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APPROXIMATE MODAL CHARACTERISTICS OF SHELL–PLATE COMBINED STRUCTURES

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1. INTRODUCTION

General purpose finite-element softwares have been widely used in industrial applications to obtain the natural frequencies and mode shapes of plate–shell combined structures for various combinations of loadings and boundary conditions. However, it is generally known that the higher the order of computed natural frequencies are, the less reliable the software results will be. To assess the applicability of the numerical results, analytical results of simplified structural models are often used as a tool for verification. In addition, it is often desirable to evaluate the responses according to a specific mode of vibrational behavior. Yet, the frequencies and mode shapes obtained by the finite-element softwares are mixtures of at least three different modes, namely, the axial, the tangential and the transverse mode. Thus, there is considerable interest in obtaining the fundamental modal characteristics of complicated structures by an analytical procedure.

Researchers have applied various analytical approaches for the modal analysis of the plate–shell combined structures. Huang and Soedel [1, 2] implemented the receptance concept and applied the component mode synthesis technique to study the effects of changes in geometry and joining position on vibration behavior of a multiple plates–shell combination, explored structural coupling behavior between the shell and the plate. Lately, Huang [3] applied the receptance method to study the influences of small curvatures on free vibration of the plate–shell combinations, consisting of a circular barrel-like shell and two spherically curved end plates.

Applying proper analytical approaches can effectively solve eigenvalue problems of certain structural configurations and obtain vibration characteristics. From a practical engineering point of view, there is considerable interest in understanding the modal characteristics of a plate–shell combination with free–free boundary conditions. However, research reports for free–free shells, such as the one published by Cheng and Nicolas [4], are rarely seen in the literature. It is understood that if the shell boundary condition is not simply supported, it is difficult to obtain closed-form solutions for natural frequencies and mode shapes, proper approximate approaches should be used. Thus, the primary objective of this study is to present a simple yet accurate procedure in obtaining modal characteristics of a free–free plate–shell combined structure by using approximate approaches. It also shows similarities and differences between the mode shapes, and leads to an understanding of the trend of evolution of vibration modes.

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2. SOLUTION APPROACH

To preserve the insight that a simplified approach provides, the Donnell–Mushrari–Vlasov equations [5] were used since they neglect neither membrane nor bending effects, and they apply to shell that is loaded normal to its surfaces. Due to simplicity and solvability, the beam functions were used as the weighing functions of the Galerkin method to derive the approximate natural frequency equations of a free–free shell with attached end plates. To understand the physics of a combined structure, a component mode synthesis technique, or a receptance method, was introduced as the solution procedure. The 4-point Gauss Legendre quadrature numerical integration scheme was applied to calculate the modal mass of the plate and the shell. The secant method was used to search for the zeroes of the system frequency equation, which are the natural frequencies of the plate–shell combination. Finally, a finite-element program was used to compute the modal characteristics of the same model to verify the results obtained by the analytical methods.

3. SHELL EQUATIONS AND APPROXIMATE METHODS

If a shell is not subjected to a simply supported boundary condition, it is difficult to formulate closed-form solutions for computing its natural frequencies and mode shapes, unless approximate approaches and simplified shell equations were used. The Donnell–Mushtari–Vlasov equations are used widely in shell vibration. It neither neglects bending nor membrane effects. It applies to shells that are loaded normal to their surfaces and concentrates on transverse deflection behavior. The equations of motion derived from the Donnell–Mushtari–Vlasov equations for a closed circular cylindrical shell are

$$D\nabla^4 u_3 + \nabla_k^2 \phi + \rho h_s \frac{\partial^2 u_3}{\partial t^2} = q_3 \tag{1}$$

and

$$E h_s \nabla_k^2 u_3 - \nabla^4 \phi = 0, \tag{2}$$

where ϕ is a stress function, D, E, q_3, u_3, ρ and h_s are the bending stiffness, elastic modulus, forcing function, transverse deflection, density and thickness of the shell respectively. To obtain the eigenvalues of equations (1) and (2), which are the equations of motion, it is necessary to substitute $q_3 = 0$, and

$$u_3(x, \theta, t) = U^s_{3mn}(x, \theta)\mathrm{e}^{\mathrm{j}\omega t}, \qquad \phi(x, \theta, t) = \Phi(x, \theta)\mathrm{e}^{\mathrm{j}\omega t} \tag{3, 4}$$

into equations (1) and (2), which results in the governing equation

$$D\left(\frac{n^2}{a^2} - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^4 U_{3m}(x) + \frac{E h_s}{a^2}\frac{\mathrm{d}^4 U_{3m}(x)}{\mathrm{d}x^4} - \rho h_s (\omega^s_{mn})^2 \left(\frac{n^2}{a^2} - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^2 U_{3m}(x) = 0, \tag{5}$$

where a is the radius and ω^s_{mn} $(m = 1, 2, \ldots, n = 0, 1, 2, \ldots)$ are the natural frequencies of the shell. For the case where the shell is closed in θ direction, the transverse displacement function of the shell is of the form

$$U^s_{3mn}(x, \theta) = U_{3m}(x)\cos n(\theta - \psi), \tag{6}$$

where ψ is an arbitrary angle.

