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APPROXIMATE DETERMINATION OF DISPERSION RELATIONS AND DISPLACEMENT FIELDS ASSOCIATED WITH ELASTIC WAVES IN BARS

Method Based on Matrix Formulation of Hamilton's Principle

S. WIDEHAMMAR AND P. A. GRADIN

Mid Sweden University, SE-851 70 Sundsvall, Sweden

AND

B. LUNDBERG

The Ångström Laboratory, Uppsala University, P.O. Box 534, SE-751 21 Uppsala, Sweden

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A method for approximate determination of dispersion relations and displacement fields associated with elastic waves in long prismatic bars is established. The method is based on a matrix formulation of Hamilton's principle and utilizes co-ordinate functions to model the displacements in the bar. The method, which involves the solution of an eigenvalue problem, is suitable for use in a mathematical toolbox. It is applied to wave propagation in bars with square and circular cross-sections. The results obtained agree well with the known exact and approximate solutions. The convergence rate is illustrated by systematically changing the order of the co-ordinate functions.

1. INTRODUCTION

Elastic wave propagation in bars has been a topic of interest to scientists and engineers for a long period of time, see, e.g., the book by Kolsky [1] where the exact solutions for a bar with circular cross-section are presented for different modes of wave propagation such as transverse, longitudinal and torsional. These solutions can also be found, e.g., in references [2–7]. Also, for elliptical cross-sections, there are exact solutions in terms of infinite series [8, 9]. For other shapes of the cross-section, exact solutions do not seem to exist. Different approaches have been used to find the approximate solutions for bars with circular as well as arbitrary cross-sections. Examples are variational methods [10–12], finite element methods [13–15], collocation methods [16, 17] and series solutions [18].

In the recent work done by Volovoi *et al.* [14] and Taweel *et al.* [15], a matrix formulation of Hamilton's principle is used to formulate a finite element problem. In this paper, a similar matrix formulation is used, but instead of finite elements, use is made of co-ordinate functions valid in the whole cross-section. The co-ordinate functions are polynomials or Fourier series in terms of the cross-sectional co-ordinates, similar to what has been used by, e.g., Nigro [10] and Boström [18]. This leads to an eigenvalue problem where the wavenumber is considered to be known and the angular frequency is determined as an eigenvalue, or *vice versa*. The eigenvectors contain the generalized co-ordinates which are coupled to the co-ordinate functions and determine the distribution of displacements over the cross-section.

The salient feature of the method presented in this paper is that it combines the matrix formulation characteristic of the finite element method with classical series solutions. This feature makes it easy to set-up and solve a problem with the aid of a mathematical toolbox such as Matlab [19] to an arbitrary order of the co-ordinate functions without use of a finite element grid. Also, the solution can be interpreted and understood from the properties of the eigenvectors. Another advantage is that a specific mode of wave propagation, e.g., longitudinal, can be studied by using co-ordinate functions which have appropriate symmetry. In this way, the size of the matrices, i.e., of the problem, becomes smaller than for general co-ordinate functions which cover all modes of wave propagation.

The method will be demonstrated for bars with isotropic material and square or circular cross-section. First, co-ordinate functions for general waves, i.e., all modes of wave propagation, will be used. Then, the results will be specialized to longitudinal waves. The results will be compared to a finite element solution obtained by Aalami [13] for the bar with square cross-section, and to exact solutions for the bar with circular cross-section.

2. MATRIX FORMULATION OF HAMILTON'S PRINCIPLE

2.1. GENERAL CASE FOR LINEARLY ELASTIC MATERIAL

In the absence of external forces, Hamilton's principle states that

$$\delta\left(\int_{t_1}^{t_2}(T-U)\,\mathrm{d}t\right)=0, \tag{1}$$

where T is the kinetic energy, U is the elastic strain energy and t_1 and t_2 are two arbitrary instants of time t. The kinetic energy is given by

$$T=\iiint_V\left(\frac{1}{2}\rho\dot{\mathbf{u}}^{\mathrm{T}}\dot{\mathbf{u}}\right)\mathrm{d}V, \tag{2}$$

where ρ is the mass density,

$$\mathbf{u}=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} \tag{3}$$

is the displacement vector, V is the volume of the body, and a 'dot' denotes partial differentiation with respect to time t. For a linearly elastic material, the generalized Hooke's law is given by

$$\boldsymbol{\tau}=\mathbf{C}\boldsymbol{\varepsilon}, \tag{4}$$

where $\mathbf{C}$ is a symmetric 6×6 matrix containing elastic constants, and

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{12} \\ \tau_{23} \\ \tau_{31} \end{pmatrix}, \quad \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{pmatrix} \tag{5}$$

are vectors containing the components of stress and strain respectively. In terms of the components of strain, the elastic strain energy of the body is

$$U = \iiint_V \left(\frac{1}{2} \boldsymbol{\varepsilon}^{\mathrm{T}} \mathbf{C} \boldsymbol{\varepsilon} \right) \mathrm{d}V. \tag{6}$$

The vector of strain components $\boldsymbol{\varepsilon}$ and the displacement vector $\mathbf{u}$ are related through the kinematic relation

$$\boldsymbol{\varepsilon} = \boldsymbol{\nabla} \mathbf{u}, \tag{7}$$

where $\boldsymbol{\nabla}$ is a 6×3 partial differential operator matrix. Substitution of equation (7) into equation (6) gives

$$U = \iiint_V \left(\frac{1}{2} (\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}} \mathbf{C} \boldsymbol{\nabla}\mathbf{u} \right) \mathrm{d}V. \tag{8}$$

Through a combination of equations (1), (2) and (8), Hamilton's principle can be expressed as

$$\delta \left(\int_{t_1}^{t_2} \iiint_V \left(\frac{1}{2} \rho \dot{\mathbf{u}}^{\mathrm{T}} \dot{\mathbf{u}} - \frac{1}{2} (\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}} \mathbf{C} \boldsymbol{\nabla}\mathbf{u} \right) \mathrm{d}V \, \mathrm{d}t \right) = 0. \tag{9}$$

Performing the variational operation, and carrying out a partial integration with respect to time, one obtains

$$\int_{t_1}^{t_2} \iiint_V (-\rho \, \delta\mathbf{u}^{\mathrm{T}} \ddot{\mathbf{u}} - (\boldsymbol{\nabla}\delta\mathbf{u})^{\mathrm{T}} \mathbf{C} \boldsymbol{\nabla}\mathbf{u}) \, \mathrm{d}V \, \mathrm{d}t = 0. \tag{10}$$

Here, use has been made of the conditions $\delta\mathbf{u} = 0$ at $t = t_1$ and t_2, the fact that the transpose of a scalar is the scalar itself, and the symmetry of $\mathbf{C}$.

2.2. ELASTIC WAVES IN A PRISMATIC BAR

Elastic waves in an infinitely long prismatic bar with axis x_3 are now considered. It is assumed that the displacements can be expressed as the product

$$\mathbf{u}(x_1, x_2, x_3, t) = \boldsymbol{\Phi}(x_1, x_2) \mathbf{h}(x_3, t), \tag{11}$$

of a dimensionless $3 \times m$ matrix $\mathbf{\Phi}(x_1, x_2)$ with the given co-ordinate functions as elements, and a vector $\mathbf{h}(x_3, t)$ with m elements which have the dimension of length. This separation leads to $\ddot{\mathbf{u}} = \mathbf{\Phi}\ddot{\mathbf{h}}$, and $\delta\mathbf{u} = \mathbf{\Phi}\,\delta\mathbf{h}$. It is also assumed that the operator matrix $\mathbf{\nabla}$ be expressed as the sum

$$\mathbf{\nabla} = \mathbf{\nabla}_{12} + \mathbf{\nabla}_3 \frac{\partial}{\partial x_3}, \tag{12}$$

where $\mathbf{\nabla}_{12}$ contains partial differential operators in the variables x_1 and x_2 and $\mathbf{\nabla}_3$ contains constant elements. This gives

$$\mathbf{\nabla}\mathbf{u} = \mathbf{\nabla}_{12}\mathbf{\Phi}\mathbf{h} + \mathbf{\nabla}_3\mathbf{\Phi}\mathbf{h}', \qquad \mathbf{\nabla}\delta\mathbf{u} = \mathbf{\nabla}_{12}\mathbf{\Phi}\,\delta\mathbf{h} + \mathbf{\nabla}_3\mathbf{\Phi}\delta\mathbf{h}', \tag{13}$$

where a 'prime' denotes partial differentiation with respect to x_3. Inserting equations (11) and (13) into equation (10), one obtains

$$\begin{aligned}\int_{t_1}^{t_2}\iiint_V \big(&-\rho\delta\mathbf{h}^{\mathrm{T}}\mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi}\ddot{\mathbf{h}} \\ &- \delta\mathbf{h}^{\mathrm{T}}(\mathbf{\nabla}_{12}\mathbf{\Phi})^{\mathrm{T}}\mathbf{C}\mathbf{\nabla}_{12}\mathbf{\Phi}\mathbf{h} - (\delta\mathbf{h}')^{\mathrm{T}}\mathbf{\Phi}^{\mathrm{T}}\mathbf{\nabla}_3^{\mathrm{T}}\mathbf{C}\mathbf{\nabla}_3\mathbf{\Phi}\mathbf{h}' \\ &- \delta\mathbf{h}^{\mathrm{T}}(\mathbf{\nabla}_{12}\mathbf{\Phi})^{\mathrm{T}}\mathbf{C}\mathbf{\nabla}_3\mathbf{\Phi}\mathbf{h}' - (\delta\mathbf{h}')^{\mathrm{T}}\mathbf{\Phi}^{\mathrm{T}}\mathbf{\nabla}_3^{\mathrm{T}}\mathbf{C}\mathbf{\nabla}_{12}\mathbf{\Phi}\mathbf{h}\big)\,\mathrm{d}V\,\mathrm{d}t = 0.\end{aligned} \tag{14}$$

