Journal of Sound and Vibration (2001) **247**(1), 165–173
doi:10.1006/jsvi.2001.3622, available online at http://www.idealibrary.com on IDEAL®

LETTERS TO THE EDITOR

AN AUTOMATIC DETERMINATION OF SMOOTHING BANDWIDTH IN B–T METHOD FOR POWER SPECTRAL ESTIMATION

K. SASAKI

Institute of Engineering Mechanics and Systems, University of Tsukuba, 1-1-1 *Ten-Ohdai, Tsukuba-Shi, Ibaraki* 305-8573, *Japan. E-mail*: *sasaki@esys.tsukuba.ac.jp*

(*Received* 4 *September* 2000, *and in final form* 2 *January* 2001)

1. INTRODUCTION

Precise power spectral estimation of stationary time series plays an important role in many applications in broad areas. From non-parametric viewpoints, standard methods of the estimation called the Blackman–Tukey (B–T) or smoothing periodogram ones are widely used. For their practice, however, there is no good means for systematically determining the maximum lag of lag windows or equivalently the smoothing bandwidth of spectral windows.

By considering the fundamental reverse relation between bias and variance of spectral estimate with respect to the smoothing bandwidth, an automatic determination of the smoothing bandwidth was proposed in the previous paper [1], on the basis of minimizing a proposed index for evaluating mean-square error (MSE) of the spectral estimate. But, the proposed constituent bias index is rather complex such that it requires the difference operation between spectral estimates based on two windows of different bandwidths, while the variance index includes an *a priori* parameter setting, which requires extra pre-calculation.

In this article, a practical and simple method for automatically determining the smoothing bandwidth is proposed, by newly introducing evaluation indices of these statistics only from an observed time series.

After a brief review of the B–T method, the fundamental reverse relation between bias and variance of the estimate with respect to the smoothing bandwidth is made clear in section 2. Based on the characteristics, in section 3, a practical index for evaluating the associated MSE only from an observed time series is proposed. In section 4, the effectiveness and the fundamental characteristics of the proposed method are illustrated by basic results of computer simulations. In section 5, the obtained main results are summarized.

2. A BRIEF REVIEW OF B–T METHOD

Power spectral estimate of a given time series $x(n)$ by the B–T method is equivalently obtained by smoothing the periodogram $P_N(f)$ with an even and positive spectral window $W(f)$. That is,

$$\hat{P}_W(f) = \int_{-f_s/2}^{f_s/2} W(\eta) P_N(f-\eta)\, \mathrm{d}\eta, \tag{1}$$

0022-460X/01/410165 + 09 $35.00/0

where $f_s = 1/\Delta t$ denotes sampling frequency. Inverse Fourier transforming equation (1) gives a Fourier transform representation of the time domain version,

$$\hat{P}_W(f) = \Delta t \sum_{m=-M}^{M} w(m) R(m) \exp(-\mathrm{j}2\pi f m \Delta t), \tag{2}$$

where $w(m)$ and $R(m)$ are defined by inverse Fourier transforms of $W(f)$ and $P_N(f)$ respectively. $w(m)$ is called a lag window with maximum lag M, and $R(m)$ is a sample auto-correlation function, i.e., a biased estimate of the auto-correlation function of the given time series [1].

Under the regular mixing condition, which assumes the smoothness of the spectrum to be estimated [2], and sufficient data length compared with inverse of bandwidth of the narrowest spectral peak, bias and variance of the smoothed estimate are evaluated as follows [3]:

$$Bias[\hat{P}_W(f)] = \frac{P''(f)}{2} \int_{-f_s/2}^{f_s/2} f^2 W(f)\,\mathrm{d}f,$$

$$= \frac{P''(f)}{2} B_{W2}, \tag{3}$$

$$Var[\hat{P}_W(f)] = (1 + \delta_{f,0}) P^2(f) E_W / T, \tag{4}$$

where $\delta_{i,j}$ denotes a Kronecker's delta, $P''(f)$ does the second order derivative of the signal spectrum $P(f)$, E_W is the energy of the window, and $T = N\Delta t$ is the observed data length.

When integral of the spectral window is normalized to unity, B_{w2} in equation (3) increases with bandwidth of $W(f)$, while the energy E_W in equation (4) decreases with it. This reflects the fundamental reverse relation between bias and variance of the smoothed power spectral estimate with respect to smoothing bandwidth as mentioned above.

3. PROPOSED EVALUATION INDICES OF BIAS, VARIANCE, AND MSE OF SMOOTHED POWER SPECTRAL ESTIMATE

Equations (3) and (4) imply that variance of the normalized estimate is given only by a ratio of window energy E_W to data length T, while the bias is by a half of a product of the second order moment of spectral window and the second order derivative of the normalized signal spectrum with respect to frequency. Thus, replacing the true spectral values by the estimated, and approximating the second order derivative with the corresponding difference give the following indices for evaluating the normalized statistics:

$$NB(W,f) = B_{W2} \frac{\hat{P}_W(f + \Delta f_d) - 2\hat{P}_W(f) + \hat{P}_W(f - \Delta f_d)}{2\hat{P}_W(f)(\Delta f_d)^2}, \tag{5}$$

$$NV(W,f) = (1 + \delta_{f,0}) E_W / T, \tag{6}$$

where Δf_d denotes a half-valued bandwidth of the spectral window.