3.1. THE WEIGHING FUNCTION

The equation of motion of a beam is solvable no matter what boundary conditions it is subjected to. In order to use the Galerkin method to solve the above governing equation, equation (7), to obtain the natural frequencies of a shell with the free–free boundary conditions, a beam function, which satisfies the boundary conditions approximately, is selected as the weighing function. The free–free boundary conditions of a beam are formulated as the moment $M_{xx}(x, t) = 0$ and the shear force $Q_{x3}(x, t) = 0$ at both ends, namely, at $x = 0$ and L_s. The free–free beam function, which is the mode shape equation of a beam with free–free boundary conditions, can be obtained by solving the equation of motion of a transversely vibrating free–free beam. It is derived as

$$U_{3m}(x) = [\cosh(\lambda_m x) + \cos(\lambda_m x)] - \left[\frac{\cosh(\lambda_m L_s) - \cos(\lambda_m L_s)}{\sinh(\lambda_m L_s) - \sin(\lambda_m L_s)}\right][\sinh(\lambda_m x) + \sin(\lambda_m x)], \tag{7}$$

where λ_m are the roots of the frequency equation of the free–free beam, which is of the form

$$\cosh(\lambda_m L_s)\cos(\lambda_m L_s) - 1 = 0. \tag{8}$$

3.2. APPROXIMATE SHELL NATURAL FREQUENCIES

To obtain the approximate natural frequencies of a cylindrical shell, a specific mode function, $U_{3m}(x)$ (equation (7)), is selected as the weighing function, and the λ_m's are calculated by the corresponding frequency equation (equation (8)). Applying the Galerkin approach, the approximate natural frequencies of a cylindrical shell with free–free boundary conditions can be obtained and expressed in terms of the following equation:

$$\omega^s_{mn} = \sqrt{\frac{D[(n/a)^8 - 4(n/a)^6 Z_m + 6(n/a)^4\lambda_m^4 - 4(n/a)^2\lambda_m^4 Z_m + \lambda_m^8] + (Eh_s/a^2)\lambda_m^4}{\rho h_s[(n/a)^4 - 2(n/a)^2 Z_m + \lambda_m^4]}}, \tag{9}$$

where

$$Z_m = \frac{\int_0^{L_s}[\mathrm{d}^2 U_{3m}(x)/\mathrm{d}x^2]\, U_{3m}(x)\,\mathrm{d}x}{\int_0^{L_s}[U_{3m}(x)]^2\,\mathrm{d}x}. \tag{10}$$

3.3. FORMULATION OF SHELL MODAL MASS

The inextensional approximation [6] applies sometimes to shells that have developable surfaces, but mainly to any shell with transverse modes of a wavelength which is one order of magnitude smaller than the smallest shell surface dimension. Using the inextensional approximation in the circumferential direction, by setting the membrane strain $\varepsilon^0_{\theta\theta} = 0 = (\partial U^s_{\theta mn}/\partial\theta) + U^s_{3mn}$ and assuming negligible in-plane membrane strain ($\varepsilon^0_{xx} = 0 = \partial U^s_{xmn}/\partial x$), the longitudinal displacement function of the shell, $U^s_{xmn}(x, \theta)$ is a constant and can be set to zero. The shell modal mass is then expressed as

$$N^s_{mn} = \int_0^{2\pi}\int_0^{L_s}\left\{[U_{3m}(x)\cos n(\theta - \psi)]^2 + \left[\frac{1}{n}\, U_{3m}(x)\sin n(\theta - \psi)^2\right]^2\right\} a\,\mathrm{d}x\,\mathrm{d}\theta. \tag{11}$$

The shell modal mass for the case of $\psi = 0$ can be derived as

$$N^s_{mn} = \begin{cases} a\pi(1 + 1/n^2) \displaystyle\int_0^{L_s} [U_{3m}(x)]^2 \, \mathrm{d}x, & m \neq 0, \quad n \neq 0, \\ 2a\pi \displaystyle\int_0^{L_s} [U_{3m}(x)]^2 \, \mathrm{d}x, & m \neq 0, \quad n = 0. \end{cases} \tag{12}$$

The zero in-plane deflection approximation works for very shallow shells and bending-dominated modes. For the plate–shell combination considered in this study, the shell modal mass calculated based on the above formulation provides useful information for evaluating vibration characteristics.