Partial integration in the axial direction x_3 leads to

$$\begin{aligned}\int_{t_1}^{t_2}\int_a^b\iint_A &\delta\mathbf{h}^{\mathrm{T}}\big(-\rho\mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi}\ddot{\mathbf{h}} - (\mathbf{\nabla}_{12}\mathbf{\Phi})^{\mathrm{T}}\mathbf{C}\mathbf{\nabla}_{12}\mathbf{\Phi}\mathbf{h} + \mathbf{\Phi}^{\mathrm{T}}\mathbf{\nabla}_3^{\mathrm{T}}\mathbf{C}\mathbf{\nabla}_3\mathbf{\Phi}\mathbf{h}'' \\ &- (\mathbf{\nabla}_{12}\mathbf{\Phi})^{\mathrm{T}}\mathbf{C}\mathbf{\nabla}_3\mathbf{\Phi}\mathbf{h}' + \mathbf{\Phi}^{\mathrm{T}}\mathbf{\nabla}_3^{\mathrm{T}}\mathbf{C}\mathbf{\nabla}_{12}\mathbf{\Phi}\mathbf{h}'\big)\,\mathrm{d}A\,\mathrm{d}x_3\,\mathrm{d}t \\ &- \int_{t_1}^{t_2}\iint_A \big[\delta\mathbf{h}^{\mathrm{T}}\mathbf{\Phi}^{\mathrm{T}}\mathbf{\nabla}_3^{\mathrm{T}}\mathbf{C}(\mathbf{\nabla}_3\mathbf{\Phi}\mathbf{h}' + \mathbf{\nabla}_{12}\mathbf{\Phi}\mathbf{h})\big]_a^b\,\mathrm{d}A\,\mathrm{d}t = 0,\end{aligned} \tag{15}$$

where A is the cross-sectional area of the bar. The last integral would provide boundary conditions at $x_3 = a$ and b for a finite bar, but will be left out hence-forth as the bar considered has infinite length. Furthermore, since $\delta\mathbf{h}(x_3, t)$ is arbitrary and $\mathbf{h}(x_3, t)$ is independent of the cross-sectional co-ordinates x_1 and x_2, equation (15) yields

$$\begin{aligned}&\left[\rho\iint_A \mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi}\,\mathrm{d}A\right]\ddot{\mathbf{h}} + \left[\iint_A (\mathbf{\nabla}_{12}\mathbf{\Phi})^{\mathrm{T}}\mathbf{C}\mathbf{\nabla}_{12}\mathbf{\Phi}\,\mathrm{d}A\right]\mathbf{h} - \left[\iint_A \mathbf{\Phi}^{\mathrm{T}}\mathbf{\nabla}_3^{\mathrm{T}}\mathbf{C}\mathbf{\nabla}_3\mathbf{\Phi}\,\mathrm{d}A\right]\mathbf{h}'' \\ &+ \left[\iint_A ((\mathbf{\nabla}_{12}\mathbf{\Phi})^{\mathrm{T}}\mathbf{C}\mathbf{\nabla}_3\mathbf{\Phi} - \mathbf{\Phi}^{\mathrm{T}}\mathbf{\nabla}_3^{\mathrm{T}}\mathbf{C}\mathbf{\nabla}_{12}\mathbf{\Phi})\,\mathrm{d}A\right]\mathbf{h}' = \mathbf{0},\end{aligned} \tag{16}$$

where $\mathbf{0}$ is the null vector. This represents a system of m partial differential equations for the m elements of $\mathbf{h}(x_3, t)$.

2.3. DIMENSIONLESS VARIABLES

Reference quantities are now introduced in order to make all the variables and equations dimensionless. For length, a characteristic transverse dimension x_0 such as the radius of a circular cross-section, or half the side length of a square cross-section, is used. The reference modulus E_0 is taken to be a characteristic modulus with dimension of stress such as Young's modulus if the material is isotropic. For phase velocity, the reference velocity

$$c_0 = \sqrt{E_0/\rho} \tag{17}$$

is introduced. From these variables, the reference time, angular frequency and wavenumber are formed by

$$t_0 = x_0/c_0, \quad \omega_0 = c_0/x_0, \quad k_0 = 1/x_0 \tag{18}$$

respectively.

By using these reference quantities, equation (16) can be rewritten in the dimensionless form

$$\left[\iint_{\bar{A}} \mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi}\,\mathrm{d}\bar{A}\right]\ddot{\bar{\mathbf{h}}} + \left[\iint_{\bar{A}} (\bar{\mathbf{V}}_{12}\mathbf{\Phi})^{\mathrm{T}}\,\bar{\mathbf{C}}\bar{\mathbf{V}}_{12}\mathbf{\Phi}\,\mathrm{d}\bar{A}\right]\bar{\mathbf{h}} - \left[\iint_{\bar{A}} \mathbf{\Phi}^{\mathrm{T}}\mathbf{V}_3^{\mathrm{T}}\bar{\mathbf{C}}\mathbf{V}_3\mathbf{\Phi}\,\mathrm{d}\bar{A}\right]\bar{\mathbf{h}}''$$

$$+ \left[\iint_{\bar{A}} ((\bar{\mathbf{V}}_{12}\mathbf{\Phi})^{\mathrm{T}}\bar{\mathbf{C}}\mathbf{V}_3\mathbf{\Phi} - \mathbf{\Phi}^{\mathrm{T}}\mathbf{V}_3^{\mathrm{T}}\bar{\mathbf{C}}\bar{\mathbf{V}}_{12}\mathbf{\Phi})\,\mathrm{d}\bar{A}\right]\bar{\mathbf{h}}' = \mathbf{0}, \tag{19}$$

where an overbar indicates a dimensionless quantity. The partial differentiations are to be carried out with respect to the dimensionless quantities. *From here on, all variables will be dimensionless, and therefore, the overbars will be left out.*

2.4. HARMONIC WAVES

Consider now the harmonic waves

$$\mathbf{h} = \mathbf{d}\mathrm{e}^{\mathrm{i}(\omega t + kx_3)}, \tag{20}$$

where $\mathbf{d}$ is a constant vector of complex amplitudes corresponding to the co-ordinate functions in $\mathbf{\Phi}$, and k and ω are the wavenumber and the angular frequency, respectively. Inserting $\mathbf{h}$ into equation (19) and dividing through by the exponential factor, one obtains the eigenvalue problem

$$\Bigg[-\omega^2\iint_A \mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi}\,\mathrm{dA} + \iint_A (\mathbf{\nabla}_{12}\mathbf{\Phi})^{\mathrm{T}}\mathbf{C}\mathbf{\nabla}_{12}\mathbf{\Phi}\,\mathrm{dA} + k^2\iint_A \mathbf{\Phi}^{\mathrm{T}}\mathbf{\nabla}_3^{\mathrm{T}}\mathbf{C}\mathbf{\nabla}_3\mathbf{\Phi}\,\mathrm{d}A$$

$$+\,\mathrm{i}k\iint_A ((\mathbf{\nabla}_{12}\mathbf{\Phi})^{\mathrm{T}}\mathbf{C}\mathbf{\nabla}_3\mathbf{\Phi} - \mathbf{\Phi}^{\mathrm{T}}\mathbf{\nabla}_3^{\mathrm{T}}\mathbf{C}\mathbf{\nabla}_{12}\mathbf{\Phi})\,\mathrm{d}A\Bigg]\mathbf{d} = \mathbf{0}, \tag{21}$$

or

$$[\mathbf{K}_0 + \mathrm{i}k\mathbf{K}_1 + k^2\mathbf{K}_2 - \omega^2\mathbf{M}]\,\mathbf{d} = \mathbf{0}, \tag{22}$$

where

$$\mathbf{K}_0 = \iint_A (\mathbf{\nabla}_{12}\mathbf{\Phi})^{\mathrm{T}}\mathbf{C}\mathbf{\nabla}_{12}\mathbf{\Phi}\,\mathrm{d}A, \tag{23}$$

$$\mathbf{K}_1 = \iint_A ((\mathbf{\nabla}_{12}\mathbf{\Phi})^{\mathrm{T}}\mathbf{C}\mathbf{\nabla}_3\mathbf{\Phi} - \mathbf{\Phi}^{\mathrm{T}}\mathbf{\nabla}_3^{\mathrm{T}}\mathbf{C}\mathbf{\nabla}_{12}\mathbf{\Phi})\,\mathrm{d}A, \tag{24}$$

$$\mathbf{K}_2 = \iint_A \mathbf{\Phi}^{\mathrm{T}}\mathbf{\nabla}_3^{\mathrm{T}}\mathbf{C}\mathbf{\nabla}_3\mathbf{\Phi}\,\mathrm{d}A \tag{25}$$

and

$$\mathbf{M} = \iint_A \mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi}\,\mathrm{d}A. \tag{26}$$

This eigenvalue problem relates the wavenumber k to the angular frequency ω, one of them being given and the other being the eigenvalue to be solved for. If k is given, equation (22) is a generalized eigenvalue problem with m real eigenvalues ω^2. If instead ω is given, equation (22) is a quadratic eigenvalue problem with $2m$ eigenvalues k. In the latter case, the eigenvalues are generally complex valued, and non-real eigenvalues correspond to end modes [14]. These modes decay exponentially at the ends of a finite bar and will not be considered here.

