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THE PROBLEM OF VIBRATION FIELD RECONSTRUCTION: STATEMENT AND GENERAL PROPERTIES

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The paper addresses the problem of reconstructing the vibration field of a structure or the acoustic field of a bounded fluid from limited data. The problem relates to practical situations when there is a need to know the dynamic stress and displacement distribution over the entire structure, e.g., for estimating its remaining service life, but the vibration may be measured only on accessible parts of the structure surface. In the paper, this problem is mathematically formulated and its general properties—solvability, uniqueness, and continuity are studied. Most attention is paid to the analysis of the error of reconstruction. One of the main results obtained is the proof of existence of the optimal vibration model of the structure,which renders minimum to the reconstruction error. The application of the results to discrete systems is discussed.

1. INTRODUCTION

The problem of reconstructing continuous vibration fields in elastic structures or sound fields in bounded fluids from the data collected at a finite number of discrete points often arises in structural dynamics and acoustics. In vibration and noise control, a solution of this problem is necessary to place optimally a limited number of sensors and actuators in order to control the global field best—see, e.g., references [1–3]. In ocean acoustics, it is called the problem of mode filtering; it consists of finding the complex normal mode coefficients and, hence, the total acoustic field, of an oceanic waveguide from the pressure amplitudes sampled by a hydrophone array [4–6]. Reconstruction problems of this type are also very important in planning modal tests and vibration experiments on structures [7–9], identification of force loading on a structure [10, 11], and in many other problems—see, e.g., references [12–15]. The references presented above are only a small part of the vast literature on the subject. In most of these papers, the structure or medium under study is considered as accessible for measurement, and the question is how to choose appropriate measurement points to reconstruct the continuous field of the entire region.

In the present paper, a field reconstruction problem of a somewhat different type is investigated. It relates to the situation when a part of a structure or fluid is accessible for direct measurement while the remaining parts of it remain inaccessible. This is the case of many engineering structures into which insertion of sensors is undesirable or impossible, and a part of the outer surface is often the only place where the vibration amplitudes may be measured. So the reconstruction problem of this paper is stated as expansion of the continuous vibration field measured (e.g., by an optical method) on one part of a structure to adjacent unmeasured parts. In mathematical language, it is a problem of analytical continuation or extrapolation while most problems studied in the literature are interpolation problems. For brevity, the problem of this paper will be referred to as the field reconstruction problem or FR-problem. One of the main modifications of the FR-problem has been formulated and studied by the present author in references [16, 17] with regard to structural intensity: it was shown how to reconstruct the intensity field inside a structure from surface measurements. Later, this formulation was extended to the general case of reconstructing the vibration stress-and-displacement field in arbitrary linear structures [18–20]. Among the works of other authors that are close to the present one by formulation, though different by results, are the papers on modal shape expansion [9, 21], on prediction of the vibration amplitudes at unmeasured points of an Nd.o.f. system [22], and reconstruction of static stress fields of machine members [23].

The objectives of the present work are to formulate rigorously the FR-problem, study its main properties, and suggest some new results. Most attention will be paid to the analysis of the reconstruction error. In particular, it will be shown that the random error in the input data and complexity of the model used for describing the structure play a key role in forming the reconstruction error. Some peculiarities and even paradoxes, important for practice, connected with the error of reconstruction will be revealed and explained.

The layout of the present paper is as follows. In section 2, several modifications of the FR-problem as well as a general solution and its properties (existence, uniqueness, and continuity) are given. In section 3, is presented one of the main results of the paper—a proof of the existence of the optimal model of the structure under study that renders the minimum reconstruction error. In section 4, the results obtained for continous structures are applied to discrete systems with a finite number of degrees of freedom. In section 5, the results are summarized. In Appendix A, is proved the three-dimensional dynamic Almansi theorem on which uniqueness of the solution is based and Appendix B is devoted to asymptotics of the singular values of continuous elastic structures.

2. FORMULATION AND GENERAL PROPERTIES OF THE FR-PROBLEM

2.1. BASIC MODIFICATION OF THE FR-PROBLEM

Let there be a bounded continuous linear elastic structure (body) Ω with boundary $\partial\Omega$ vibrating harmonically (time-dependence $\exp(-\mathrm{i}\omega t)$ is implied) under the action of an external load applied to the boundary (see Figure 1(a)). The geometry and material parameters of Ω are assumed to be known. Further, only elastic structures will be considered though all the results obtained are valid for bounded fluid media as well. The boundary of the structure $\partial\Omega$ is assumed to consist of two parts, $\partial\Omega = \partial\Omega_{ac} \oplus \partial\Omega_{in}$, where part $\partial\Omega_{ac}$ is free of traction and is accessible for direct measurement of the vibration displacement amplitude, while part $\partial\Omega_{in}$ where external forces may be applied is inaccessible for measuring vibration and hence neither displacement nor external force is known at $\partial\Omega_{in}$. The FR-problem for the structure Ω can be formulated as follows. Find a solution to the homogeneous governing differential equation

$$\mathbf{Lu}(x) = 0, \quad x \in \Omega, \tag{1}$$

which satisfies the following boundary conditions on the accessible part $\partial\Omega_{ac}$:

$$\mathbf{u}(s) = \mathbf{u}_0(s), \quad \mathbf{f}(s) = 0, \quad s \in \partial\Omega_{ac}. \tag{2}$$

Here, $\mathbf{u} = \{u_1, u_2, u_3\}^{\mathrm{T}}$ is the displacement vector, $\mathbf{f} = \{f_1, f_2, f_3\}^{\mathrm{T}}$ the vector of the external force, $\mathbf{u}_0(s)$ the vector of known (measured) displacement amplitudes, $\mathbf{L}$ a linear differential operator of an elliptic type, T means transposition. The operator $\mathbf{L}$ is not specified in this section: for an elastic body equation (1) is the well-known Lame-vector equation of the elasticity theory, for a flexurally vibrating plate it is the Germain–Lagrange equation, etc. (see, e.g., reference [24]).

From a mathematical point of view, FR-Problem (1), (2) is not a conventional problem for an operator of the elliptic type because not a field quantity is specified on the inaccessible part $\partial\Omega_{in}$ of the boundary, while the boundary conditions on the accessible part $\partial\Omega_{ac}$ are over-determined, i.e., both the displacement and traction are specified. As a consequence, the FR-problem belongs, according to the definition of Hadamard [25], to the class of ill-posed problems. Solutions to such problems are extremely sensitive to errors in the input data $\mathbf{u}_0(s)$.

The boundary value problem (1), (2) can be reformulated as a Fredholm integral equation of the first kind

$$\mathbf{u}_0 = \mathbf{Gf}, \tag{3}$$

where $\mathbf{f}$ is the unknown vector force acting on the inaccessible boundary $\partial\Omega_{in}$, $\mathbf{G}$ the integral operator over $\partial\Omega_{in}$ with the matrix Green function (which is assumed to be known) as the kernel [17]. The integral formulation is useful for investigating the general properties. It is also used in the identification of a force loading on a structure via the remote structure response [10, 11].

2.2. OTHER MODIFICATIONS

Another practical modification of the FR-problem of this type is that when the vibration amplitudes are prescribed (measured) not only on a part of the boundary but also on a part Ω_{ac} of the structure Ω (Figure 1(b)). This is mostly the case of two- and one-dimensional structures (shells, plates, beams, rods, etc.).