4. FORMULATION OF RECEPTANCES

A point receptance was defined as the ratio of a displacement or slope response at a certain point to a harmonic force or moment input at the same or a different point. When structures are joined along lines, line receptances may be defined and treated the same as point receptances under the condition that the spatial dependencies of the receptance can be eliminated. For the case where two circular plates are joined to a circular cylindrical shell as shown in Figure 1, in general five displacement or slope connections and five force or

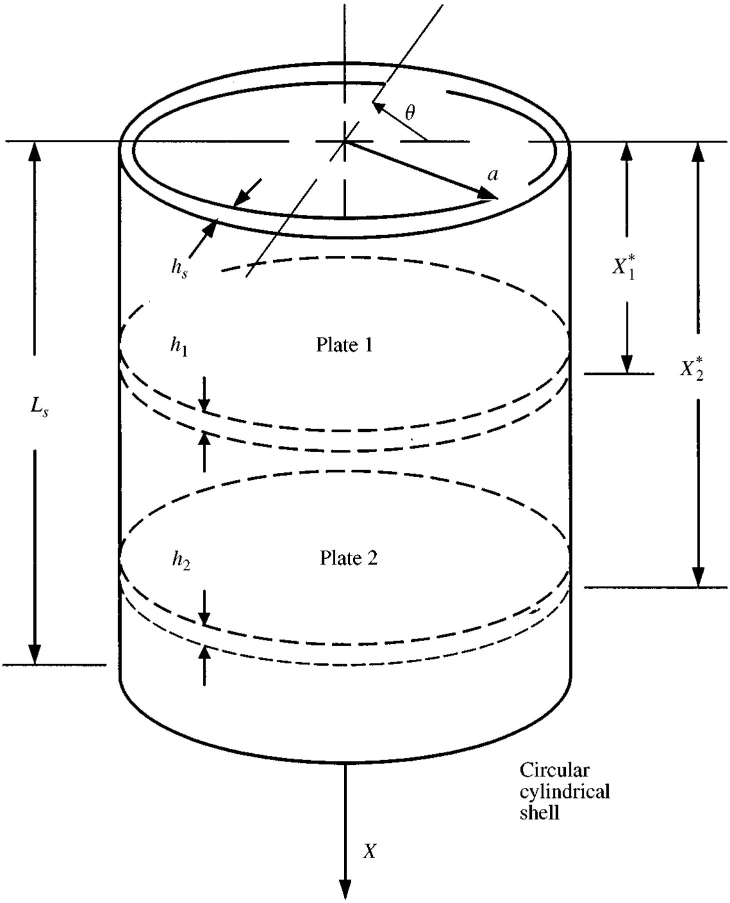


Figure 1. Two circular plates are joined to a cylindrical shell at arbitrary axial locations.

moment connections could be considered for each plate–shell pair. The possibilities are displacements in three directions and slopes in two directions (consistent with classical thin shell theory) for the shell and corresponding displacements or slopes for the plate, together with appropriate force or moment connections. However, the full treatment of using five connections is not necessary, if one contains the analysis to the lower frequencies and modes of the system. For these modes, it is reasonable to assume that the circular plate is inelastic in its plane and admits only to transverse deflections and a rigid-body motion in its plane. Also, compared to typical expected transverse amplitudes of the connected system, possible influences of small connection deflections in the tangential planes of the shell can be neglected. The number of connections of each plate–shell joint pair can be reduced to two, namely the radial slope changes at the plate boundary coupled to axial slope changes of the shell and the radial rigid-body motion of the plate coupled to a transverse motion of the shell. The system frequency equation of a two plate–shell combination was derived [2] as

$$\begin{vmatrix} \alpha_{11}+\beta_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22}+\beta_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}+\gamma_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44}+\gamma_{44} \end{vmatrix} = 0. \tag{13}$$

4.1. SHELL RECEPTANCES

When two circular plates are joined to a shell at its axial locations x_1^* and x_2^*, as shown in Figure 1, the receptances of the shell are defined in a similar way as discussed by Huang [2, 3]. Using a suitable beam function, the receptances of the shell under specified boundary conditions, which are in different forms from those reported previously, can be defined as

(1) The receptance due to the displacement response of the shell at x_i^* $(i = 1, 2)$ to the coupling line force at $x_j^*(j = 1, 2)$ is

$$\alpha_{bd} = \begin{cases} \displaystyle\sum_{m=1}^{\infty} \frac{U_{3m}(x_i^*)U_{3m}(x_j^*)}{\rho h_s(1+1/n^2)\{\int_0^{L_s}[U_{3m}(x)]^2\,\mathrm{d}x\}[(\omega_{mn}^s)^2-\omega^2]}, & n = P \neq 0, \\ \displaystyle\sum_{m=1}^{\infty} \frac{U_{3m}(x_i^*)U_{3m}(x_j^*)}{\rho h_s\{\int_0^{L_s}[U_{3m}(x)]^2\,\mathrm{d}x\}[(\omega_{mn}^s)^2-\omega^2]}, & n = P = 0, \end{cases} \tag{14}$$

where $b = 2i - 1$ and $d = 2j - 1$.