3. COMPUTATIONAL DETAILS

Techniques to solve the eigenvalue problems represented by equation (22) and a thorough discussion of the solutions can be found in, e.g., the papers by Volovoi *et al.* [14] and Taweel *et al.* [15]. In this paper, the standard commands in Matlab [19], i.e., '*eig*' and '*polyeig*' were used. The area integrations in equations (23)–(26) were carried out exactly. For simple cross-section geometries, this can be done for the arbitrary order of the co-ordinate functions and can be implemented in Matlab. For more complicated geometries of the cross-section, it is recommended to use numerical integration.

Examples of the matrix $\mathbf{\Phi}$ for square and circular cross-sections will follow. Corresponding explicit expressions for the matrices $\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2$ and $\mathbf{M}$ can be found in Appendix A.

4. ISOTROPIC MATERIAL

So far, it has been assumed that the material is generally anisotropic. If the material is isotropic, then the matrix $\mathbf{C}$, normalized with respect to Young's modulus E, becomes

$$\mathbf{C} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_2 & 0 & 0 & 0 \\ \alpha_2 & \alpha_1 & \alpha_2 & 0 & 0 & 0 \\ \alpha_2 & \alpha_2 & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_3 \end{pmatrix}, \tag{27}$$

where

$$\alpha_1 = \frac{1-v}{(1+v)(1-2v)}, \quad \alpha_2 = \frac{v}{(1+v)(1-2v)}, \quad \alpha_3 = \frac{1}{2(1+v)} \tag{28}$$

and v is the Poisson ratio.

5. SQUARE CROSS-SECTION

5.1. CARTESIAN CO-ORDINATES

For bars with square cross-section, Cartesian co-ordinates, $x_1 = x, x_2 = y$ and $x_3 = z$, normalized with respect to half the side length of the cross-section, will be used. In this case, the operator matrices are

$$\mathbf{\nabla}_{12} = \begin{pmatrix} \dfrac{\partial}{\partial x} & 0 & 0 \\ 0 & \dfrac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial y} \\ 0 & 0 & \dfrac{\partial}{\partial x} \end{pmatrix} \tag{29}$$

and

$$\mathbf{V}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{30}$$

5.2. GENERAL WAVES

Polynomials in x and y are used to form a complete set of co-ordinate functions which can be used to express an arbitrary, unique and continuous displacement field. Thus, the matrix $\mathbf{\Phi}$ of co-ordinate functions is expressed as

$$\mathbf{\Phi} = (\mathbf{\Phi}_0 \;\; \mathbf{\Phi}_1 \;\; \mathbf{\Phi}_2 \;\; \ldots \;\; \mathbf{\Phi}_n), \tag{31}$$

where

$$\mathbf{\Phi}_k = x^k \begin{pmatrix} \boldsymbol{\phi}^{\mathrm{T}}(y) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\phi}^{\mathrm{T}}(y) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\phi}^{\mathrm{T}}(y) \end{pmatrix} \tag{32}$$

with

$$\boldsymbol{\phi}^{\mathrm{T}}(y) = (1 \quad y \quad y^2 \quad \ldots \quad y^n). \tag{33}$$

In Figure 1, dispersion curves (angular frequency ω versus given wavenumber k) for a bar with square cross-section obtained with $n = 3$ are compared with the results of Aalami [13]. The displacement fields

$$\mathbf{u} = \Re\{\mathbf{\Phi}\mathbf{d}\mathrm{e}^{\mathrm{i}(\omega t + kx_3)}\} \tag{34}$$

($\Re$ denoting the real part) for the corresponding modes are illustrated in Figure 2.

5.3. LONGITUDINAL WAVES

If a special kind of waves are of interest, inherent symmetries can be used in order to reduce the number of unknowns. As an example, longitudinal waves with a major displacement in the axial direction, and symmetry for the displacement vector with regard to the four planes $x = 0$, $y = 0$, $x = y$ and $x = -y$, will be considered. The matrix $\mathbf{\Phi}$ can then be written as

$$\mathbf{\Phi} = (\mathbf{\Phi}_0 \;\; \mathbf{\Phi}_2 \;\; \mathbf{\Phi}_4 \;\; \ldots \;\; \mathbf{\Phi}_n), \tag{35}$$

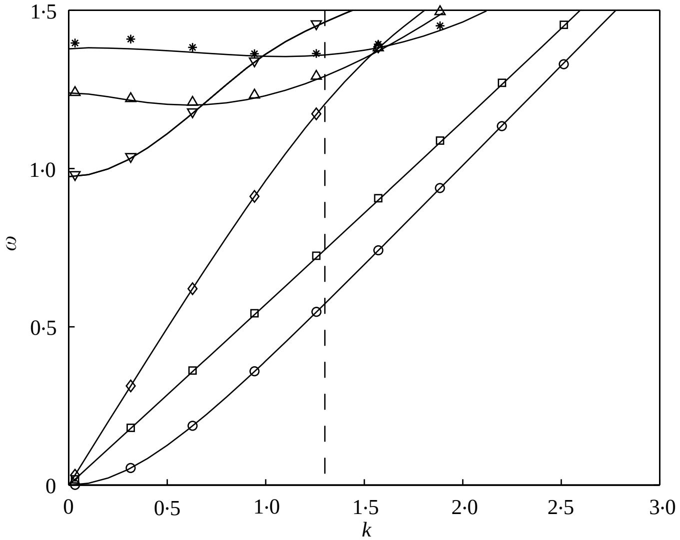

Figure 1. Angular frequency ω versus wavenumber k for general waves in a bar with square cross-section. First six non-duplicate modes, $n = 3$ (solid lines). Comparison with Aalami (Aalami's notation: ○-B1; □-T1; ◇-L1; △-O1: $*$-S1; ▽-B2). The Poisson ratio $\nu = 0{\cdot}3$. (Dashed line is related to Figure 2).

where n is an even number. The sub matrices are

$$\boldsymbol{\Phi}_k = \begin{pmatrix} \mathbf{0} & x^{k+1}\boldsymbol{\phi}_n^{\mathrm{T}}(y) \\ \mathbf{0} & y^{k+1}\boldsymbol{\phi}_n^{\mathrm{T}}(x) \\ x^k\boldsymbol{\phi}_k^{\mathrm{T}}(y) + y^k\boldsymbol{\phi}_k^{\mathrm{T}}(x) & \mathbf{0} \end{pmatrix}, \tag{36}$$

where the vectors $\boldsymbol{\phi}_k$ and $\boldsymbol{\phi}_n$ are

$$\boldsymbol{\phi}_k^{\mathrm{T}}(x) = (1 \;\; x^2 \;\; x^4 \;\; \dots \;\; x^k), \quad \boldsymbol{\phi}_n^{\mathrm{T}}(x) = (1 \;\; x^2 \;\; x^4 \;\; \dots \;\; x^n). \tag{37}$$

In Figure 3, the angular frequency ω is shown as a function of the wavenumber k for the first mode with $n = 0, 2, 4$ and 6. The convergence rate of the method can be judged from the figure.