From a mathematical point of view, the data in the region Ω_{ac} are excessive. For constructing a unique solution, it is sufficient to know the vibration amplitudes only on the corresponding part of the boundary. So, this modification of the FR-problem is reduced to the basic modification as shown in Figure 1(b). However, from a practical point of view, the data in Ω_{ac} are not redundant: when the measured values are contaminated by noise, the reconstruction error is strongly dependent on the amount of data: the more the data, the higher the accuracy of reconstruction.

One more modification of the FR-problems relates to large and non-uniform structures that are too complicated to be reasonably treated analytically or numerically. In this case, the structure is divided into several more simple and tractable substructures. For each of them the corresponding FR-problem is formulated. To reconstruct the stress-and-displacement field in such structures one has, thus, to solve independently several problems (1)–(3) and to collect the results into one solution (see Figure 1(c)).

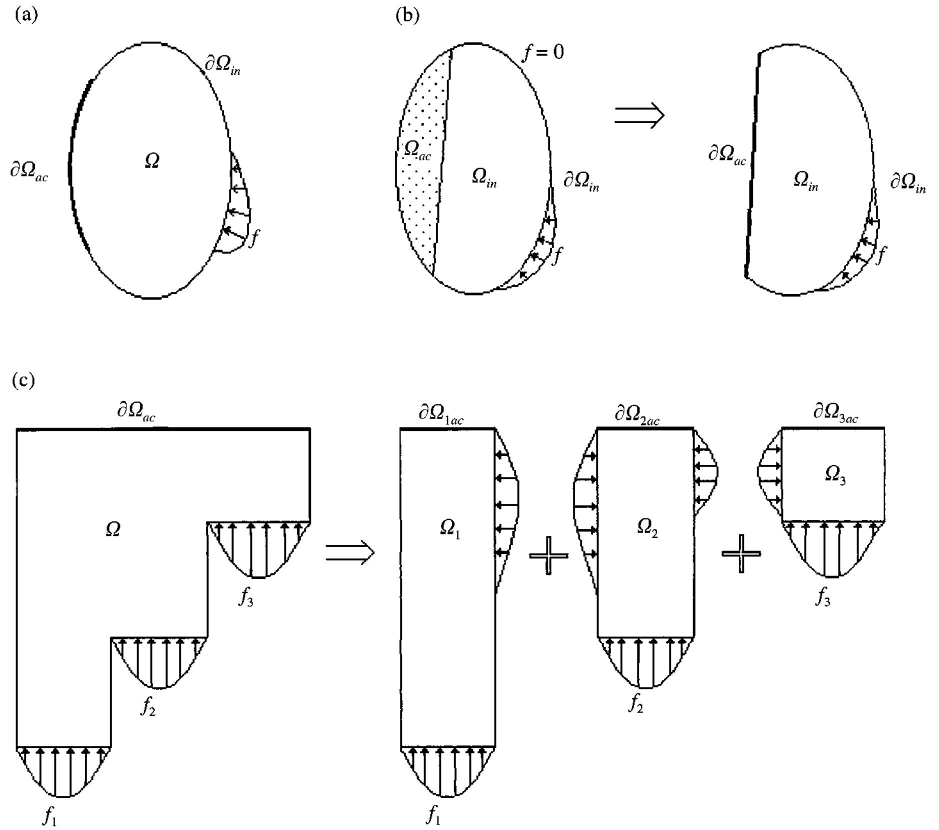

Figure 1. The field reconstruction problem for an elastic structure Ω with boundary surface $\partial\Omega = \partial\Omega_{ac} \oplus \partial\Omega_{in}$. (a) Basic modification: given the displacement vector $\mathbf{u}_0$ on a free of traction accessible part $\partial\Omega_{ac}$ of the boundary; it is required to reconstruct via $\mathbf{u}_0$ the stress-and-displacement field in Ω and, if needed, unknown external force $\mathbf{f}$ applied to the inaccessible part $\partial\Omega_{in}$ of the boundary. (b) Modification of the FR-problem (displacement is prescribed in a part Ω_{ac} of the structure) is reduced to the basic modification. (c) The FR-problem for a complicated structure may be split into several (here three) more simple FR-problems for substructures Ω_1, Ω_2 and Ω_3.

2.3. FORMAL SOLUTION AND GENERAL PROPERTIES

In this section, the main general properties of the FR-problem are briefly documented (for more details see reference [17]). However, first an exact formal solution based on the singular-value decomposition (SVD) technique is presented.

Let $\sigma_1 > \sigma_2 > \cdots$ be the singular values of the integral operator $\mathbf{G}$ in equation (3), and $\{\mathbf{f}_1, \mathbf{f}_2, \cdots\}$ and $\{\mathbf{u}_1, \mathbf{u}_2 \cdots\}$ be the singular pair, i.e., two sets of orthonormal functions (note that σ_m^2 are the eigenvalues of the Hermitian operators $\mathbf{G}^*\mathbf{G}$ and $\mathbf{G}\mathbf{G}^*$, $\mathbf{f}_m$ and $\mathbf{u}_m$ being their eigenfunctions (see, e.g., reference [26]). Representing the given function $\mathbf{u}_0$ as a series in $\mathbf{u}_m$

$$\mathbf{u}_0(s) = \sum_{m=1}^{\infty} b_m \mathbf{u}_m(s) \tag{4}$$

and seeking the solution for $\mathbf{f}$ as a series in $\mathbf{f}_m$, one can obtain, after substitution into equation (3), the following exact formal solution for the unknown forces and, hence, for the

FR-problem:

$$\mathbf{f}_{ex}(q) = \sum_{m=1}^{\infty} \frac{b_m}{\sigma_m} \mathbf{f}_m(q). \tag{5}$$

Here, co-ordinates s and q relate to the accessible and inaccessible regions of the structure, $s \in \partial\Omega_{ac}$, $q \in \partial\Omega_{in}$ respectively.

Existence of solution. It follows from equation (5) that a bounded solution of the FR-problem exists (the series converges) if the coefficients b_m in expansion (4) of the given vector function $\mathbf{u}_0(s)$ tend to zero more rapidly than the singular values σ_m do. As shown in Appendix B, the singular values decrease exponentially with the index m. Therefore, the prescribed function $\mathbf{u}_0(s)$ must contain only few first spatial harmonics $\mathbf{u}_m(s)$ with lower indexes m. In other words, a solution of the reconstruction problem exists if the measured function $\mathbf{u}_0(s)$ is sufficiently smooth, i.e., only first coefficients b_m in its spatial expansion (4) are not zeros. If the displacement $\mathbf{u}_0(s)$ could be known mathematically exactly and without errors, the solution of the FR-problem would, in principle, exist for an arbitrarily small, finite-continuous accessible part $\partial\Omega_{ac}$ of the structure boundary.

In reality, the prescribed function $\mathbf{u}_0(s)$ is always contaminated with random noise that, as a rule, has a wide spatial spectrum. Expansion (4) of such data contains non-zero components of large indexes which, after amplifying by inversion, i.e., by factors σ_m^{-1} in equation (5), may give an arbitrarily large error in the solution, making it unstable. Thus, for noisy data, an exact solution of the FR-problem does not generally exist, and only approximate solutions may be found in that case (see section 3).

