_{1212}(w_2, w_1, w_2) = 0,$$

$$\mathbf{C}_{1221}(w_2, w_2, w_1) = \mathbf{C}_{1222}(w_2, w_2, w_2) = \mathbf{C}_{2111}(w_1, w_1, w_1) = \mathbf{C}_{2112}(w_1, w_1, w_2) = 0,$$

$$\mathbf{C}_{2121}(w_1, w_2, w_1) = \mathbf{C}_{2122}(w_1, w_2, w_2) = \mathbf{C}_{2212}(w_2, w_1, w_2) = \mathbf{C}_{2221}(w_2, w_2, w_1) = 0.$$

Assuming expansions (4) for the displacements, solutions (10) are obtained for the linear problem, where eigenfunctions $Y_n(x)$ satisfy equations (11), or substituting the specific forms of the linear operators from equation (28) gives

$$v_t^2 Y_1'' + \omega_1^2 Y_1 - \frac{v_l^2}{v_t^4}\int_0^1 Y_1\,\mathrm{d}x = 0, \tag{29}$$

$$v_t^2 Y_2'' + \omega_2^2 Y_2 = 0, \qquad Y_{1,2}(0) = Y_{1,2}(0) = 0. \tag{30, 31}$$

Equation (29) with boundary conditions possesses two types of solutions, namely the symmetric and the antisymmetric in-plane solutions with respect to the mid-span of the cable. The symmetric in-plane solution is

$$Y_1(x) = C\left\{1 - \tan\left(\frac{\omega_1}{2v_t}\right)\sin\left(\frac{\omega_1}{v_t}x\right) - \cos\left(\frac{\omega_1}{v_t}x\right)\right\}, \tag{32}$$

where ω_1 satisfies the equation

$$\frac{\omega_1^3 v_t^3}{v_l^2} - \frac{\omega_1}{v_t} + 2\tan\left(\frac{\omega_1}{2v_t}\right) = 0. \tag{33}$$

Constant C should be chosen such that $\int_0^1 Y_1^2\,\mathrm{d}x = 1$. The antisymmetric in-plane solution is

$$Y_1(x) = \sqrt{2}\sin\left(\frac{\omega_1}{v_t}x\right), \qquad \frac{\omega_1}{v_t} = 2k\pi, \quad k = 1, 2, 3, \ldots . \tag{34}$$

The symmetric in-plane solution can account for stretching of the cable whereas the anti-symmetric solution corresponds to the zero stretching case [10].

The out-of-plane solution is

$$Y_2(x) = \sqrt{2}\sin\left(\frac{\omega_2}{v_t}x\right), \qquad \frac{\omega_2}{v_t} = k\pi, \quad k = 1, 2, 3, \ldots . \tag{35}$$

The next step is to find solutions at order ε^2. Equations (12) are the solutions at this order where $\xi_{nmp}(x)$ and $\eta_{nmp}(x)$ satisfy equations (13) and (14) or using the specific forms of the operators,

$$v_t^2\xi_{111}'' - \frac{v_l^2}{v_t^4}\int_0^1 \xi_{111}\,\mathrm{d}x + 4\omega_1^2\xi_{111} = \frac{v_l^2}{v_t^2}\left(-\frac{1}{2}\int_0^1 Y_1'^2\,\mathrm{d}x + Y_1''\int_0^1 Y_1\,\mathrm{d}x\right),$$

$$v_t^2\xi_{112}'' - \frac{v_l^2}{v_t^4}\int_0^1 \xi_{112}\,\mathrm{d}x + (\omega_1+\omega_2)^2\xi_{112} = 0,$$

$$v_t^2\xi_{121}'' - \frac{v_l^2}{v_t^4}\int_0^1 \xi_{121}\,\mathrm{d}x + (\omega_1+\omega_2)^2\xi_{121} = 0,$$