6. CIRCULAR CROSS-SECTION

6.1. CYLINDRICAL CO-ORDINATES

For bars with circular cross-section, cylindrical co-ordinates $x_1 = r$, $x_2 = \varphi$ and $x_3 = z$ will be used. The co-ordinates r and z are normalized with respect to the radius of the

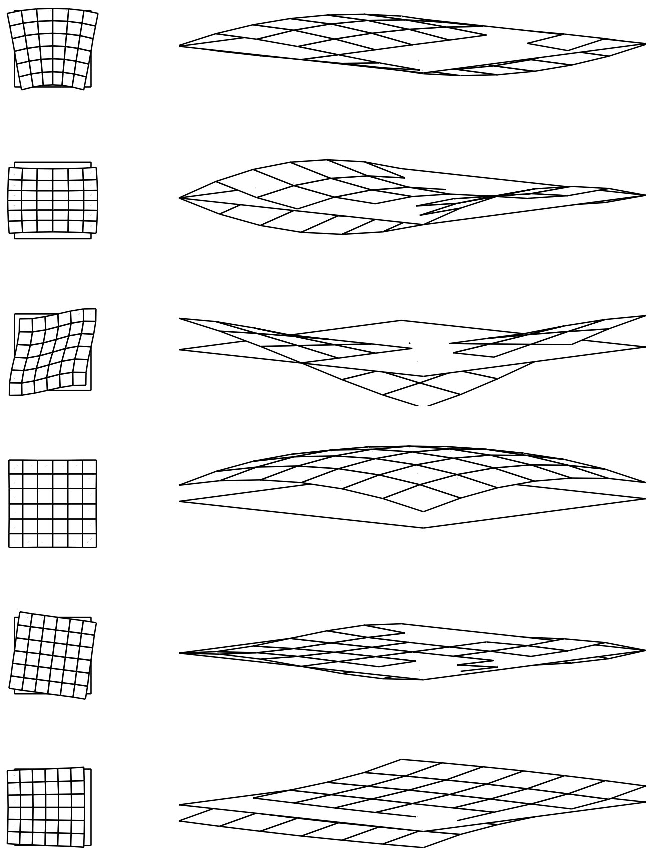

Figure 2. Transverse (left) and axial (right) displacement fields corresponding to $k = 1{\cdot}3$ for the modes in Figure 1. Increasing angular frequency from bottom to top.

cross-section. In this case, the operator matrices are

$$\mathbf{\nabla}_{12} = \begin{pmatrix} \dfrac{\partial}{\partial r} & 0 & 0 \\ \dfrac{1}{r} & \dfrac{1}{r}\dfrac{\partial}{\partial \varphi} & 0 \\ 0 & 0 & 0 \\ \dfrac{1}{r}\dfrac{\partial}{\partial \varphi} & \dfrac{\partial}{\partial r} - \dfrac{1}{r} & 0 \\ 0 & 0 & \dfrac{1}{r}\dfrac{\partial}{\partial \varphi} \\ 0 & 0 & \dfrac{\partial}{\partial r} \end{pmatrix} \tag{38}$$

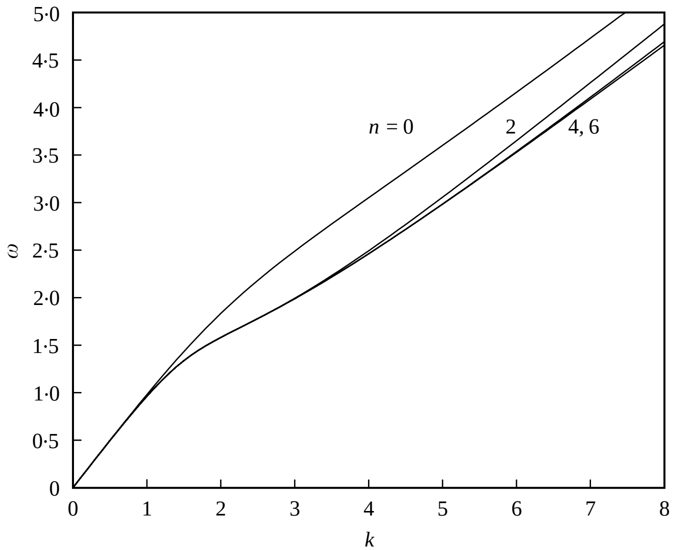

Figure 3. Angular frequency ω versus wavenumber k for longitudinal waves in a bar with square cross-section. First mode and $n = 0$, 2, 4, 6. The Poisson ratio $\nu = 0{\cdot}3$.

and

$$\mathbf{V}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{39}$$

6.2. GENERAL WAVES

For cylindrical co-ordinates, co-ordinate functions which are polynomials in the radial co-ordinate r and Fourier series in the angular co-ordinate φ are used to form a complete set of co-ordinate functions which can be used to express an arbitrary unique and continuous displacement field. In this case, the matrix $\mathbf{\Phi}$ can be written as

$$\mathbf{\Phi} = (\mathbf{\Phi}_0 \;\; \mathbf{\Phi}_{c0} \;\; \mathbf{\Phi}_{s1} \;\; \mathbf{\Phi}_{c1} \;\; \mathbf{\Phi}_{s2} \;\; \ldots \;\; \mathbf{\Phi}_{sn} \;\; \mathbf{\Phi}_{cn}), \tag{40}$$

where the sub matrix

$$\mathbf{\Phi}_0 = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{41}$$

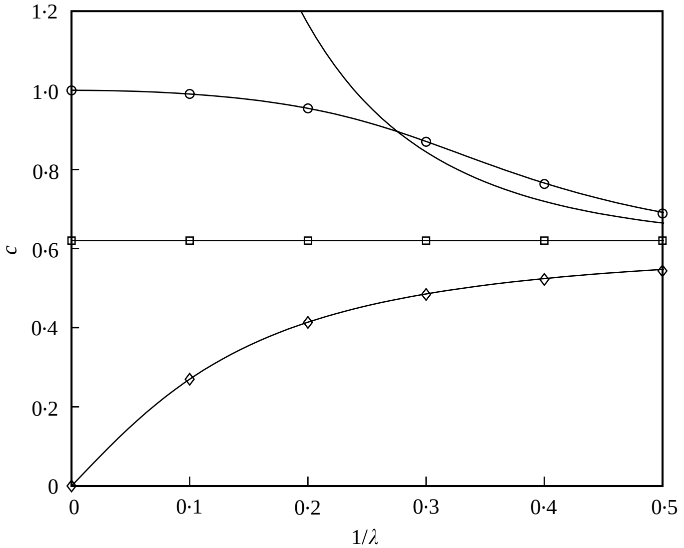

Figure 4. Phase velocity c $(=\omega/k)$ versus inverse wavelength λ^{-1} $(=k/2\pi)$ for the general waves in a bar with circular cross-section. First four non-duplicate modes, $n=2$ (solid lines). Comparison with exact solutions from Pao (◇-transversal), Love (□-torsional) and Bancroft (○-longitudinal). For the fourth mode, similar to O1 and S1 in Figures 1 and 2, no exact solution has been found. The Poisson ratio $\nu = 0{\cdot}3$.

corresponds to rigid body motion of the cross-section. The sub matrices

$$\mathbf{\Phi}_{ck} = \cos(k\varphi)\begin{pmatrix} \boldsymbol{\phi}^{\mathrm{T}}(r) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\phi}^{\mathrm{T}}(r) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\phi}^{\mathrm{T}}(r) \end{pmatrix} \tag{42}$$

and

$$\mathbf{\Phi}_{sk} = \sin(k\varphi)\begin{pmatrix} \boldsymbol{\phi}^{\mathrm{T}}(r) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\phi}^{\mathrm{T}}(r) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\phi}^{\mathrm{T}}(r) \end{pmatrix}, \tag{43}$$

with

$$\boldsymbol{\phi}^{\mathrm{T}}(r) = (r \;\; r^2 \;\; r^3 \;\; \ldots \;\; r^n), \tag{44}$$

are also continuous in the cross-section.

Dispersion curves (phase velocity $c=\omega/k$ versus the given inverse wavelength $\lambda^{-1}=k/2\pi$) for $n=2$, are shown in Figure 4. Comparisons are made with exact solutions for transversal waves from Pao [7], torsional waves from Love [4], and longitudinal waves from Bancroft [6]. The fourth curve corresponds to a mode similar to mode O1 or S1 in Figure 1 (cf., also Figure 2) for which no exact solution has been found.

6.3. LONGITUDINAL WAVES

For longitudinal waves in a bar with circular cross-section, there is no dependence on φ and the circumferential displacement u_φ is zero. Therefore, the dimensions of vectors and

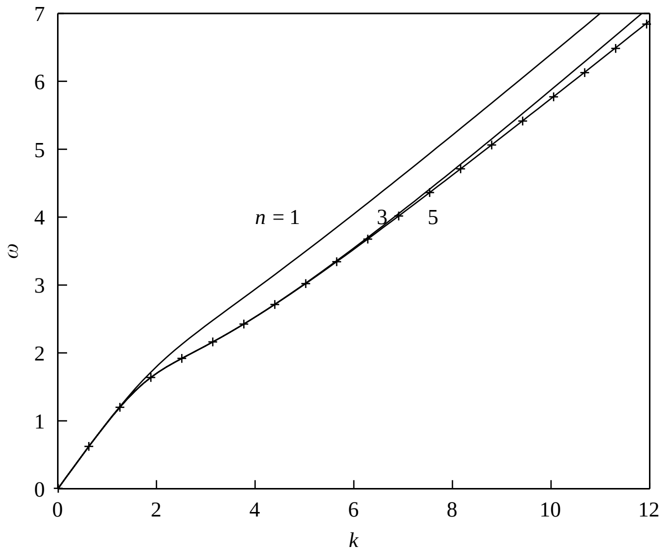

Figure 5. Angular frequency ω versus wave number k for longitudinal waves in a bar with circular cross-section. First mode and $n = 1, 3, 5$ (solid lines). Comparison with exact solution from Bancroft (+). The Poisson ratio $\nu = 0{\cdot}3$.

matrices can be reduced. Thus, the displacement vector and those of stresses and strains become

$$\mathbf{u} = \begin{pmatrix} u_r \\ u_z \end{pmatrix}, \qquad \boldsymbol{\tau} = \begin{pmatrix} \tau_{rr} \\ \tau_{\varphi\varphi} \\ \tau_{zz} \\ \tau_{zr} \end{pmatrix}, \qquad \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{rr} \\ \varepsilon_{\varphi\varphi} \\ \varepsilon_{zz} \\ \gamma_{zr} \end{pmatrix}. \tag{45}$$

Also, the operator matrices become

$$\boldsymbol{\nabla}_{12} = \begin{pmatrix} \dfrac{\partial}{\partial r} & 0 \\ \dfrac{1}{r} & 0 \\ 0 & 0 \\ 0 & \dfrac{\partial}{\partial r} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\nabla}_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{46}$$

Finally, for an isotropic material,

$$\mathbf{C} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_2 & 0 \\ \alpha_2 & \alpha_1 & \alpha_2 & 0 \\ \alpha_2 & \alpha_2 & \alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_3 \end{pmatrix} \tag{47}$$

with α_1, α_2 and α_3 given by equations (28) as previously.