Uniqueness of solution. A solution to the FR-problem, when it exists, is unique. This property is a consequence of the fact that the boundary conditions of both types, kinematic and force, are imposed on a continuous region $\partial\Omega_{ac}$ of the structure boundary (see equation (2)). A proof of uniqueness can be carried out with the help of the Almansi theorem [27] in the same manner as is done in reference [17]. According to this theorem, a finite elastic body is at rest and stress-free if on a part of its surface, even on very small one, the displacements and stresses are simultaneously equal to zero. If two different solutions to the FR-problem, $\mathbf{u}_1$ and $\mathbf{u}_2$, are allowed to exist, the difference $\mathbf{u}_1 - \mathbf{u}_2$ satisfies the conditions of the Almansi theorem, and hence $\mathbf{u}_1 - \mathbf{u}_2 = 0$ everywhere in the body. The proof of the Almansi theorem for three-dimensional vibrating bodies, as well as a discussion of its validity for discrete systems, is given in Appendix A of this paper.

Continuity of solution. A solution of the FR-problem, when it exists, does not continuously depend on the input data $\mathbf{u}_0(s)$. This property follows from the Riemann lemma [28]: small variations in the data of a Fredholm integral equation of the first kind with an analytical kernel can lead to arbitrarily large variations in the solution. Integral equation (3) has an analytical kernel, since the accessible and inaccessible parts of the boundary do not intersect, therefore its solutions are not stable with respect to the input data $\mathbf{u}_0(s)$. Consequently, the field reconstruction problem belongs to ill-posed, in the Hadamard sense [25], problems of mathematical physics. Such problems are often met with in various fields of science. In fact, all physical problems where a cause is determined from consequences are ill-posed. There is a lot of literature on the subject with many techniques developed for treating them—see, e.g., reference [28]. The principal idea of all these techniques is that such an ill-posed problem is replaced by a certain well-posed problem that is close, in some sense, to the initial one. One of them based on the truncated SVD is used in the next section.

3. EXISTENCE OF THE OPTIMAL VIBRATION MODEL

In this section, the FR-problem with the input data containing a random noise component is considered. As was established in the previous section, the problem in this case generally does not have an exact solution and should therefore be reshaped to become solvable approximately. Section 3 shows how this is done in the frame of the SVD approach. Most attention is paid to the reconstruction error of approximate solutions. It is proved that there exists an optimal model of the vibrating structure that yields an approximate solution with the minimal reconstruction error. The remarkable feature is that, though the structure under study is continuous, its optimal model has a finite number of model parameters (and d.o.f.s) which is determined mostly by the SNR of the input data. This result (existence of the optimal model), owing to the generality of its derivation, is valid for all modifications of the reconstruction problem in any linear structure or medium.

3.1. TRUNCATED SVD-SOLUTION

Let the displacement amplitudes $\mathbf{u}_0(s)$ in formulations (1)–(3) be measured with an additive noise $\mathbf{n}(s)$ which has a wide singular spectrum $\{\delta_m\}$, i.e., the amplitudes of the spatial noise components in the expansion

$$\mathbf{n}(s) = \sum_{m=1}^{\infty} \delta_m \mathbf{u}_m(s) \tag{6}$$

decrease slowly compared to the singular values, $|\delta_m/\sigma_m| > 1$. (Note that the primary noise is a random function of time and space, $\mathbf{n}(t, s)$. Here, it is assumed that the function $\mathbf{n}(s)$ is the ω-component of the finite Fourier transform of a particular sample of $\mathbf{n}(t, s)$). Examples of such a noise are the instrumentation noise or the noise due to the rounding errors of the digital representation of the input data in a computer. For them, series (5) diverges and the exact solution of the FR-problem does not exist.

One way to attack the FR-problem in that situation is to seek an approximate solution in the form of a truncated SVD

$$\mathbf{f}(q) = \sum_{n=1}^{N} a_n \mathbf{f}_n(q) \tag{7}$$

containing a finite number N of terms. The main idea of such a method (used also in other works, e.g., references [6, 9, 17]) is that, with a judicious choice of the number N of singular modes, it does not differ much from the exact solution (5) because it does not contain spatial components of high indexes. On the other hand, this solution cuts off higher components thereby reducing the inaccuracy due to the instrumentation and computer noise.

It should be emphasized, however, that the truncation of the SVD-solution actually means that the continuous elastic medium of the problem (having infinite number of d.o.f.s) is replaced with an elastic system with a finite number of generalized d.o.f.s defined by the set of N singular modes. (Perhaps, it could be proved that such a system is equivalent to a certain FE-model.) The problem of finding approximate solution (7) to the initial FR-problem is, thus, equivalent to the problem of finding an exact solution of the same FR-problem for a more simple (discrete) elastic system. This new system, being described by the first N "long-wavelength" singular modes, has smooth response functions and is insensitive to "short-wavelength" external loading at the boundary $\partial\Omega_{in}$. Therefore, the solution to the new FR-problem can be proved to be stable with respect to the input data variations and, besides, satisfies the other two conditions of Hadamard [25]. So the FR-problem for the simplified structure (obtained as a result of truncation of the SVD) is a well-posed mathematical problem and its solutions may be obtained by usual techniques. The question of how these solutions relate to solutions of the initial FR-problem will be discussed in the next two subsections.

Substituting equations (4), (6) and (7) into equation (3) with noise added, $\mathbf{u}_0 + \mathbf{n} = \mathbf{G}\mathbf{f}$, and using the orthogonality property of the singular modes, one can obtain the sought approximate solution of the initial FR-problem in the form

$$\mathbf{f}(q, N) = \sum_{n=1}^{N} (b_n + \delta_n)\,\mathbf{f}_n(q)/\sigma_n. \tag{8}$$

3.2. RECONSTRUCTION ERROR

The goal now is to examine how closely solution (8) approximates to the actual vibration field in the inaccessible part of the structure. The difference between this solution and exact solution (5) is

$$\Delta\mathbf{f} = \mathbf{f}(q, N) - \mathbf{f}_{ex}(q) = \sum_{n=1}^{N} (\delta_n/\sigma_n)\,\mathbf{f}_n(q) - \sum_{n=N+1}^{\infty} (b_n/\sigma_n)\,\mathbf{f}_n(q),$$

$q \in \Omega_{in}$. The squared Euclidean norm of the difference in the inaccessible part of the structure

$$\|\Delta\mathbf{f}\|_{in}^2 = \int_{\Omega_{in}} |\Delta\mathbf{f}|^2\,\mathrm{d}q = \sum_{n=N+1}^{\infty} |b_n|^2/\sigma_n^2 + \sum_{n=1}^{N} |\delta_n|^2/\sigma_n^2 \tag{9}$$

characterizes the absolute reconstruction error. After introducing the norm of the exact solution

$$\|\mathbf{f}_{ex}\|_{in}^2 = \int_{\Omega_{in}} |\mathbf{f}_{ex}|^2\,\mathrm{d}q = \sum_{n=1}^{\infty} |b_n|^2/\sigma_n^2,$$

the relative field reconstruction error can be defined as

$$\Delta_{rec} = \|\Delta\mathbf{f}\|_{in}/\|\mathbf{f}_{ex}\|_{in}. \tag{10}$$

Similarly, the absolute and relative errors of the approximate solution (8) in the input (measured) data can be introduced as

$$\Delta_{data} = \|\mathbf{u}_0 + \mathbf{n} - \mathbf{G}\mathbf{f}(q, N)\|_{ac}/\|\mathbf{u}_0\|_{ac}, \tag{11}$$

where the Euclidean norm is taken over the accessible part of the structure

$$\|\mathbf{u}_0\|_{ac}^2 = \int_{\Omega_{ac}} \|\mathbf{u}_0(s)\|^2\,\mathrm{d}s = \sum_{n=1}^{\infty} |b_n|^2, \qquad \|\mathbf{u}_0 + \mathbf{n} - \mathbf{G}\mathbf{f}(q, N)\|_{ac}^2 = \sum_{n=N+1}^{\infty} |b_n + \delta_n|^2. \quad (12, 13)$$