$$v_t^2\xi_{122}'' - \frac{v_l^2}{v_t^4}\int_0^1 \xi_{122}\,\mathrm{d}x + 4\omega_2^2\xi_{122} = -\frac{v_l^2}{2v_t^2}\int_0^1 Y_2'^2\,\mathrm{d}x,$$

$$v_t^2\eta_{111}'' - \frac{v_l^2}{v_t^4}\int_0^1 \eta_{111}\,\mathrm{d}x = \frac{v_l^2}{v_t^2}\left(-\frac{1}{2}\int_0^1 Y_1'^2\,\mathrm{d}x + Y_1''\int_0^1 Y_1\,\mathrm{d}x\right),$$

$$v_t^2\eta_{112}'' - \frac{v_l^2}{v_t^4}\int_0^1 \eta_{112}\,\mathrm{d}x + (\omega_1-\omega_2)^2\eta_{112} = 0,$$

$$v_t^2\eta_{121}'' - \frac{v_l^2}{v_t^4}\int_0^1 \eta_{121}\,\mathrm{d}x + (\omega_2-\omega_1)^2\eta_{121} = 0,$$

$$v_t^2\eta_{122}'' - \frac{v_l^2}{v_t^4}\int_0^1 \eta_{122}\,\mathrm{d}x = -\frac{v_l^2}{2v_t^2}\int_0^1 Y_2'^2\,\mathrm{d}x,$$

$$v_t^2\xi_{211}'' + 4\omega_1^2\xi_{211} = 0, \qquad v_t^2\xi_{212}'' + (\omega_1+\omega_2)^2\xi_{212} = 0, \tag{36}$$

$$v_t^2\xi''_{221} + (\omega_2+\omega_1)^2\xi_{221} = \frac{v_l^2}{v_t^2}Y''_2\int_0^1 Y_1\,\mathrm{d}x,$$

$$v_t^2\xi''_{222} + 4\omega_2^2\xi_{222} = 0, \qquad v_t^2\eta''_{211} = 0, \qquad v_t^2\eta''_{212} + (\omega_1-\omega_2)^2\eta_{212} = 0,$$

$$v_t^2\eta''_{221} + (\omega_2-\omega_1)^2\eta_{221} = \frac{v_l^2}{v_t^2}Y''_2\int_0^1 Y_1\,\mathrm{d}x, \qquad v_t^2\eta''_{222} = 0,$$

$$\xi_{nmp}(0) = \xi_{nmp}(1) = \eta_{nmp}(0) = \eta_{nmp}(1) = 0.$$

Solving the above set of equations,

$$\xi_{111} = \left(\frac{v_l^2}{2v_t^2}\int_0^1 Y_1'^2\,\mathrm{d}x - C^2\omega_1^4\right)\left\{\frac{v_l^2}{v_t^4}\left(1-\frac{v_t}{\omega_1}\tan\left(\frac{\omega_1}{v_t}\right)\right)-4\omega_1^2\right\}^{-1}$$

$$\times\left\{1-\tan\left(\frac{\omega_1}{v_t}\right)\sin\left(\frac{2\omega_1}{v_t}x\right)-\cos\left(\frac{2\omega_1}{v_t}x\right)\right\}-\frac{1}{3}C\omega_1^2Y_1(x),$$

$$\xi_{122} = \frac{\omega_2^2v_l^2}{2(4\omega_2^2v_t^4-v_l^2)}\left\{\cos\left(\frac{2\omega_2}{v_t}x\right)-1\right\},$$

$$\eta_{111} = \frac{1}{12v_t^6+v_l^2}\left\{6C^2\omega_1^4v_t^4 - 3v_t^2v_l^2\int_0^1 Y_1'^2\,\mathrm{d}x\right\}(x^2-x) + C\omega_1^2Y_1(x),$$