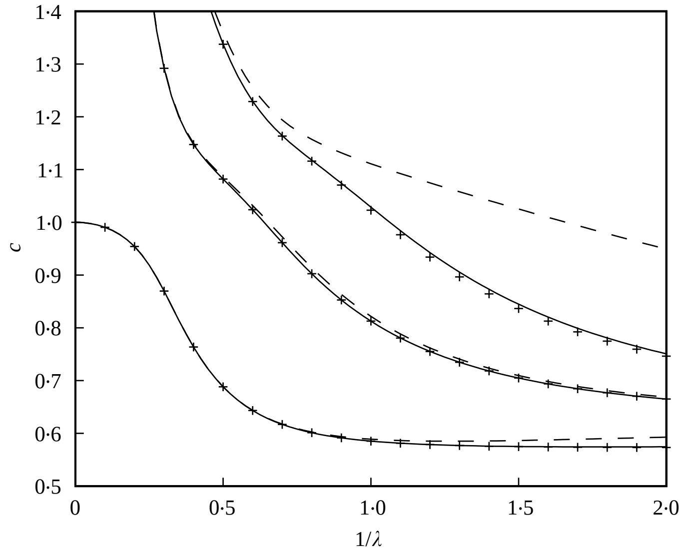

Figure 6. Phase velocity $c\ (=\omega/k)$ versus inverse wavelength $\lambda^{-1}\ (=k/2\pi)$ for longitudinal waves in a bar with circular cross-section. First three modes and $n=3$ (dashed lines) and $n=5$ (solid lines). Comparison with exact solutions from Bancroft $(+)$. The Poisson ratio $\nu=0{\cdot}3$.

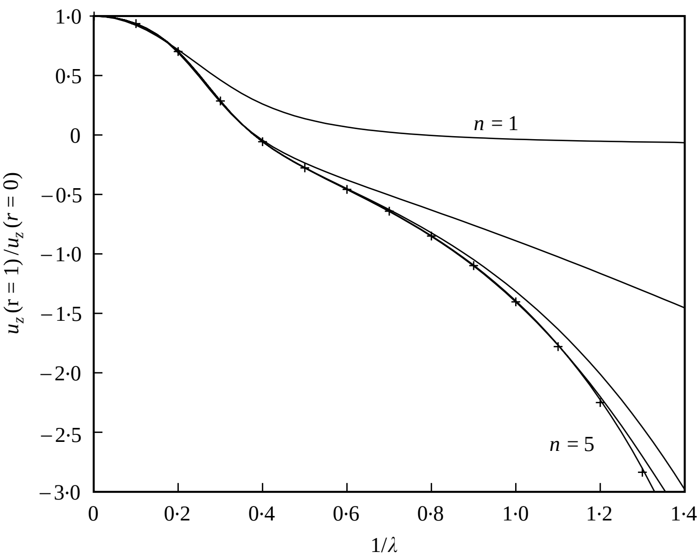

Figure 7. Axial surface displacement $u_z(r=1)$, normalized to axial displacement at centerline $u_z(r=0)$, versus inverse wavelength $\lambda^{-1}(=k/2\pi)$ for longitudinal waves in a bar with circular cross section. First mode and $n=1-5$ (solid lines). Comparison with exact solution from Davies $(+)$. The Poisson ratio $\nu=0{\cdot}3$.

Suitable co-ordinate functions, which assure unique and continuous displacements, can be written as

$$\boldsymbol{\Phi}=\begin{pmatrix} 0 & \boldsymbol{\phi}^{\mathrm{T}}(r) & \mathbf{0} \\ 1 & \mathbf{0} & \boldsymbol{\phi}^{\mathrm{T}}(r) \end{pmatrix} \tag{48}$$

with $\boldsymbol{\phi}^{\mathrm{T}}(r)$ given by equation (44) as previously.

Figure 5 shows dispersion curves (angular frequency ω as a function of wavenumber k) for the first mode and $n = 1$, 3 and 5. The results are compared with the exact solution from Bancroft [6]. Figure 6 shows phase velocities versus inverse wavelength for the first three modes for $n = 3$ and 5, also compared with exact solutions from Bancroft. On the basis of equation (34), Figure 7 shows axial displacements at the boundary ($r = 1$), normalized with respect to axial displacements at the centerline ($r = 0$), for $n = 1$–5. The results are compared with exact results from Davies [5].

Figures 5–7 show the precision and convergence rate for the dispersion relations as well as the displacements, using this method.

7. DISCUSSION

A matrix formulation of Hamilton's principle has been used to find approximate dispersion relations and corresponding displacement fields associated with elastic waves in long prismatic bars. The method fits well into mathematical toolboxes such as Matlab [19]. The matrix formulation makes it easy to implement different co-ordinate systems or co-ordinate functions. Different kinds of linearly elastic materials, such as isotropic or orthotropic, and different shapes of the cross-section can be handled in a straightforward manner.

The convergence rate of the method has been demonstrated by solving problems for increasing orders of co-ordinate functions. The results obtained for relatively low orders of such functions have been found to agree well with approximate results from the literature for square cross-sections, and with exact solutions for circular cross-sections.

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REFERENCES

1. H. Kolsky 1963 *Stress Waves in Solids*. New York: Dover Publications.
2. L. Pochhammer 1876 *Journal fur die Reine und Angewandte Mathematik* **81**, 324–336. Ueber die Fortpflanzungs geschwindigkeiten kleiner Schwingungen in einem unbegrenzten isotropen Kreiscylinder.
3. C. Chree 1889 *Proc. Camb. Soc.* **14**, 250–369. The equations of an isotropic elastic solid in polar and cylindrical coordinates, their solution and application.
4. A. E. H. Love 1944 *A Treatise on the Mathematical Theory of Elasticity*. New York: Dover Publications; Fourth edition, 278–309.
5. E. D. H. Davies 1948 *Philosophical Transactions A* **240**, 375–457. A critical study of the Hopkinson pressure bar.
6. D. Bancroft 1941 *Physical Review* **59**, 588–593. The velocity of longitudinal waves in cylindrical bars.
7. Y-H. Pao and R. D. Mindlin 1960 *Journal of Applied Mechanics* **27**, 513–520. Dispersion of flexural waves in an elastic, circular cylinder.
8. P. K. Wong, J. Milkowitz and R. A. Scott 1966 *The Journal of the Acoustical Society of America* **40**, 393–398. Propagation of harmonic flexural waves in an infinite elastic rod of elliptical cross section.
9. K. Sato 1978 *Bulletin of the Japanese Society of Mechanical Engineering* **21**, 203–209. Elastic wave propagation in an infinite bar of elliptical cross section.