3.3. MINIMUM RECONSTRUCTION ERROR

Since series (12) and the analogous series for noise (6), $\|\mathbf{n}\|_{ac}^2 = \sum_{n=1}^{\infty} |\delta_n|^2$, converges absolute error (13) as well as relative error (11) in the data decrease monotonically with N. If one is interested in a good description of the input (measured) data only, one should choose as complex a singular model as possible: the finer the model, i.e., the larger the N the better

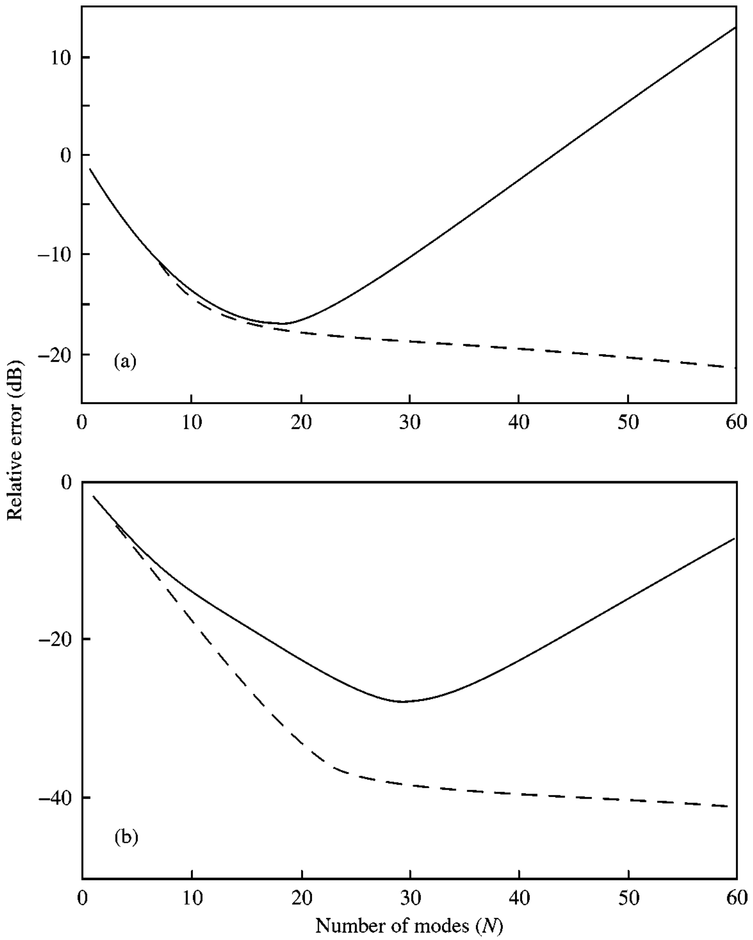

Figure 2. Reconstruction error (solid lines) and error in the input data (dashed lines) as functions of the model complexity (a) $SNR = 16$ dB and (b) $SNR = 36$ dB.

the approximation, so that the error in the data tends to zero, $\lim \Delta_{data} = 0$, when N tends to infinity.

The reconstruction error behaves quite differently. From equation (9) it is seen that the absolute reconstruction error consists of two components. The first component is the model error, i.e., the error due to removing singular modes of high indexes. It is represented by the first summation on the right hand side of equation (9). This component of the error diminishes monotonically with N: the finer the model, the smaller the model error. The second component of the reconstruction error, represented by the second summation in equation (9), is the noise error. It is induced by noise and, owing to inequality $|\delta_n|/\sigma_n > 1$—see preceeding section—increases monotonically with N: the finer the model, i.e., the larger the N, the greater the noise error.

When N is small, the first (decreasing) component of the error dominates. When N is large, the second (increasing) component becomes dominant. Mathematically, a sum of two variables, one of which monotonically decreases and another increases, must have a minimum. Thus, the reconstruction error, at a certain number $N = N_0$ has a minimum. The model containing N_0 singular modes is just the optimal model of the problem that provides the minimum reconstruction error. Figure 2 depicts reconstruction error (10) as well as the error in data (11) as functions of model complexity, i.e., of the number N of the model parameters. The results correspond to the structure (bar) and frequency for which the first 10 singular values are equal to unity and the rest decay exponentially with the index n as $\sigma_n = \exp[-\alpha(n-10)]$ with $\alpha = 0{\cdot}1$ (see Appendix B). The data are given by expansion (4) with coefficients decaying faster than the singular values, $b_n = \exp(-\beta n)$ with $\beta = 0{\cdot}2$, while noise is taken such that the coefficients of its expansion (6) tend to zero rather slowly, $\delta_n = \delta_0 \exp(-\gamma n)$ with $\gamma = 0{\cdot}01$, and the exact solution of the FR-problem does not exist ($\delta_n/\sigma_n > 1$). The constant δ_0 depends on the signal-to-noise-ratio defined as

$$SNR = 20 \log \|\mathbf{u}_0\|_{ac}/\|\mathbf{n}\|_{ac}. \tag{14}$$

It is seen in Figure 2 that the error in the data (dashed lines) diminishes monotonically while the reconstruction error (solid lines) decreases for small N (coarse models) and increases for large N (fine models) having a distinct minimum. The optimal number N_0 of the model parameters depends on the signal-to-noise-ratio (14): the higher the SNR the larger the N_0. In Figure 2, the values $SNR = 16$ and 36 dB correspond to the optimal models with $N_0 = 17$ and 30. In the limit, when the data are noiseless, the reconstruction error decreases monotonically for all N, so that the finer the model the better the reconstruction is.

Figure 3 illustrates the structure of the errors shown in Figure 2: both components of the errors, the model error component and the error component due to noise (see equations (9) and (13)), are represented by dashed lines. In case of the data error (Figure 3(a)), both components are decreasing and therefore the total error also decreases monotonically—rapidly for small N (as the model component) and then changes its slope and decreases (as the noise component) for large N. In the case of the reconstruction error (Figure 3(b)), the model error components decreases rapidly with N, while the noise component increases, so that the total reconstruction error tends in turn to these two curves having a minimum at the point of their intersection: $N = N_0 = 30$ ($SNR = 36$ dB).

As can be clearly seen in Figures 2 and 3, the minimum of the reconstruction error is situated near the region where the curve of the data error changes its slope. In other words, the minimum corresponds to models for which the model error component becomes comparable with the noise error component. This observation is useful in practice: in reconstructing the vibration field in a real structure, when the optimal model is unknown *a priori*, this property allows one to find the optimal number N_0 using the input data only [19].

The fact that finer models ($N > N_0$) may have larger reconstruction error looks like a paradox being at variance with physical intuition, i.e., with the statement: "More exact models must give better description of reality." However, this statement is not true always. The FR-problem is just the case when the opposite is valid.