(2) The receptance due to the displacement response of the shell at $x_i^*(i = 1, 2)$ to the coupling line moment at $x_j^*(j = 1, 2)$ is

$$\alpha_{bg} = \begin{cases} \displaystyle\sum_{m=1}^{\infty} \frac{-U_{3m}(x_i^*)U'_{3m}(x_j^*)}{\rho h_s(1+1/n^2)\{\int_0^{L_s}[U_{3m}(x)]^2\,\mathrm{d}x\}[(\omega_{mn}^s)^2-\omega^2]}, & n = P \neq 0, \\ \displaystyle\sum_{m=1}^{\infty} \frac{-U_{3m}(x_i^*)U'_{3m}(x_j^*)}{\rho h_s\{\int_0^{L_s}[U_{3m}(x)]^2\,\mathrm{d}x\}[(\omega_{mn}^s)^2-\omega^2]}, & n = P = 0, \end{cases} \tag{15}$$

where $b = 2i - 1$ and $g = 2j$.

(3) The receptance due to the slope response of the shell at $x_i^*(i=1,2)$ to the coupling line force at $x_j^*(j=1,2)$ is

$$\alpha_{fd}=\begin{cases}\sum_{m=1}^{\infty}\dfrac{-U'_{3m}(x_i^*)U_{3m}(x_j^*)}{\rho h_s(1+1/n^2)\{\int_0^{L_s}[U_{3m}(x)]^2\,\mathrm{d}x\}[(\omega_{mn}^s)^2-\omega^2]}, & n=P\neq 0,\\ \sum_{m=1}^{\infty}\dfrac{-U'_{3m}(x_i^*)U_{3m}(x_j^*)}{\rho h_s\{\int_0^{L_s}[U_{3m}(x)]^2\,\mathrm{d}x\}[(\omega_{mn}^s)^2-\omega^2]}, & n=P=0,\end{cases}\tag{16}$$

where $f=2i$ and $d=2j-1$.

(4) The receptance due to the slope response of the shell at $x_i^*(i=1,2)$ to the coupling line moment at $x_j^*(j=1,2)$ is

$$\alpha_{fg}=\begin{cases}\sum_{m=1}^{\infty}\dfrac{U'_{3m}(x_i^*)U'_{3m}(x_j^*)}{\rho h_s(1+1/n^2)\{\int_0^{L_s}[U_{3m}(x)]^2\,\mathrm{d}x\}[(\omega_{mn}^s)^2-\omega^2]}, & n=P\neq 0,\\ \sum_{m=1}^{\infty}\dfrac{U'_{3m}(x_i^*)U'_{3m}(x_j^*)}{\rho h_s\{\int_0^{L_s}[U_{3m}(x)]^2\,\mathrm{d}x\}[(\omega_{mn}^s)^2-\omega^2]}, & n=P=0,\end{cases}\tag{17}$$

where $f=2i$ and $g=2j$. In the above formulation of shell receptances, $U_{3m}(x)$ and $U'_{3m}(x)$ will be different if the boundary conditions change.

4.2. PLATE RECEPTANCES

Unlike the shell receptances, the plate receptances are not affected by changes of the shell boundary conditions, and can be defined following a similar way as discussed by Huang [2, 3]. The plate receptances of a multiple plates–shell combination (k plates connected to a shell) are systematized as follows:

(1) The receptance due to the radial displacement response of plate i ($i=1,2,3,\ldots,k$) to the tangential loading F_{ri} ($i=1,2,3,\ldots,k$) is

$$\beta_{rr}=\begin{cases}-1/(a\rho h_{pi}\omega^2), & P=1,\\ 0, & P\neq 1,\end{cases}\tag{18}$$

where $r=1,3,5,\ldots,2k-1$.