10. N. J. NIGRO 1966 *Journal of Acoustical Society of America* **40**, 1501–1508. Steady-state wave propagation in infinite bars of noncircular cross section.
11. M. A. MEDICK 1966 *Journal of Applied Mechanics* **33**, 489–495, *Transactions of American Society of Mehanical Engineers*, *Series E* **88**. One-dimensional theories of wave propagation and vibrations in elastic bars of rectangular cross section.
12. A. D. S. BARR 1962 *Journal of Mechanical Engineering Science* **4**, 127–135. Torsional waves in uniform rods of non-circular section.
13. B. AALAMI 1973 *Journal of Applied Mechanics* **40**, 1067–1072, *Transactions of American Society of Mehanical Engineers*, *Series E* **95**. Waves in prismatic guides of arbitrary cross section.
14. V. V. VOLOVOI, D. H. HODGES, V. L. BERDICHEVSKY and V. G. SUTYRIN 1998 *Journal of Sound and Vibration* **215**, 1101–1120. Dynamic dispersion curves for non-homogeneous, anisotropic beams with cross-sections of arbitrary geometry.
15. H. TAWEEL, S. B. DONG and M. KAZIC 2000 *International Journal of Solids and Structures* **37**, 1701–1726. Wave reflection from the free end of a cylinder with an arbitrary cross-section.
16. W. B. FRASER 1969 *International Journal of Solids Structures* **5**, 379–397. Stress wave propagation in rectangular bars.
17. K. NAGAYA 1982 *Journal of Applied Mechanics* **49**, 157–164. Stress wave propagation in a bar of arbitrary cross section.
18. A. BOSTRÖM 2000 *Zeitschrift Angewandte Mathematik und Mechanik* **80**, 245–251. On wave equations for elastic rods.
19. MATLAB www.mathworks.com

APPENDIX A: EXPLICIT EXPRESSIONS FOR MATRICES $\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2$ AND $\mathbf{M}$

The facts that $\mathbf{K}_0$, $\mathbf{K}_2$ and $\mathbf{M}$ is symmetric and that $\mathbf{K}_1$ is antisymmetric might be used when programming, though not always clearly pointed out in this appendix. Throughout the Appendix, $\mathbf{0}$ denotes a null-matrix of suitable size. The parameters α_1,α_2 and α_3 are the elastic constants from the matrix $\mathbf{C}$, as defined in equations (28). The parameter n is the order of the coordinate functions as defined in sections 5.2, 5.3, 6.2 and 6.3. The letters k, l, p and q are indices.

A.1. SQUARE CROSS-SECTION, GENERAL WAVES

A.1.1. Matrix $\mathbf{K}_0$

$$\mathbf{K}_0 = \begin{bmatrix} \mathbf{K}_0^{00} & \mathbf{K}_0^{01} & \mathbf{K}_0^{02} & \cdots & \mathbf{K}_0^{0n} \\ \mathbf{K}_0^{10} & \mathbf{K}_0^{11} & \mathbf{K}_0^{12} & \cdots & \mathbf{K}_0^{1n} \\ \mathbf{K}_0^{20} & \mathbf{K}_0^{21} & \mathbf{K}_0^{22} & \cdots & \mathbf{K}_0^{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{K}_0^{n0} & \mathbf{K}_0^{n1} & \mathbf{K}_0^{n2} & \cdots & \mathbf{K}_0^{nn} \end{bmatrix}, \tag{A.1}$$

$$\mathbf{K}_0^{kl} = \begin{bmatrix} \mathbf{K}_{0,1}^{kl} & \mathbf{K}_{0,3}^{kl} & \mathbf{0} \\ \mathbf{K}_{0,2}^{kl} & \mathbf{K}_{0,4}^{kl} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{0,5}^{kl} \end{bmatrix}, \tag{A.2}$$

$$\mathbf{K}_{0,1}^{kl} = \begin{bmatrix} K_{0,1}^{kl,00} & K_{0,1}^{kl,01} & K_{0,1}^{kl,02} & \cdots & K_{0,1}^{kl,0n} \\ K_{0,1}^{kl,10} & K_{0,1}^{kl,11} & K_{0,1}^{kl,12} & \cdots & K_{0,1}^{kl,1n} \\ K_{0,1}^{kl,20} & K_{0,1}^{kl,21} & K_{0,1}^{kl,22} & \cdots & K_{0,1}^{kl,2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{0,1}^{kl,n0} & K_{0,1}^{kl,n1} & K_{0,1}^{kl,n2} & \cdots & K_{0,1}^{kl,nn} \end{bmatrix}, \tag{A.3}$$

$$K_{0,1}^{kl,pq} = \alpha_1 kl \frac{(1-(-1)^{k+l-1})}{(k+l-1)} \frac{(1-(-1)^{p+q+1})}{(p+q+1)} + \alpha_3 pq \frac{(1-(-1)^{k+l+1})}{(k+l+1)} \frac{(1-(-1)^{p+q-1})}{(p+q-1)}. \tag{A.4}$$

The other sub matrices in matrix $\mathbf{K}_0^{kl}$ have the same structure as $\mathbf{K}_{0,1}^{kl}$ (i.e., numbering of p and q), but the elements are

$$K_{0,2}^{kl,pq} = (\alpha_2 lp + \alpha_3 kq) \frac{(1-(-1)^{k+l})}{(k+l)} \frac{(1-(-1)^{p+q})}{(p+q)}, \tag{A.5}$$

$$\mathrm{K}_{0,3}^{kl,pq} = (\alpha_2 kq + \alpha_3 lp) \frac{(1-(-1)^{k+l})}{(k+l)} \frac{(1-(-1)^{p+q})}{(p+q)}, \tag{A.6}$$

$$K_{0,4}^{kl,pq} = \alpha_1 pq \frac{(1-(-1)^{k+l+1})}{(k+l+1)} \frac{(1-(-1)^{p+q-1})}{(p+q-1)} + \alpha_3 kl \frac{(1-(-1)^{k+l-1})}{(k+l-1)} \frac{(1-(-1)^{p+q+1})}{(p+q+1)}, \tag{A.7}$$

$$K_{0,5}^{kl,pq} = \alpha_3 pq \frac{(1-(-1)^{k+l+1})}{(k+l+1)} \frac{(1-(-1)^{p+q-1})}{(p+q-1)} + \alpha_3 kl \frac{(1-(-1)^{k+l-1})}{(k+l-1)} \frac{(1-(-1)^{p+q+1})}{(p+q+1)}. \tag{A.8}$$

A.1.2. Matrix $\mathbf{K}_1$

$$\mathbf{K}_1 = \begin{bmatrix} \mathbf{K}_1^{00} & \mathbf{K}_1^{01} & \mathbf{K}_1^{02} & \cdots & \mathbf{K}_1^{0n} \\ \mathbf{K}_1^{10} & \mathbf{K}_1^{11} & \mathbf{K}_1^{12} & \cdots & \mathbf{K}_1^{1n} \\ \mathbf{K}_1^{20} & \mathbf{K}_1^{21} & \mathbf{K}_1^{22} & \cdots & \mathbf{K}_1^{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{K}_1^{n0} & \mathbf{K}_1^{n1} & \mathbf{K}_1^{n2} & \cdots & \mathbf{K}_1^{nn} \end{bmatrix} - \begin{bmatrix} \mathbf{K}_1^{00} & \mathbf{K}_1^{01} & \mathbf{K}_1^{02} & \cdots & \mathbf{K}_1^{0n} \\ \mathbf{K}_1^{10} & \mathbf{K}_1^{11} & \mathbf{K}_1^{12} & \cdots & \mathbf{K}_1^{1n} \\ \mathbf{K}_1^{20} & \mathbf{K}_1^{21} & \mathbf{K}_1^{22} & \cdots & \mathbf{K}_1^{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{K}_1^{n0} & \mathbf{K}_1^{n1} & \mathbf{K}_1^{n2} & \cdots & \mathbf{K}_1^{nn} \end{bmatrix}^{\mathbf{T}}, \tag{A.9}$$

$$\mathbf{K}_1^{kl} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{K}_{1,3}^{kl} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{1,4}^{kl} \\ \mathbf{K}_{1,1}^{kl} & \mathbf{K}_{1,2}^{kl} & \mathbf{0} \end{bmatrix}. \tag{A.10}$$

The sub matrices have the same structure as $\mathbf{K}_{0,1}^{kl}$ in equation (A.3), (i.e., the numbering of p and q), but the elements are

$$K_{1,1}^{kl,pq} = \alpha_3 k \frac{(1-(-1)^{k+l})}{(k+l)} \frac{(1-(-1)^{p+q+1})}{(p+q+1)}, \tag{A.11}$$

$$K_{1,2}^{kl,pq} = \alpha_3 p \frac{(1-(-1)^{k+l+1})}{(k+l+1)} \frac{(1-(-1)^{p+q})}{(p+q)}, \tag{A.12}$$

$$K_{1,3}^{kl,pq} = \alpha_2 k \frac{(1-(-1)^{k+l})}{(k+l)} \frac{(1-(-1)^{p+q+1})}{(p+q+1)}, \tag{A.13}$$

$$K_{1,4}^{kl,pq} = \alpha_2 p \frac{(1-(-1)^{k+l+1})}{(k+l+1)} \frac{(1-(-1)^{p+q})}{(p+q)}. \tag{A.14}$$

A.1.3. Matrix $\mathbf{K}_2$

Matrix $\mathbf{K}_2$ is similar to matrix $\mathbf{K}_0$ in equation (A.1), but the sub matrices are

$$\mathbf{K}_2^{kl} = \begin{bmatrix} \mathbf{K}_{2,1}^{kl} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{2,1}^{kl} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{2,2}^{kl} \end{bmatrix}. \tag{A.15}$$

The sub matrices of the matrices $\mathbf{K}_{2,1}^{kl}$ and $\mathbf{K}_{2,2}^{kl}$ have the same structure as $\mathbf{K}_{0,1}^{kl}$ in equation (A.3), (i.e., the numbering of p and q), but the elements are

$$K_{2,1}^{kl,pq} = \alpha_3 \frac{(1-(-1)^{k+l+1})}{(k+l+1)} \frac{(1-(-1)^{p+q+1})}{(p+q+1)}, \tag{A.16}$$

$$K_{2,2}^{kl,pq} = \alpha_1 \frac{(1-(-1)^{k+l+1})}{(k+l+1)} \frac{(1-(-1)^{p+q+1})}{(p+q+1)}. \tag{A.17}$$

A.1.4. Matrix $\mathbf{M}$

Matrix $\mathbf{M}$ is the same as matrix $\mathbf{K}_2$ in section A.1.3 if the elastic constants α_1 and α_3 are set to one.