$$\eta_{122} = -\frac{3\omega_2^2v_l^2}{12v_t^6+v_l^2}(x^2-x), \qquad \xi_{221} = \frac{\sqrt{2}C\omega_1^2\omega_2^2}{\omega_2^2-(\omega_1+\omega_2)^2}\sin\left(\frac{\omega_2}{v_t}x\right), \tag{37}$$

$$\eta_{221} = \frac{\sqrt{2}C\omega_1^2\omega_2^2}{\omega_2^2-(\omega_1-\omega_2)^2}\sin\left(\frac{\omega_2}{v_t}x\right),$$

$$\xi_{112} = \xi_{121} = \eta_{112} = \eta_{121} = \xi_{211} = \xi_{212} = \xi_{222} = \eta_{211} = \eta_{212} = \eta_{222} = 0.$$

At order ε^3, the solvability conditions given in equations (19) are obtained, where the α_{nmrq} coefficients are defined in equations (20). Substituting the specific form of the operators into equations (20), gives the coefficients

$$\alpha_{1111} = -\left\{\frac{v_l^2}{v_t^2}(b_2b_6+2b_1b_8+2b_3b_6+4b_1b_9)-\frac{3}{2}v_l^2b_6^2\right\},$$

$$\alpha_{1122} = -\left\{\frac{v_l^2}{v_t^2}(b_4b_6+2b_1b_{10}+b_1b_{13})-\frac{1}{2}v_l^2b_6b_7\right\},$$

$$\alpha_{1212} = -\left\{\frac{v_l^2}{v_t^2}\left(\frac{1}{2}b_1b_{12}+\frac{1}{2}b_1b_{13}+b_5b_6+2b_1b_{11}\right)-\frac{1}{2}v_l^2b_6b_7\right\},$$

$$\alpha_{1221} = -\left\{\frac{v_l^2}{v_t^2}\left(\frac{1}{2}b_1 b_{12} + b_5 b_6 + 2b_1 b_{11} + \frac{1}{2}b_1 b_{13}\right) - \frac{1}{2}v_l^2 b_6 b_7\right\},$$

$$\alpha_{2112} = -\left\{\frac{v_l^2}{v_t^2}(b_1 b_{12} + b_1 b_{13} + b_3 b_7) - \frac{1}{2}v_l^2 b_6 b_7\right\}, \tag{38}$$

$$\alpha_{2121} = -\left\{\frac{v_l^2}{v_t^2}(b_3 b_7) - \frac{1}{2}v_l^2 b_6 b_7\right\}, \qquad \alpha_{2211} = -\left\{\frac{v_l^2}{v_t^2}(b_2 b_7 + b_1 b_{13}) - \frac{1}{2}v_l^2 b_6 b_7\right\},$$

$$\alpha_{2222} = -\left\{\frac{v_l^2}{v_t^2}(b_4 b_7 + 2b_5 b_7) - \frac{3}{2}v_l^2 b_7^2\right\},$$

$$\alpha_{1112} = \alpha_{1121} = \alpha_{1211} = \alpha_{1222} = \alpha_{2111} = \alpha_{2122} = \alpha_{2212} = \alpha_{2221} = 0,$$

where

$$b_1 = \int_0^1 Y_1\,\mathrm{d}x, \qquad b_2 = \int_0^1 \xi_{111}\,\mathrm{d}x, \qquad b_3 = \int_0^1 \eta_{111}\,\mathrm{d}x,$$

$$b_4 = \int_0^1 \xi_{122}\,\mathrm{d}x, \qquad b_5 = \int_0^1 \eta_{122}\,\mathrm{d}x, \qquad b_6 = \int_0^1 Y_1'^2\,\mathrm{d}x$$

$$b_7 = \int_0^1 Y_2'^2\,\mathrm{d}x, \qquad b_8 = \int_0^1 Y'_1\xi'_{111}\,\mathrm{d}x, \qquad b_9 = \int_0^1 Y'_1\eta'_{111}\,\mathrm{d}x, \qquad b_{10} = \int_0^1 Y'_1\xi'_{122}\,\mathrm{d}x,$$

$$b_{11} = \int_0^1 Y'_1\eta'_{122}\,\mathrm{d}x, \qquad b_{12} = \int_0^1 Y'_2\xi'_{221}\,\mathrm{d}x, \qquad b_{13} = \int_0^1 Y'_2\eta'_{221}\,\mathrm{d}x. \tag{39}$$