An explanation of the paradox is the following. In the reconstruction problem, the model parameters are identified from the input data measured with unavoidable noise. Finer models, containing the singular modes of high indexes, $N > N_0$ describe mostly the noise component instead of the signal. It is evident, e.g., in Figure 3(a) that for $N > 20$, signal components are too small to matter and the coefficients of the modes are determined by the noise components. These higher index modes give a better description of the measured data, but they have nothing to do with the actual vibration field signal in the inaccessible parts of the structure and therefore may, and actually do, give large errors. Thus, physically, it is not reasonable to approximate the noisy input data very accurately. There should exist the optimal accuracy of describing measured data, and, hence, the optimal vibration model of the structure, which is the best for reconstructing the vibration field in unmeasured parts of the structure.

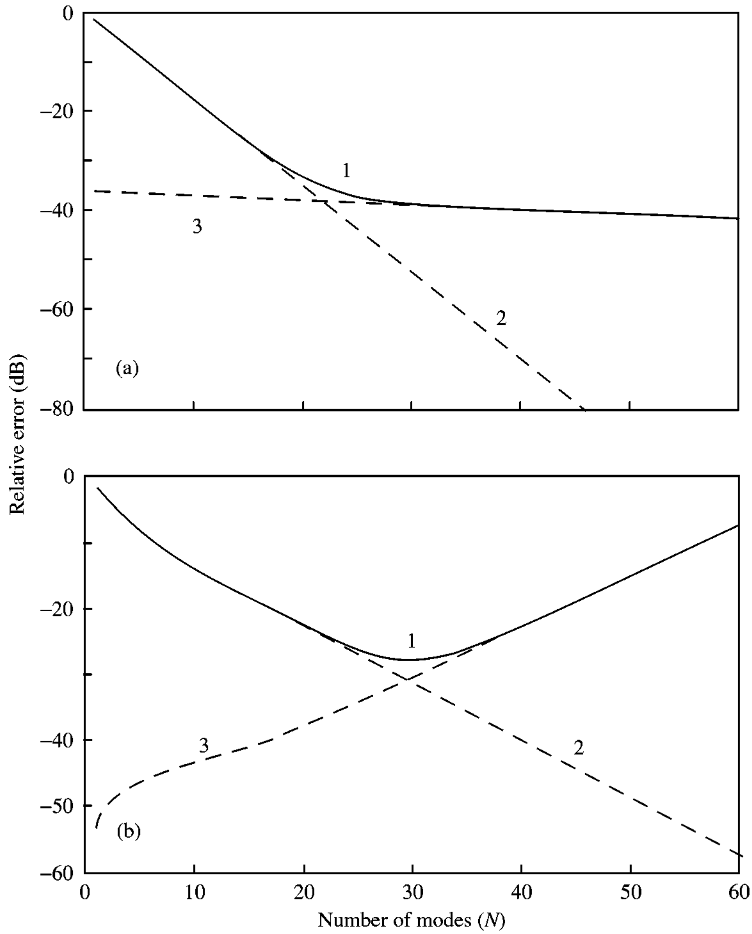

Figure 3. Error in the input data (a) and reconstruction error (b) versus model complexity: total error 1 and its components—error of modelling 2 and error due to noise 3. $SNR = 36$ dB.

4. FR-PROBLEM IN *N*d.o.f. SYSTEMS

It is common practice nowadays to represent an elastic structure as a discrete mechanical system, e.g., as a finite-element model. Therefore, in this section it will be shown what the above results, established for continuous structures, mean when applied to systems with a finite number N degrees of freedom (Nd.o.f. systems).

Relating to the field reconstruction problem, the main difference between continuous structures and discrete systems is in the number of d.o.f.s. In a continuous structure, the number of measurement points in any of its accessible parts is arbitrarily large (mathematically, infinite). In Nd.o.f. systems, the number of measured points is always finite. Another difference is that a general Nd.o.f system may be much more complex than a typical continuous structure: a discrete model of such a continuous structure has sparse mass and stiffness matrices, while the structure matrices of Nd.o.f. systems may, generally, be arbitrary. As a result, in a continuous structure, it is possible to expand the vibration field from a small measurement part to much larger unmeasured adjacent parts of the structure, while in a general Nd.o.f. system this possibility is severely restricted.

4.1. GENERAL Nd.o.f. SYSTEM

Consider an arbitrary linear system with Nd.o.f. executing harmonic vibration of angular frequency ω under the action of external forces applied to N_f d.o.f. Let the rest of the $N - N_f$ undriven d.o.f.s consist of N_m measured d.o.f.s and N_u unmeasured d.o.f.s so that $N = N_m + N_u + N_f$. The field reconstruction problem is formulated as follows.

Given the structural matrices of the system and N_m complex amplitudes $\mathbf{x}_m$ of the vibration displacement of the measured d.o.f.s, find $N_u + N_f$ unmeasured displacement amplitudes, $\mathbf{x}_u$ and $\mathbf{x}_f$, and N_f amplitudes of the external forces, $\mathbf{f}_1$.

The equations of motion of this system can be written as

$$\mathbf{A}\mathbf{x} = \mathbf{f}, \tag{15}$$

where N-vector $\mathbf{x}$ consists of the above three subvectors, $\mathbf{x} = [\mathbf{x}_m^{\mathrm{T}}, \mathbf{x}_u^{\mathrm{T}}, \mathbf{x}_f^{\mathrm{T}}]^{\mathrm{T}}$, the first $N_m + N_u$ components of the N-vector $\mathbf{f}$ are equal to zero, $\mathbf{f} = [0, 0, \mathbf{f}_1^{\mathrm{T}}]^{\mathrm{T}}$; $\mathbf{A} = \mathbf{K} - \mathrm{i}\omega\mathbf{B} - \omega^2\mathbf{M}$ is an $(N \times N)$-matrix of the complex dynamic stiffnesses, with $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{B}$ being real symmetric mass, static stiffness and damping matrices, T denotes transposition. By splitting the matrix $\mathbf{A}$ into nine submatrices, according to the splitting the N-vectors into three subvectors, one can represent set (15) of N equations in the form

$$\begin{aligned}
&\mathbf{A}_{mm}\mathbf{x}_m + \mathbf{A}_{mu}\mathbf{x}_u + \mathbf{A}_{mf}\mathbf{x}_f = 0 \quad (N_m \text{ equations}),\\
&\mathbf{A}_{um}\mathbf{x}_m + \mathbf{A}_{uu}\mathbf{x}_u + \mathbf{A}_{uf}\mathbf{x}_f = 0 \quad (N_u \text{ equations}),\\
&\mathbf{A}_{fm}\mathbf{x}_m + \mathbf{A}_{fu}\mathbf{x}_u + \mathbf{A}_{ff}\mathbf{x}_f = \mathbf{f}_1 \quad (N_f \text{ equations}).
\end{aligned} \tag{16}$$

As N_m measured amplitudes $\mathbf{x}_m$ are given while other displacement amplitudes and forces are unknown, it is reasonable to rewrite set (16) as

$$\begin{bmatrix} \mathbf{A}_{mu} & \mathbf{A}_{mf} & 0 \\ \mathbf{A}_{uu} & \mathbf{A}_{uf} & 0 \\ \mathbf{A}_{fu} & \mathbf{A}_{ff} & -\mathbf{E}_f \end{bmatrix} \begin{bmatrix} \mathbf{x}_u \\ \mathbf{x}_f \\ \mathbf{f}_1 \end{bmatrix} = - \begin{bmatrix} \mathbf{A}_{mm} & \mathbf{x}_m \\ \mathbf{A}_{um} & \mathbf{x}_m \\ \mathbf{A}_{fm} & \mathbf{x}_m \end{bmatrix}, \tag{17}$$

where $\mathbf{E}_f$ is the unity matrix of order N_f. Set (17) represents $N = N_m + N_u + N_f$ linear algebraic equations with $N_u + 2N_f$ unknowns.