A.2. SQUARE CROSS-SECTION, LONGITUDINAL WAVES

A.2.1. Matrix $\mathbf{K}_0$

$$\mathbf{K}_0 = \begin{bmatrix} \mathbf{K}_0^{00} & \mathbf{K}_0^{02} & \mathbf{K}_0^{04} & \cdots & \mathbf{K}_0^{0n} \\ \mathbf{K}_0^{20} & \mathbf{K}_0^{22} & \mathbf{K}_0^{24} & \cdots & \mathbf{K}_0^{2n} \\ \mathbf{K}_0^{40} & \mathbf{K}_0^{42} & \mathbf{K}_0^{44} & \cdots & \mathbf{K}_0^{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{K}_0^{n0} & \mathbf{K}_0^{n2} & \mathbf{K}_0^{n4} & \cdots & \mathbf{K}_0^{nn} \end{bmatrix}, \tag{A.18}$$

$$\mathbf{K}_0^{kl} = 8 \begin{bmatrix} \mathbf{K}_{0,1}^{kl} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{0,2}^{kl} \end{bmatrix}, \tag{A.19}$$

$$\mathbf{K}_{0,1}^{kl} = \begin{bmatrix} K_{0,1}^{kl,00} & K_{0,1}^{kl,02} & K_{0,1}^{kl,04} & \cdots & K_{0,1}^{kl,0l} \\ K_{0,1}^{kl,20} & K_{0,1}^{kl,22} & K_{0,1}^{kl,24} & \cdots & K_{0,1}^{kl,2l} \\ K_{0,1}^{kl,40} & K_{0,1}^{kl,42} & K_{0,1}^{kl,44} & \cdots & K_{0,1}^{kl,4l} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{0,1}^{kl,k0} & K_{0,1}^{kl,k2} & K_{0,1}^{kl,k4} & \cdots & K_{0,1}^{kl,kl} \end{bmatrix}, \tag{A.20}$$

$$K_{0,1}^{kl,pq} = \alpha_3\left(\frac{pq}{(k+l+1)(p+q-1)} + \frac{lp}{(k+q+1)(l+p-1)} + \frac{kq}{(l+p+1)(k+q-1)} + \frac{kl}{(p+q+1)(k+l-1)}\right), \tag{A.21}$$

$$\mathbf{K}_{0,2}^{kl} = \begin{bmatrix} K_{0,2}^{kl,00} & K_{0,2}^{kl,02} & K_{0,2}^{kl,04} & \cdots & K_{0,2}^{kl,0n} \\ K_{0,2}^{kl,20} & K_{0,2}^{kl,22} & K_{0,2}^{kl,24} & \cdots & K_{0,2}^{kl,2n} \\ K_{0,2}^{kl,40} & K_{0,2}^{kl,42} & K_{0,2}^{kl,44} & \cdots & K_{0,2}^{kl,4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{0,2}^{kl,n0} & K_{0,2}^{kl,n2} & K_{0,2}^{kl,n4} & \cdots & K_{0,2}^{kl,nn} \end{bmatrix}, \tag{A.22}$$

$$K_{0,2}^{kl,pq} = \alpha_1 \frac{(k+1)(l+1)}{(k+l+1)(p+q+1)} + \alpha_2 \frac{(k+1)(l+1)}{(l+p+1)(k+q+1)} + \alpha_3\left(\frac{pq}{(k+l+3)(p+q-1)} + \frac{pq}{(k+q+1)(l+p+1)}\right). \tag{A.23}$$

A.2.2. Matrix $\mathbf{K}_1$

$$\mathbf{K}_1 = \begin{bmatrix} \mathbf{K}_1^{00} & \mathbf{K}_1^{02} & \mathbf{K}_1^{04} & \cdots & \mathbf{K}_1^{0n} \\ \mathbf{K}_1^{20} & \mathbf{K}_1^{22} & \mathbf{K}_1^{24} & \cdots & \mathbf{K}_1^{2n} \\ \mathbf{K}_1^{40} & \mathbf{K}_1^{42} & \mathbf{K}_1^{44} & \cdots & \mathbf{K}_1^{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{K}_1^{n0} & \mathbf{K}_1^{n2} & \mathbf{K}_1^{n4} & \cdots & \mathbf{K}_1^{nn} \end{bmatrix} - \begin{bmatrix} \mathbf{K}_1^{00} & \mathbf{K}_1^{02} & \mathbf{K}_1^{04} & \cdots & \mathbf{K}_1^{0n} \\ \mathbf{K}_1^{20} & \mathbf{K}_1^{22} & \mathbf{K}_1^{24} & \cdots & \mathbf{K}_1^{2n} \\ \mathbf{K}_1^{40} & \mathbf{K}_1^{42} & \mathbf{K}_1^{44} & \cdots & \mathbf{K}_1^{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{K}_1^{n0} & \mathbf{K}_1^{n2} & \mathbf{K}_1^{n4} & \cdots & \mathbf{K}_1^{nn} \end{bmatrix}^{\mathrm{T}}, \tag{A.24}$$

$$\mathbf{K}_1^{kl} = 8\begin{bmatrix} \mathbf{0} & \mathbf{K}_{1,2}^{kl} \\ \mathbf{K}_{1,1}^{kl} & \mathbf{0} \end{bmatrix}, \tag{A.25}$$

$$\mathbf{K}_{1,1}^{kl} = \begin{bmatrix} K_{1,1}^{kl,00} & K_{1,1}^{kl,02} & K_{1,1}^{kl,04} & \cdots & K_{1,1}^{kl,0l} \\ K_{1,1}^{kl,20} & K_{1,1}^{kl,22} & K_{1,1}^{kl,24} & \cdots & K_{1,1}^{kl,2l} \\ K_{1,1}^{kl,40} & K_{1,1}^{kl,42} & K_{1,1}^{kl,44} & \cdots & K_{1,1}^{kl,4l} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{1,1}^{kl,n0} & K_{1,1}^{kl,n2} & K_{1,1}^{kl,n4} & \cdots & K_{1,1}^{kl,nl} \end{bmatrix}, \tag{A.26}$$

$$K_{1,1}^{kl,pq} = \alpha_2(k+1)\left(\frac{1}{(k+l+1)(p+q+1)} + \frac{1}{(k+q+1)(l+p+1)}\right), \tag{A.27}$$

$$\mathbf{K}^{kl}_{1,2} = \begin{bmatrix} K^{kl,00}_{1,2} & K^{kl,02}_{1,2} & K^{kl,04}_{1,2} & \cdots & K^{kl,0n}_{1,2} \\ K^{kl,20}_{1,2} & K^{kl,22}_{1,2} & K^{kl,24}_{1,2} & \cdots & K^{kl,2n}_{1,2} \\ K^{kl,40}_{1,2} & K^{kl,42}_{1,2} & K^{kl,44}_{1,2} & \cdots & K^{kl,4n}_{1,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K^{kl,k0}_{1,2} & K^{kl,k2}_{1,2} & K^{kl,k4}_{1,2} & \cdots & K^{kl,kn}_{1,2} \end{bmatrix}, \tag{A.28}$$

$$K^{kl,pq}_{1,2} = \alpha_3\left(\frac{p}{(k+q+1)(l+p+1)} + \frac{k}{(p+q+1)(k+l+1)}\right). \tag{A.29}$$

A.2.3. Matrix $\mathbf{K}_2$

Matrix $\mathbf{K}_2$ is similar to matrix $\mathbf{K}_0$ in section A.2.1, but the elements in the sub matrices are

$$K^{kl,pq}_{2,1} = \alpha_1\left(\frac{1}{(k+l+1)(p+q+1)} + \frac{1}{(k+q+1)(l+p+1)}\right), \tag{A.30}$$

$$K^{kl,pq}_{2,2} = \frac{\alpha_3}{(p+q+1)(k+l+3)}. \tag{A.31}$$

A.2.4. Matrix $\mathbf{M}$

Matrix $\mathbf{M}$ is the same as matrix $\mathbf{K}_2$ in section A.2.3 if the elastic constants α_1 and α_3 are set to one.