(2) The receptance due to the slope response of plate i ($i=1,2,3,\ldots,k$) to the coupling line moment loading M_{ri} ($i=1,2,3,\ldots,k$) is

$$\beta_{ss}=\sum_{m=1}^{\infty}\frac{\pi\lambda_{mn}^2 a[J_{n+1}(\lambda_{mn}a)-J_n(\lambda_{mn}a)I_{n+1}(\lambda_{mn}a)/I_n(\lambda_{mn}a)]^2}{\rho h_{pi}N_{mn}^p[(\omega_{mn}^p)^2-\omega^2]},\tag{19}$$

where $s=2,4,6,\ldots,2k$, and the plate modal mass is

$$N_{mn}^p=\pi\int_0^a[J_n(\lambda_{mn}r)-J_n(\lambda_{mn}a)I_n(\lambda_{mn}r)/I_n(\lambda_{mn}a)]^2 r\,\mathrm{d}r.\tag{20}$$

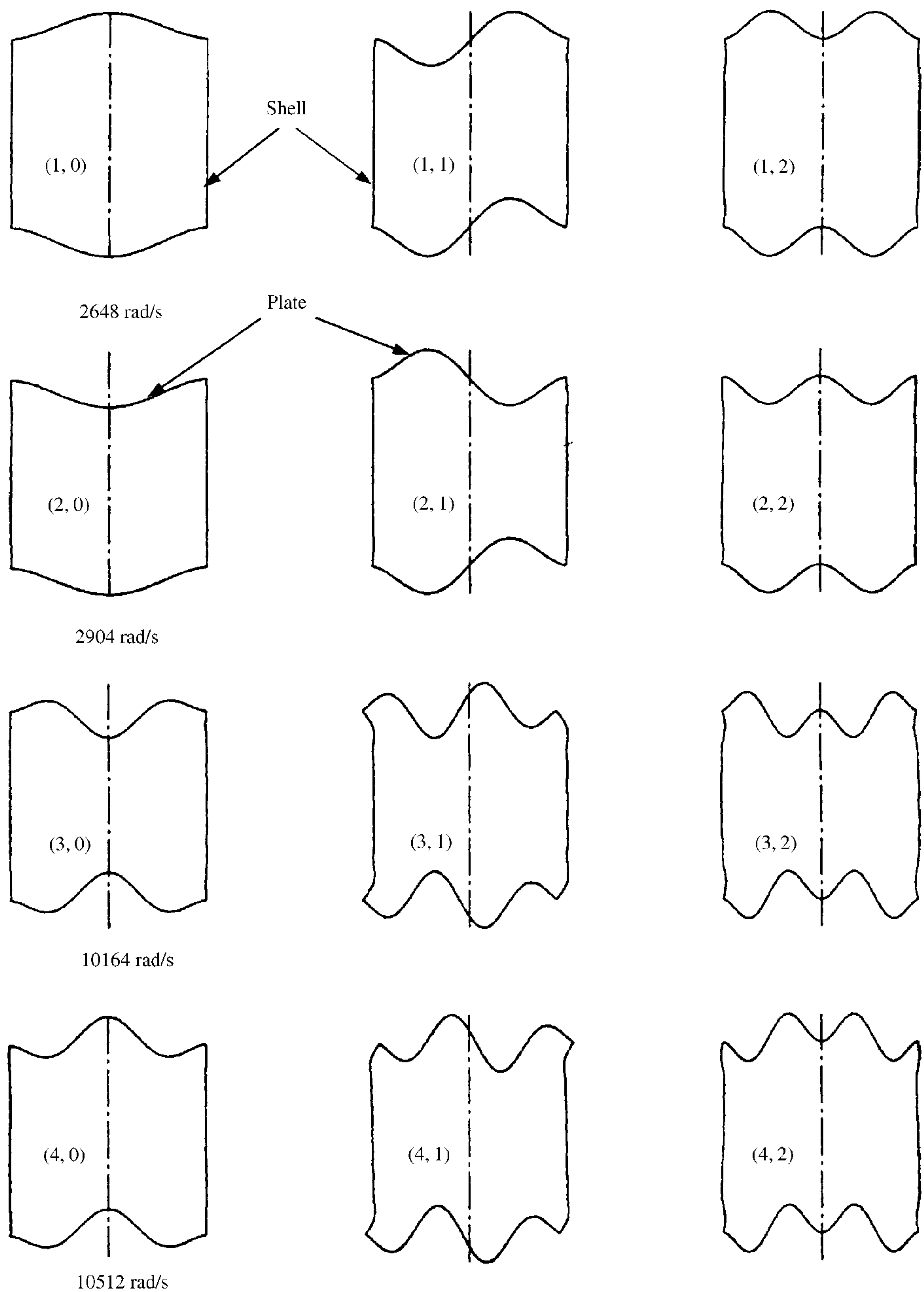
The cross receptances $\beta_{12},\beta_{21},\beta_{34},\beta_{43},\ldots,\beta_{2k-1,2k}$ are all zero from the reciprocity theorem.