If the number N_m of the measured d.o.f.s is less than the number N_f of the driven d.o.f.s, the number of equations in set (17) is less than the number of unknowns and therefore the set has an infinite number of solutions. It means that the full reconstruction of the field is impossible in this general case. So, the condition of existence of the unique solution to set (17) is the inequality

$$N_m \geqslant N_f, \tag{18}$$

which means that, for reconstructing the vibration field in a general Nd.o.f. system, the number of the measured d.o.f.s must be greater than the number of the driven d.o.f.s. When the inequality holds and the rank of the matrix is equal to the number of unknowns, all the displacement and force amplitudes can be computed uniquely through the measured data by any method of linear algebra, e.g., by the SVD techniques [29].

Uniqueness condition (18) is new compared to that established in the previous section. The uniqueness theorem for the field reconstruction in continuous structures does not put any restrictions on the size of the measured (accessible) region, except that it must be finite, and the area of this region may be much less than the area of the unmeasured (inaccessible) driven region. When such a continuous structure is modelled by a Nd.o.f. system, inequality (18) may be broken; nevertheless, the reconstruction is possible. Actually, there is no contradiction in this. The point is in the specific, sparse and very often band-like structural matrices of engineering continuous structures that allows one to expand the vibration field from measured d.o.f.s to the unmeasured undriven d.o.f.s. The situation is illustrated in more detail in the following example.

4.2. CHAIN-LIKE N.d.o.f. SYSTEM

Consider a spring–mass chain in Figure 4, i.e., one-dimensional discrete system with Nd.o.f. which models a non-uniform rod longitudinally vibrating in horizontal direction. The first L masses are free of forcing; external forces act on the rest of the $N-L$ masses. The harmonic forced vibration of the system is described by equation (15) with the displacement vector $\mathbf{x} = [x_1, \dots, x_N]^{\mathrm{T}}$, force vector $\mathbf{f} = [0, \dots, 0, f_{L+1}, \dots, f_N]^{\mathrm{T}}$, and the following matrix of the dynamic stiffness:

$$\mathbf{A} = \begin{bmatrix} a_1 & -k_1 & 0 & 0 & * & 0 & 0 & 0 \\ -k_1 & a_2 & -k_2 & 0 & * & 0 & 0 & 0 \\ 0 & -k_2 & a_3 & -k_3 & * & 0 & 0 & 0 \\ 0 & 0 & -k_3 & a_4 & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & a_{N-2} & -k_{N-2} & 0 \\ 0 & 0 & 0 & 0 & * & -k_{N-2} & a_{N-1} & -k_{N-1} \\ 0 & 0 & 0 & 0 & * & 0 & -k_{N-1} & a_N \end{bmatrix}, \qquad (19)$$

where $a_1 = k_1 - m_1\omega^2$, $a_j = k_{j-1} + k_j - m_j\omega^2$, $j = 2, \dots, N$. The matrix is tri-diagonal and this allows one to reconstruct the vibration field in all the undriven d.o.f.s, though inequality (18) may not be satisfied. Namely, according to the results of section 3, to reconstruct the field in the rod, it is sufficient to know only one quantity, the displacement amplitude at the free end, i.e., the amplitude x_1 of the first mass of the discrete model in Figure 4. From the first equation (15), (19) one can find the amplitude x_2 of the second mass, $x_2 = x_1 a_1 k_1$. Then, knowing amplitudes x_1 and x_2, one can, from the second equation (15), compute the displacement amplitude x_3 of the third mass. Continuing this process, one can find the vibration amplitudes of all the d.o.f.s that are free of the external loading, x_j with

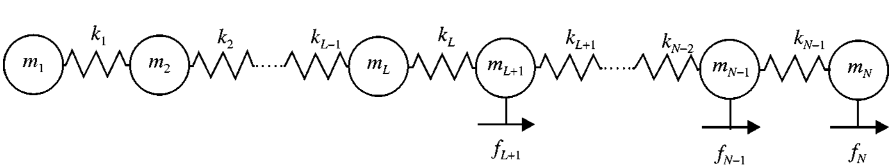

Figure 4. A discrete chain-like model of a non-uniform longitudinally vibrating thin rod.

$j = 2, \ldots, L$, and even the amplitude x_{L+1} of the first driven mass. However, the amplitudes of the other driven masses as well as the force amplitudes may not be obtained because inequality (18) is not satisfied when the number of the forces is greater than one. Only when there is one external force acting on the last mass of the chain, requirement (18) is met ($N_m = N_f = 1$), and all the displacement amplitudes and force amplitude f_N may be reconstructed uniquely.

4.3. OTHER EXAMPLES

The chain system in Figure 4 is one of the simplest Nd.o.f. systems. Each of its d.of.s interacts only with the two nearest d.o.f.s and its dynamic stiffness matrix is tri-diagonal. These features make it possible to expand the vibration field from one measured d.o.f.s to all unmeasured undriven d.o.f.s independently of the number of external forces.

In a similar manner one can study a chain in which every mass interacts with the first and second neighbours and its structural matrices are five-diagonal. This chain models a flexurally vibrating beam. According to the results of section 3, for reconstructing the field in the beam one needs to measure two quantities, the displacement and slope amplitudes of the free end or, equivalently, the amplitudes x_1 and x_2 of the first two masses of the chain. Using equation (15) or (17), one can compute, through x_1 and x_2, the amplitudes of all adjacent undriven d.o.f.s and the nearest two driven d.o.f.s.

The structural matrices of the finite-element models and finite-difference models for engineering structures are sparse and many of them are band-like [30, 31]. For a model with $(2k + 1)$ diagonal matrices, knowing the displacement amplitudes of $N_m = k$ undriven d.o.f.s permits the expansion of the vibration field to all adjacent undriven d.o.f.s without reconstructing all the external forces. However, if k is comparable with the number N_f of the driven d.o.f.s so that inequality (18) holds it may be preferable to compute all the unknown displacement and force amplitudes from equation (17) like in the case of general Nd.o.f systems (see section 4.1.).

Thus, reconstruction of the vibration field in Nd.o.f. systems with simple (sparse, band-like) structural matrices may be performed by expansion of the field from a small number of undriven measured d.o.f.s to a larger number of unmeasured undriven d.o.f.s without reconstructing external forces, as is done in continuous structures. For complex Nd.o.f. systems with filled dynamic stiffness matrices, the reconstruction is possible if the number of measured d.o.f.s is greater than the number of external forces, and it does not much matter in this case the amplitudes of which d.o.f.s, driven or undriven, are actually measured.

5. MAIN RESULTS AND CONCLUSION

A problem of reconstructing the harmonic vibration field in a linear structure from limited data measured on its accessible part is considered. This problem is mathematically formulated for continuous elastic structures; its general properties have been investigated. It is shown that the problem has a unique solution even when the data are measured on a small continuous part of the structure surface. However, the problem is ill-posed in the Hadamard sense and is unstable with respect to variations in the input data. When the data contain random noise the problem generally does not have an exact solution and should therefore be reshaped to become solvable approximately. This is done in the frame of the truncated SVD technique. Most attention is paid to the reconstruction error of approximate solutions. It is proved that there exists an optimal model of the vibrating structure that yields an approximate solution with the minimal reconstruction error. The remarkable feature is that, though the structure under study is continuous, its optimal model has a finite number of model parameters (and d.o.f.s) which is determined mostly by the signal-to-noise-ratio of the input data. This general result (existence of the optimal model) is valid for all modifications of the reconstruction problem in any linear structure or medium.