A.3. CIRCULAR CROSS-SECTION, GENERAL WAVES

A.3.1. Matrix $\mathbf{K}_0$

$$\mathbf{K}_0 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{K}^0_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}^1_0 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{K}^n_0 \end{bmatrix}, \tag{A.32}$$

$$\mathbf{K}^0_0 = 2\pi\begin{bmatrix} \mathbf{K}_{0,1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{0,2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{0,3} \end{bmatrix} \tag{A.33}$$

and for $k > 0$:

$$\mathbf{K}^k_0 = \pi\begin{bmatrix} \mathbf{K}_{0,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{0,2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{0,3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{0,1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{0,2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{0,3} \end{bmatrix}$$

$$+\pi\begin{bmatrix} k^2\mathbf{K}_{0,4} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -k\mathbf{K}_{0,6}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & k^2\mathbf{K}_{0,5} & \mathbf{0} & k\mathbf{K}_{0,6} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & k^2\mathbf{K}_{0,4} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & k\mathbf{K}_{0,6}^{\mathrm{T}} & \mathbf{0} & k^2\mathbf{K}_{0,4} & \mathbf{0} & \mathbf{0} \\ -k\mathbf{K}_{0,6} & \mathbf{0} & \mathbf{0} & \mathbf{0} & k^2\mathbf{K}_{0,5} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & k^2\mathbf{K}_{0,4} \end{bmatrix}. \tag{A.34}$$

The sub matrices are of dimension $n \times n$ with row index p, column index q, and the elements are

$$K_{0,1}^{pq} = \alpha_1 \frac{1+pq}{p+q} + \alpha_2, \quad K_{0,2}^{pq} = \alpha_3 \frac{1-p-q+pq}{p+q}, \tag{A.35}$$

$$K_{0,3}^{pq} = \alpha_3 \frac{pq}{p+q}, \quad K_{0,4}^{pq} = \alpha_3 \frac{1}{p+q}, \tag{A.36}$$

$$K_{0,5}^{pq} = \alpha_1 \frac{1}{p+q}, \quad K_{0,6}^{pq} = \frac{\alpha_1 + \alpha_2 q - \alpha_3(p-1)}{p+q}. \tag{A.37}$$

A.3.2. Matrix $\mathbf{K}_1$

$$\mathbf{K}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{K}_1^b & \mathbf{K}_1^0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{K}_1^c & \mathbf{0} & \mathbf{K}_1^1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{K}_1^n \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{K}_1^b & \mathbf{K}_1^0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{K}_1^c & \mathbf{0} & \mathbf{K}_1^1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{K}_1^n \end{bmatrix}^{\mathrm{T}}, \tag{A.38}$$

$$\mathbf{K}_1^b = 2\pi \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{K}_{1,1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{A.39}$$

$$\mathbf{K}_1^c = \pi \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{1,2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{K}_{1,2} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{A.40}$$

$$\mathbf{K}_1^0 = 2\pi \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{K}_{1,4} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{K}_{1,3} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{A.41}$$

and for $k > 0$:

$$\mathbf{K}_1^k = \pi \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{K}_{1,4} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & k\mathbf{K}_{1,6} \\ \mathbf{K}_{1,3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & k\mathbf{K}_{1,5} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{1,4} \\ \mathbf{0} & \mathbf{0} & -k\mathbf{K}_{1,6} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -k\mathbf{K}_{1,5} & \mathbf{0} & \mathbf{K}_{1,3} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{A.42}$$

$\mathbf{K}_{1,1}$ and $\mathbf{K}_{1,2}$ are column vectors of dimension n with row index p, and the elements are (all elements are equal):

$$K_{1,1}^p = \alpha_2, \quad K_{1,2}^p = \alpha_3. \tag{A.43}$$

The other sub matrices are of dimension $n \times n$ with row index p, column index q, and the elements are

$$K_{1,3}^{pq} = \alpha_3 \frac{p}{p+q+1}, \quad K_{1,4}^{pq} = \alpha_2 \frac{p+1}{p+q+1}, \tag{A.44}$$

$$K_{1,5}^{pq} = \alpha_3 \frac{1}{p+q+1}, \quad K_{1,6}^{pq} = \alpha_2 \frac{1}{p+q+1}. \tag{A.45}$$

A.3.3. Matrix $\mathbf{K}_2$

$$\mathbf{K}_2 = \begin{bmatrix} \mathbf{K}_2^a & [\mathbf{K}_2^b]^{\mathrm{T}} & [\mathbf{K}_2^c]^{\mathrm{T}} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{K}_2^b & \mathbf{K}_2^0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{K}_2^c & \mathbf{0} & \mathbf{K}_2^1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_2^2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{K}_2^n \end{bmatrix}, \tag{A.46}$$

$$\mathbf{K}_2^a = \pi \begin{bmatrix} \alpha_3 & 0 & 0 \\ 0 & \alpha_3 & 0 \\ 0 & 0 & \alpha_1 \end{bmatrix}, \tag{A.47}$$

$$\mathbf{K}_2^b = 2\pi \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{2,1} \end{bmatrix}, \tag{A.48}$$

$$\mathbf{K}_2^c = \pi \begin{bmatrix} \mathbf{0} & \mathbf{K}_{2,2} & \mathbf{0} \\ -\mathbf{K}_{2,2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{K}_{2,2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{2,2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{A.49}$$

$$\mathbf{K}_2^0 = 2\pi \begin{bmatrix} \mathbf{K}_{2,3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{2,3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{2,4} \end{bmatrix} \tag{A.50}$$

and for $k > 0$:

$$\mathbf{K}_2^k = \pi \begin{bmatrix} \mathbf{K}_{2,3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{2,3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{2,4} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{2,3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{2,3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{2,4} \end{bmatrix}. \tag{A.51}$$

$\mathbf{K}_{2,1}$ and $\mathbf{K}_{2,2}$ are column vectors of dimension n with row index p, and the elements are

$$K_{2,1}^p = \alpha_1 \frac{1}{p+2}, \quad K_{2,2}^p = \alpha_3 \frac{1}{p+2}. \tag{A.52}$$

The other sub matrices are of dimension $n \times n$ with row index p, column index q, and the elements are

$$K_{2,3}^{pq} = \alpha_3 \frac{1}{p+q+2}, \quad K_{2,4}^{pq} = \alpha_1 \frac{1}{p+q+2}. \tag{A.53}$$

A.3.4. Matrix M

Matrix $\mathbf{M}$ is the same as matrix $\mathbf{K}_2$ in section A.3.3, if the elastic constants α_1 and α_3 are set to one.

A.4. CIRCULAR CROSS-SECTION, LONGITUDINAL WAVES

A.4.1. Matrix $\mathbf{K}_0$

$$\mathbf{K}_0 = 2\pi \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{0,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{0,2} \end{bmatrix}. \tag{A.54}$$

The sub matrices $\mathbf{K}_{0,1}$ and $\mathbf{K}_{0,2}$ are of dimension $n \times n$ with row index p, column index q, and the elements are

$$K_{0,1}^{pq} = \alpha_1 \frac{pq+1}{p+q} + \alpha_2, \quad K_{0,2}^{pq} = \alpha_3 \frac{pq}{p+q}. \tag{A.55}$$

A.4.2. Matrix $\mathbf{K}_1$

$$\mathbf{K}_1 = 2\pi \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{K}_{1,1} & \mathbf{0} & \mathbf{K}_{1,3} \\ \mathbf{0} & \mathbf{K}_{1,2} & \mathbf{0} \end{bmatrix} - 2\pi \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{K}_{1,1} & \mathbf{0} & \mathbf{K}_{1,3} \\ \mathbf{0} & \mathbf{K}_{1,2} & \mathbf{0} \end{bmatrix}^{\mathrm{T}}. \tag{A.56}$$

The column vector $\mathbf{K}_{1,1}$ is of dimension n with row index p, and the elements are (all elements are equal)

$$K_{1,1}^{p} = \alpha_2. \tag{A.57}$$

The sub matrices $\mathbf{K}_{1,2}$ and $\mathbf{K}_{1,3}$ are of dimension $n \times n$ with row index p, column index q, and the elements are

$$K_{1,2}^{pq} = \alpha_3 \frac{p}{p+q+1}, \quad K_{1,3}^{pq} = \alpha_2 \frac{p+1}{p+q+1}. \tag{A.58}$$

A.4.3. Matrix $\mathbf{K}_2$

$$\mathbf{K}_2 = 2\pi \begin{bmatrix} \frac{1}{2}\alpha_1 & \mathbf{0} & [\mathbf{K}_{2,1}]^{\mathrm{T}} \\ \mathbf{0} & \mathbf{K}_{2,2} & \mathbf{0} \\ \mathbf{K}_{2,1} & \mathbf{0} & \mathbf{K}_{2,3} \end{bmatrix}. \tag{A.59}$$

The column vector $\mathbf{K}_{2,1}$ is of dimension n with row index p, and the elements are

$$K_{2,1}^{p} = \alpha_1 \frac{1}{p+2}. \tag{A.60}$$

The sub matrices $\mathbf{K}_{2,2}$ and $\mathbf{K}_{2,3}$ are of dimension $n \times n$ with row index p, column index q, and the elements are.

$$K_{2,2}^{pq} = \alpha_3 \frac{1}{p+q+2}, \quad K_{2,3}^{pq} = \alpha_1 \frac{1}{p+q+2}. \tag{A.61}$$

A.4.4. Matrix $\mathbf{M}$

Matrix $\mathbf{M}$ is the same as $\mathbf{K}_2$ in section A.4.3, if the elastic constants α_1 and α_3 are set to one.