5. RESULTS AND DISCUSSION

Using the receptance approach formulated above, numerical results are presented for a double plates–shell structure as shown in Figure 1. The structure includes one plate at $x_1^* = 0$ and the other plate at $x_2^* = L_s$. The shell and the plate are assumed to be steel with material properties: $E = 20{\cdot}6 \times 10^4$ N/mm^2, $\rho = 7{\cdot}85 \times 10^{-9}$ Ns^2/mm^4 and $\nu = 0{\cdot}3$. The dimensions are $L_s = 200$ mm, $a = 100$ mm, $h_s = h_{p1} = h_{p2} = 2$ mm. The solutions of the system frequency equation, equation (13), are the natural frequencies of the plate–shell combination. The left-hand side of equation (13) becomes infinite (singular) at each natural frequency of the plate and the shell, with the system natural frequencies being the zeros between the infinities. An iteration method (secant method) was selected to search for these system frequencies. Since all receptances were expressed as a series of plate or shell modes, convergence of the solution, as a function of terms used in the series expressions, was studied. In general, it was found that satisfactory convergence of solutions in the frequency range of interest was achieved using 40 terms. The maximum amplitudes of the mode shapes included in this report have been normalized to be 15% of the length of the shell for easy viewing. Hence, they should not be viewed as realistic amplitudes, which would typically not exceed the order of the shell thickness.

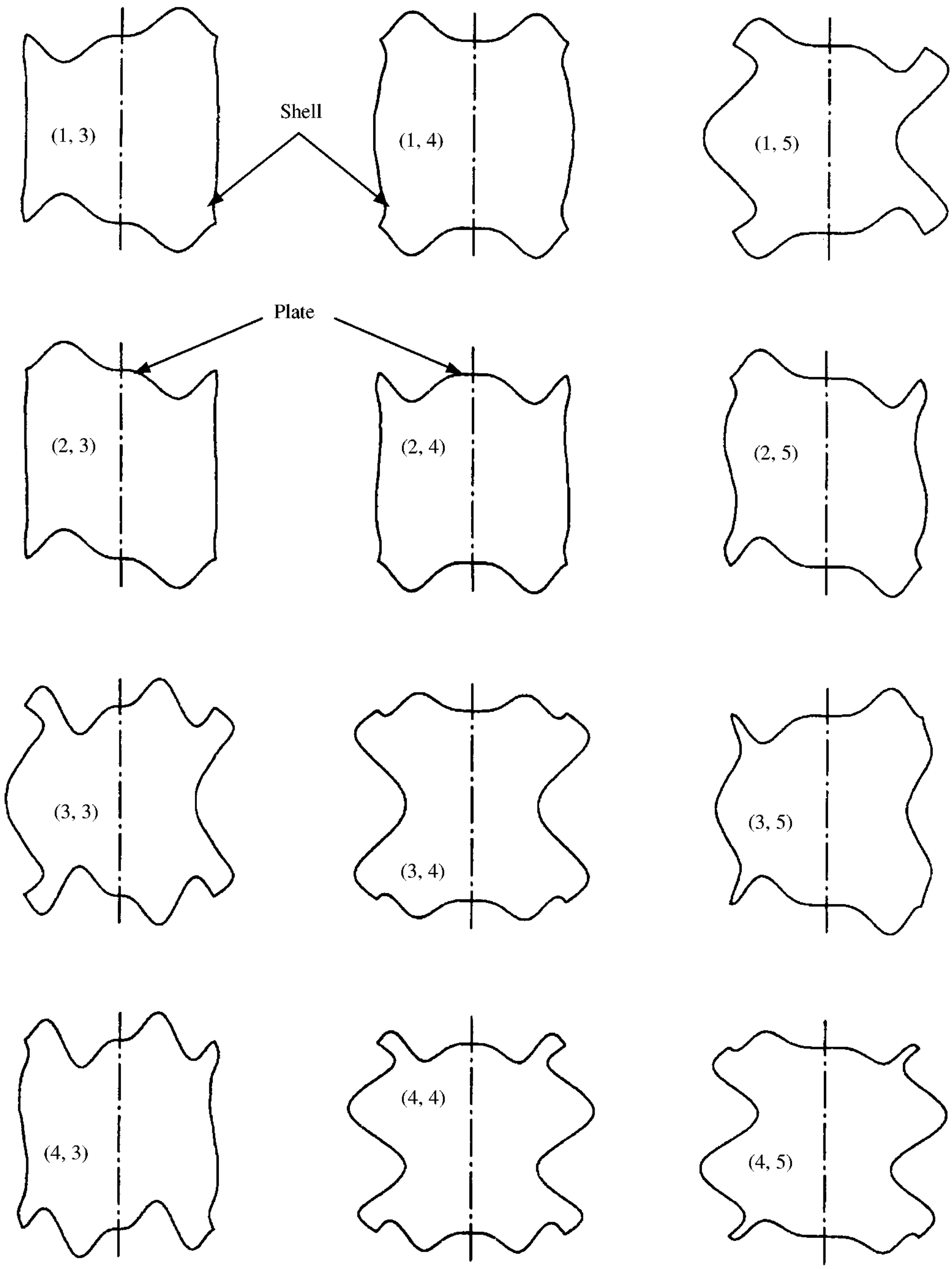
The mode shapes of the plate–shell combined structure under free–free boundary conditions are shown in Figure 2(a, b). Each mode shape is represented by an (m, n) designation, where n is the number of circumferential waves and m is the ascending frequency number for each n. These mode shapes show the normal displacements (transverse deflections) and rigid-body translations of the plate component and the radial displacements (transverse deflections) of the shell in the cross-section defined by $\theta = 0$ and π. The first four modes ($m = 1$–4) for $n = 0$–2 are shown in Figure 2(a); while those for $n = 3$–5 are shown in Figure 2(b). Since the two end plates are of the same thickness ($h_{p1} = h_{p2} = 2$ mm) and are at symmetrical positions, the in-phase and the out-of-phase mode pairs at the plate-dominated modes are as expected. It should be noted that the free–free shell with attached plates would exhibit so-called rigid-body modes of zero natural frequencies. The end plates of the joined structure with free–free boundary conditions are allowed to translate and rotate. Therefore, slope and displacement responses of the plates to the coupling line forces and line moments are expected to show up in some of the $n = 1$ modes, such as modes (3, 1) and (4, 1) in Figure 2(a). By carefully reviewing modes (1, 3), (1, 4) and (1, 5) in Figure 2(b), it is found that the shell of a free–free combined structure vibrates with less amplitude relative to the amplitudes of the two plate elements.