Applicability of the results, obtained for continuous structures, to discrete systems with finite degrees of freedom is also discussed. It is shown in particular that, in continuous structures, continuation of the vibration field from measured parts to adjacent unmeasured parts is possible owing to the very simple governing differential equations of the linear theory of elasticity, which, after discretization, give sparse or band-like stiffness and mass matrices of the corresponding discrete models. For general Nd.o.f. systems with full structural matrices, the reconstruction problem reduces to solvability of a set of linear algebraic equations.

The FR-problem has many aspects. This paper deals mostly with the mathematical formulation and general properties of the problem. Practically, a very important aspect is concerned with estimation of the minimum reconstruction error and development of the algorithms that permit to attain the lowest values of the error. This aspect, as well as its experimental implementation, is a challenging topic for future work.

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APPENDIX A: THE DYNAMIC ALMANSI THEOREM

Let a finite continuous three-dimensional elastic body be free of volume force, and the displacements and surface forces be zero on a part $\partial\Omega_{ac}$ of the body surface. Then, according to the Almansi theorem, there are no stresses anywhere in the body, and it is at rest. An equivalent formulation is: there does not exist a surface forcing that gives a non-zero response displacement at a small continuous part of the surface free of the forcing.

Almansi proved the theorem in 1907 for static elasticity [27]. Its extension to the dynamic theory of elasticity in the two-dimensional case is given in reference [17]. A proof of the theorem is presented for the general three-dimensional case of the linear theory of dynamic elasticity. The proof is followed by a discussion of its validity for discrete structures.

For the sake of simplicity, it is assumed that the body under study is made of an isotropic elastic material and a small continuous part $\partial\Omega_{ac}$ of the body surface is flat and lies in the x–y plane ($z = 0$), of Cartesian co-ordinates. The conditions of the theorem at the surface part $\partial\Omega_{ac}$ are that three components of the displacement are zero, namely,

$$u(x, y, 0) = v(x, y, 0) = w(x, y, 0) = 0 \tag{A1}$$

and three components of the stress tensor are also zero,

$$\sigma_{zz}(x, y, 0) = \sigma_{yz}(x, y, 0) = \sigma_{xz}(x, y, 0) = 0, \tag{A2}$$

where $x, y \in \partial\Omega_{ac}$ and the z-axis points into the body. The plan of the proof is first to show that *all* derivatives of the displacement components are zero in the region $\partial\Omega_{ac}$, and then extend the vibration field analytically, step by step, from $\partial\Omega_{ac}$ to the entire body. The proof is based on the use of Hook's law [32]:

$$\begin{aligned} \sigma_{xx} &= K[(1-\nu)\,\partial u/\partial x + \nu(\partial v/\partial y + \partial w/\partial z)], \quad \sigma_{xy} = G(\partial u/\partial y + \partial v/\partial x),\\ \sigma_{yy} &= K[(1-\nu)\,\partial v/\partial y + \nu(\partial u/\partial x + \partial w/\partial z)], \quad \sigma_{xz} = G(\partial u/\partial z + \partial w/\partial x),\\ \sigma_{zz} &= K[(1-\nu)\,\partial w/\partial z + \nu(\partial u/\partial x + \partial v/\partial y)], \quad \sigma_{yz} = G(\partial v/\partial z + \partial w/\partial y). \end{aligned} \tag{A3}$$

and the equation of motion [32]:

$$\begin{aligned} &\partial\sigma_{xx}/\partial x + \partial\sigma_{xy}/\partial y + \partial\sigma_{xz}/\partial z + \rho\omega^2 u = 0,\\ &\partial\sigma_{xy}/\partial x + \partial\sigma_{yy}/\partial y + \partial\sigma_{yz}/\partial z + \rho\omega^2 v = 0,\\ &\partial\sigma_{xz}/\partial x + \partial\sigma_{yz}/\partial y + \partial\sigma_{zz}/\partial z + \rho\omega^2 w = 0. \end{aligned} \tag{A4}$$

Here, $K = E/(1+\nu)(1-2\nu)$, $G = E/2(1+\nu)$; E, ν and ρ are Young's modulus, the Poisson ratio and density of the material; ω is the angular frequency (harmonic factor $\exp(-\mathrm{i}\omega t)$ is implied). Note that equations (A4) are homogeneous because, according to the theorem condition, the body is free of volume forcing. First, it will be proved that all derivatives of the displacement components with respect to the co-ordinates x, y, z are equal to zero at all internal points of the region $\partial\Omega_{ac}$. There are nine derivatives of the first order. Six of them are equal to zero,

$$\partial u/\partial x = \partial u/\partial y = \partial v/\partial x = \partial v/\partial y = \partial w/\partial x = \partial w/\partial y = 0, \tag{A5}$$

because the components u, v and w are constant (zero)in $\partial\Omega_{ac}$, see equation (A1). Three remaining derivatives (with respect to z) are zero due to three equations (A2). For example, from the first equation (A2) one obtains, using equations (A3) and (A5), that $\partial w/\partial z = 0$. Similarly, two other equations can be obtained:

$$\partial u/\partial z = \partial v/\partial z = \partial w/\partial z = 0 \quad \text{in } \partial\Omega_{ac}. \tag{A6}$$

Thus, all the derivatives of the first order are zero.

There are 18 derivatives of the second order. Equality to zero of 15 of them can be obtained by differentiation of equations (A5) and (A6) with respect to x and y. Equality to zero of the remaining three derivatives follows from three equations (A4) after substitution of equations (A3).

Suppose now that all $3n(n+1)/2$ derivatives of order $(n-1)$ are zero, then, by differentiating them with respect to x and y, one can derive that all nth derivatives but three are equal to zero in $\partial\Omega_{ac}$. Equality to zero of the remaining three ($\partial^{(n)}u/\partial z^n$, $\partial^{(n)}v/\partial z^n$, $\partial^{(n)}/w/\partial z^n$) can be obtained from three equations (A4) after differentiating them $(n-2)$ times with respect to z. Thus, if all the derivatives of order $(n-1)$ are zero, then all the derivatives of order n are also zero. By inductive reasoning one can conclude that all the derivatives of all orders of the displacement components with respect to co-ordinates x, y and z are equal to zero in the region $\partial\Omega_{ac}$.

Now choose a point, say $(0, 0, 0)$, in $\partial\Omega_{ac}$ and expand displacements u, v and w as functions of x, y, z in the Taylor series. This series converges in a spherical vicinity that does not contain singular points and besides is equal identically to zero, because all the derivatives at $(0, 0, 0)$ are zero. Hence, the displacement components and all their derivatives are zero at inner points of the vicinity. Choosing another point in this vicinity, one can construct another vicinity with the radius of convergence that equals the distance to the nearest singular point. The displacement field and all the derivatives are zero also in this vicinity. Since, owing to the absence of volume forces, there are no singular points within the body and the only singular points may lie on the boundary surface outside the region $\partial\Omega_{ac}$, the entire body, with the possible exception of the boundary points, can be covered by spherical overlapping vicinities in which all the displacement components as well as their derivatives and, hence, all the stress components are equal to zero. This proves the Almansi theorem for finite-continuous elastic bodies or structures of isotropic materials.