From equations (26), and substituting the specific forms, the approximate solutions are found to be

$$w_1(x,t) = \varepsilon a_1 \cos(\Omega t - \gamma_1) Y_1(x) + \frac{\varepsilon^2}{2}\{a_1^2[\cos(2(\Omega t - \gamma_1))\xi_{111}(x) + \eta_{111}(x)]$$

$$+ a_2^2[\cos(2(\Omega t - \gamma_2))\xi_{122}(x) + \eta_{122}(x)]\} + O(\varepsilon^3), \tag{40}$$

$$w_2(x,t) = \varepsilon a_2 \cos(\Omega t - \gamma_2) Y_2(x) + \frac{\varepsilon^2}{2}\{a_1 a_2[\cos(2\Omega t - \gamma_1 - \gamma_2)\xi_{221}(x)$$

$$+ \cos(\gamma_1 - \gamma_2)\eta_{221}(x)]\} + O(\varepsilon^3), \tag{41}$$

where the amplitudes and phases are governed by equations (23) and (24). Substituting the specific forms, the amplitude and phase modulation equations take the following form:

$$
\begin{aligned}
a_1' &= -\frac{1}{2}\mu_1 a_1 - \frac{1}{8\omega_1}\alpha_{1122} a_1 a_2^2 \sin(2(\gamma_1-\gamma_2)) + \frac{f_1}{2\omega_1}\sin\gamma_1, \\
a_2' &= -\frac{1}{2}\mu_2 a_2 - \frac{1}{8\omega_2}\alpha_{2211} a_1^2 a_2 \sin(2(\gamma_2-\gamma_1)), \\
\gamma_1' &= \rho - \frac{1}{8\omega_1}\alpha_{1111} a_1^2 - \frac{1}{8\omega_1}(\alpha_{1212}+\alpha_{1221}) a_2^2 \\
&\quad - \frac{1}{8\omega_1}\alpha_{1122} a_2^2 \cos(2(\gamma_1-\gamma_2)) + \frac{f_1}{2a_1\omega_1}\cos\gamma_1, \\
\gamma_2' &= \rho - \sigma_2 - \frac{1}{8\omega_2}(\alpha_{2112}+\alpha_{2121}) a_1^2 - \frac{1}{8\omega_2}\alpha_{2211} a_1^2 \cos(2(\gamma_2-\gamma_1)) - \frac{1}{8\omega_2}\alpha_{2222} a_2^2 .
\end{aligned}
\tag{42}
$$

With the proper transformations, the results presented here are fully compatible with those given in references [4, 11]. Reference [11] includes a detailed stability and bifurcation analysis of the amplitude and phase modulation equations.

4. CONCLUDING REMARKS

The operator notation previously developed to treat vibration problems in a general sense has been generalized to express a system of N coupled differential equations. The model is a generalization of many vibration problems in continuous systems. Approximate solutions of the model are presented in a general form so that an algorithm can be constructed for solutions of a wide range of specific problems. As an illustration, the algorithm is used to solve a non-linear cable vibration problem.

In this study only the arbitrary linear and homogenous boundary conditions are considered. Non-linear boundary conditions can be added as a next step. Different internal resonance cases other than one-to-one can be considered in a similar way. Numerical solutions can be sought when it is hard to find explicitly the functions appearing at the first and second orders of approximation. This will not involve any problem at the last level of approximation, since the coefficients of the modulation equations are presented in a suitable way to allow further numerical calculations.

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