For the purpose of verifying the results obtained by the analytical method, the first four natural frequencies and shapes of the $n = 0$ mode type, calculated by using a finite-element model with the same geometric configuration and boundary conditions, are shown in Figure 3. As the analytical results are compared to the numerical results, it is seen that there is, in general, good agreement in lower order modes. However, it should be noted that there are sizable differences between the finite-element results and the analytical results as far as the system natural frequencies are concerned. Discrepancies are due to formulations used for the finite element, assumptions made for the system receptances and the approximate approaches used for obtaining the solution. The membrane effect of the shell is considered to include the in-plane deformation in the formulation of the finite element. For simplification, the analytical model does not employ full connection, it assumes that the circular plate is inelastic in its plane and thus neglects possible influences of small connection deflections in the tangential planes of the shell in order to reduce the number of connections of each plate–shell joint. For lower frequencies and modes of the system, the



(a)

Figure 2. (a) Mode shapes of a free–free circular cylindrical shell with two circular end plates. Each mode is represented by an (m, n) designation, where n is the number of circumferential waves and m is the ascending frequency number for each n. Here, modes (m, n) are for $m = 1$–4 and $n = 0$–2. (b) Mode shapes of a free–free circular cylindrical shell with two circular end plates. Here, modes (m, n) are for $m = 1$–4 and $n = 3$–5.



(b)

Figure 2. Continued

analytical solution based on the formulation of two connections is satisfactory. However, for high frequencies and modes, full connection is necessary.

Figure 4 shows a variation of the first ($m = 1$) natural frequencies of the plate–shell combination with free–free and pinned–pinned [2] boundary conditions versus the