In a similar manner, the Almansi theorem can be proved in more complicated cases, e.g., for bodies of anisotropic materials and for structures built up of different members.

In conclusion, several remarks follow concerning the validity of the theorem for discrete systems with a finite number of degrees of freedom. For such systems the theorem can be formulated as follows: if the vibration amplitudes of N_m undriven d.o.f.s of an Nd.o.f. system are zero, the displacement of all other d.o.f.s and external forces are also zero. The necessary number N_m of immobile d.o.f.s depends on the dynamic stiffness matrix of the system. For a general Nd.o.f. system, which has filled structural matrices and for which it is not simple to identify boundary and inner d.o.f.s, the theorem reduces to the existence of the unique solution to the set of linear algebraic equations (17). A physical sense and practical usefulness the Almansi theorem has when applied to Nd.o.f. systems with sparse (band) stiffness and mass matrices for which the number N_m of immobile d.o.f.s may be comparatively small. Equality to zero of the vibration amplitudes of a small number of d.o.f.s yields immobility of almost all d.o.f.s of the system. For example, for a system with a band dynamic stiffness matrix of bandwidth k [31], this number equals k and from equality to zero of k d.o.f.s follows equality to zero of all d.o.f.s with possible exclusion of some driven d.o.f.s, see also section 4.

APPENDIX B: ASYMPTOTICS OF SINGULAR VALUES

In this appendix, the decay function of the singular values σ_n of the operator $\mathbf{G}$ in equation (3) with large indexes n is estimated in several typical situations. For an elastic body Ω that executes harmonic vibration under the action of an external force $\mathbf{f}(q)$ distributed over a part $\partial\Omega_{in}$ of the body surface $\partial\Omega$ and for which the vibration response

$\mathbf{u}_0(s)$ is measured at another surface part $\partial\Omega_{ac}$ (see Figure 1(a)), it is shown that the singular values decay with n as

$$\sigma_n \cong \sigma_0 n^a \exp(-\beta n), \tag{B1}$$

where α and σ_0 are functions of the body parameters and frequency, and β depends on the distance between the surfaces $\partial\Omega_{in}$ and $\partial\Omega_{ac}$ (β is zero when n is less than a certain number n_0, and is non-zero when $n > n_0$).

Physically, a singular value σ_n of the operator **G** has a sense of the influence coefficient of the certain load at the excitation surface $\partial\Omega_{in}$ on the response at the receiver surface $\partial\Omega_{ac}$. More exactly, when the unit external force of the nth singular form $\mathbf{f}_n(q)$ is applied to $\partial\Omega_{in}$, the displacement response at $\partial\Omega_{ac}$ is equal to $\sigma_n \mathbf{u}_n(s)$ and, thus, the amplitude of the response equals σ_n see also equations (4) and (5) of section 2.3. The orthonormal vector-functions $\mathbf{u}_n$ and $\mathbf{f}_n$ depend on physical and geometric parameters of the body. However, some of their properties are valid for all linear mechanical systems. In particular, all these functions are of oscillating character (the higher the index n is, the more rapid the oscillations are), the spatial period of oscillations being inversely proportional to index n.

Bearing this in mind, consider an elastic body (or medium) with surfaces $\partial\Omega_{in}$ and $\partial\Omega_{ac}$ that are flat, parallel, have a characteristic dimension l and lie in the planes $z = z_{in}$ and $z = z_{ac}$ of Cartesian co-ordinates. Let the singular form $\mathbf{f}_n$ be approximately represented as $f_n(q) \cong \mathbf{f}_0 \exp(2\pi i n q/l)$, $q \in \partial\Omega_{in}$. This force causes vibration at the excitation surface $\partial\Omega_{in}$ of the same form $u(q, z_{in})$ with the displacement amplitude which, for all elastic systems known to the author, depends on the index n as a power function n^α, the power α being a function of the system parameters. For example, for a thin plate flexurally vibrating under the action of a transverse force oscillating along the x-axis applied to the linear edge $z = z_{in}$, the amplitude of displacement of the edge is proportional, for large n, to n^3 and, hence, $\alpha = -3$. When the plate is excited by a bending moment, instead of the force, α is equal to $\alpha = -2$, etc.

The disturbance of the boundary $\partial\Omega_{in}$ is transmitted through the body along the co-ordinate z by a normal wave of the type

$$u(q, z) \cong u(q, z_{in})\exp\{[k_0^2 - (2\pi n/l)^2]^{1/2}(z - z_{in})\}, \tag{B2}$$

where k_0 is the biggest wavenumber of the body (in isotropic media, it is the wavenumber of shear waves). When the index n is not large, the z-component of the wavenumber, i.e., the quantity in the square brackets of equation (B2), is positive, so that the sound wave propagates along z without attenuation. Therefore, the field at the plane $z = z_{ac}$ and, at the surface $\partial\Omega_{ac}$, has the amplitude of the same order as that at the plane $z = z_{in}$. In other words, the influence coefficients and corresponding singular values are, in that case, determined by the power function. However, when the index n exceeds a certain number n_0, the z-components of all the wavenumbers of the body are purely imaginary, and the field decays exponentially along the z-axis, so that at the surface $\partial\Omega_{ac}$ the amplitude is given by

$$u(q, z_{ac}) \cong u(q, z_{in}) \exp[-2\pi n(z_{ac} - z_{in})/l].$$

Hence, the influence coefficients and singular values decay with large n as in equation (B1), the decay rate being equal to $\beta = 2\pi(z_{ac} - z_{in})/l$.

Similarly, if the excitation surface $\partial\Omega_{in}$ and response surface $\partial\Omega_{ac}$ lie on cylindrical surfaces, $r = r_{in}$ and r_{ac}, and the force is oscillatory with the polar angle φ, $\mathbf{f}_n(\varphi) = \mathbf{f}_0 \cos(n\varphi)$, the dependence of the field on the radial co-ordinate r is described by the Hankel function, so that the influence coefficients, and hence singular values, are proportional to the ratio

$H_n(k_0 r_{ac})/H_n(k_0 r_{in})$. According to the Debay expansion of the Hankel function for $n \to \infty$ [33], this ratio tends asymptotically to the power function $(r_{in}/r_{ac})^n$ that can be written in the exponential form (B1) with the decay coefficient $\beta = \ln(r_{ac}/r_{in})$.

This result also holds when $\partial\Omega_{in}$ and $\partial\Omega_{ac}$ lie on spherical surfaces, $R = R_{in}$ and R_{ac}. Assuming the angular dependence of the field as in the Legendre polynomials, it is easy to show that the influence coefficients and corresponding singular values are proportional to the ratio of the spherical Hankel functions $h_n(k_0 R_{ac})/h_n(k_0 R_{in})$ which, for large n, is equivalent to $(R_{in}/R_{ac})^{n+1}$, i.e., again as in equation (B1) with $\beta = \ln(R_{in}/R_{ac})$.

For more complex geometries of the surfaces $\partial\Omega_{in}$ and $\partial\Omega_{ac}$, the asymptotics of the singular values may differ from those presented above by expressions for α and β. However, the general character of the decay, determined by the product of power and exponential functions, remains. This is supported by examples found in the literature (see, e.g., references [14, 17, 34, 35]).