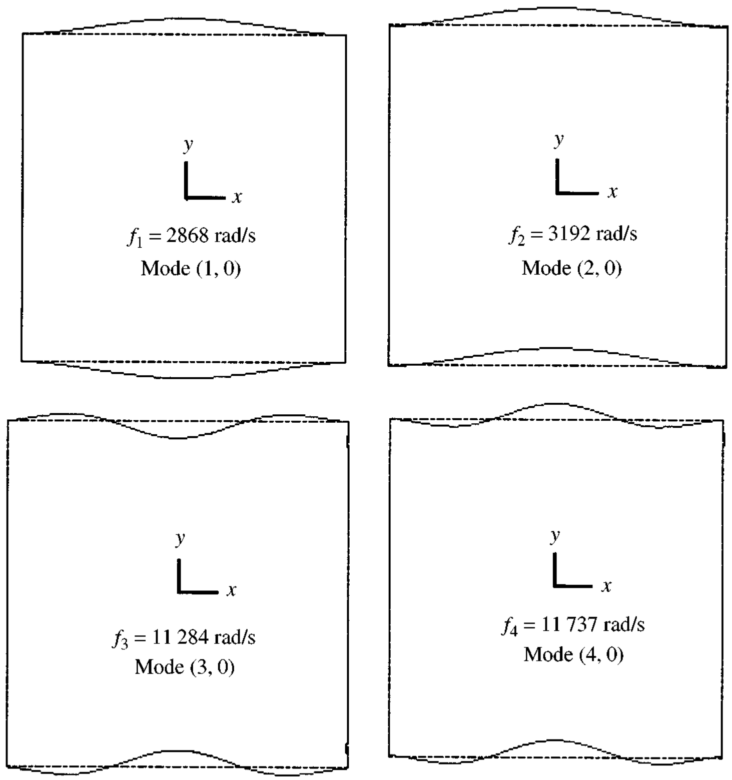


Figure 3. Natural frequencies and mode shapes of a free–free shell–plate combined structure obtained by using the finite-element program, ANSYS. They agree with those reported in Figure 2(a).

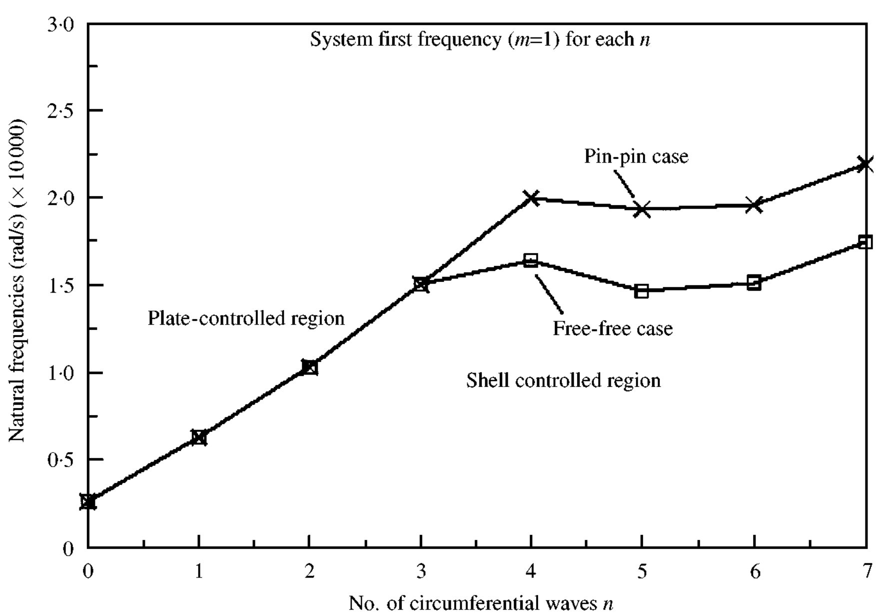


Figure 4. The first system frequencies of a free–free and a pinned–pinned shell–plate combined structures. The plate-controlled region is clearly visible.

circumferential wave number n. As it is seen in Figure 4, the $m = 1$ frequencies of the free–free case are different from those of the pinned–pinned case in the shell-controlled region. The natural frequencies of the free–free case now become the lower bounds. The shell becomes less stiff since it is free from constraints at both ends.

6. CONCLUSIONS

This study attempted to calculate approximate modal characteristics of a free–free plate–shell combined structure by applying approximate approaches. The Donnell–Mushrari–Vlasov equations were used since they neglect neither bending nor membrane effects. Due to simplicity and solvability, the beam functions were used as the weighing functions of the Galerkin method to derive the approximate natural frequency equations of the plate–shell combined structure under free–free boundary constraints. As a consequence, this study shows similarities and differences between the mode shapes, and leads to an understanding of the trend of the evolution of vibration modes. Although the calculated system natural frequencies are approximate, they agree with those obtained by the finite-element method in the lower order modes. It indicates that the approaches used in this study are appropriate. The proposed approaches can be used to calculate approximate modal characteristics of the plate–shell combined structures under various boundary conditions. The mode shapes obtained by these approaches are useful for design engineers to access the dynamic behavior of such systems.

In general, by physical reasoning, the mode shapes of a free–free shell combined with end plates are expected to be similar to those of a pinned–pinned shell combined with end plates for all circumferential wave numbers except for the case of $n = 1$. This is indeed illustrated in the results for some lower order natural modes. The reason that the results for the higher order modes do not show this as clearly is that the solution for the pinned–pinned case is an exact one, satisfying all the boundary conditions of the shell element, while the solution for the free–free case is only an approximate one. The approximation is expected to become less accurate for higher order modes. Also, since only two connections are considered in the formulation of receptance, torsional modes are hence not included in the results.

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