To evaluate them only from an observed time series, any averaging operation is necessary to suppress the sampling variation. Since the normalized spectrum may be expected to exhibit white characteristics, averaging over the whole frequency range is considered in this article, resulting in the following proposed indices for evaluating the relevant statistics:

$$Bias = A_{v,f}[NB(W,f)], \tag{7}$$

$$Var = A_{v,f}[NV(W,f)], \tag{8}$$

$$MSE = Var + Bias^2, \tag{9}$$

where $A_{v,f}$ denotes the associated frequency averaging operation.

Once each sample time series is given, with a form of the window being assumed beforehand, the minimum of the MSE index may provide an automatic determination of the optimum smoothing bandwidth or equivalently the maximum lag of the lag window, so that the optimum spectral estimate will be obtained. This is the principle of the automatic determination of the smoothing bandwidth to establish an observed data oriented power spectral estimation proposed in this article.

4. RESULTS OF COMPUTER SIMULATIONS

To ascertain the effectiveness and fundamental characteristics of the proposed automated method, computer simulations are carried out by assuming several power spectra, each of which consists of two Gaussian peaks at center frequencies 70 and 200 Hz. At that time, a half-power bandwidth of the higher frequency peak is fixed as 50 Hz, while that of the lower is varied from 4 to 48 Hz. Sample time series of the assumed power spectral characteristics are generated by filtering a physically generated band-limited white noise with the relevant shaping filters of two Gaussian peaks in frequency domain [1, 4]. For the window, Bohman's one is used, since it is positive and even in both time and frequency domains as follows [1, 5]:

$$w(m) = \frac{1}{M}\left|\sin\frac{\pi m}{M}\right| + \left(1 - \frac{|m|}{M}\right)\cos\frac{\pi m}{M}, \tag{10}$$

$$W(f) = \frac{4\pi^2 M\Delta t[1 + \cos(2\pi f M\Delta t)]}{[(2\pi f M\Delta t)^2 - \pi^2]^2}. \tag{11}$$

In the following simulations, sampling frequency f_s is fixed at 1·8 kHz. A half-power bandwidth of the lower frequency peak B_h is first set as 4–48 Hz. Then, by parametrically changing data length $T = N\Delta t$ from $N = 512$ to 4096, the characteristics of the proposed indices and their corresponding theoretical values against the smoothing bandwidth are experimentally evaluated through power spectral estimation by 100 times. Smoothed power spectrum $\hat{P}_W(f)$ is estimated according to equation (2), i.e., via FFT of the lag windowed sample auto-correlation function $w(m)R(m)$ after zero-padding up to the data length $T = N\Delta t$.

Typical examples of the obtained characteristics for $B_h = 8$, 24, and 48 Hz are shown in Figures 1, 2, and 3 respectively. In these figures, solid and dotted lines denote the values of the proposed indices and the theoretically expected respectively. (a), (b), and (c) are the results when data length parameter N is changed as 1024, 2048, and 4096 respectively.

In the case of the half-power bandwidth of the lower peak $B_h = 8$ Hz in Figure 1, the proposed squared bias indices in (a)–(c) exhibit zigzag changes, since the half-power bandwidth Δf_d for difference approximation in equation (5) takes comparable values with

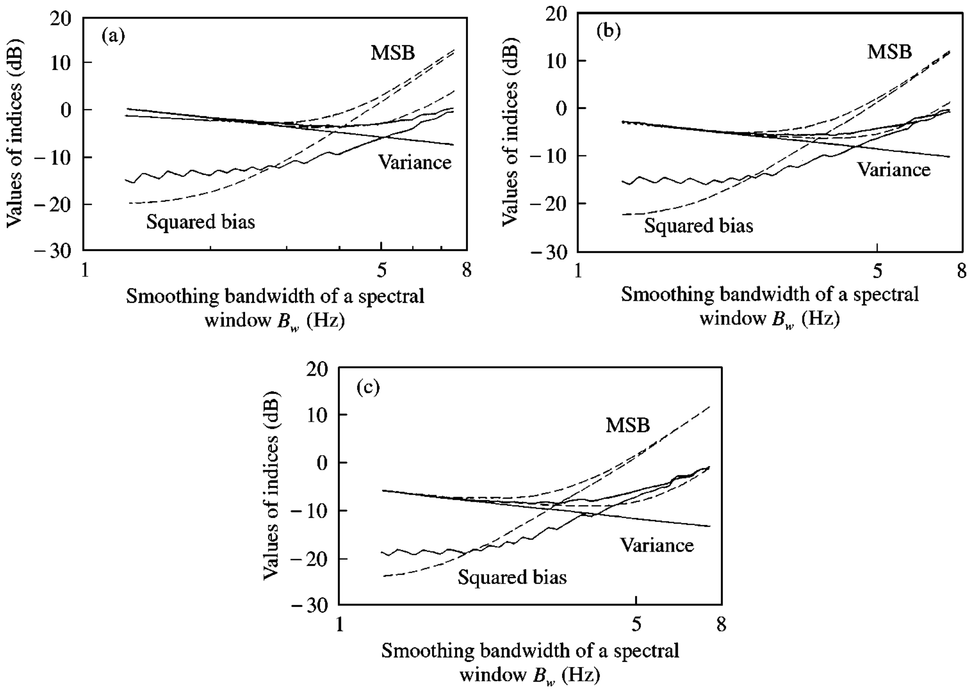

Figure 1. Characteristics of the proposed and theoretically expected indices estimated through power spectral estimation by 100 times, where sampling interval is $\Delta t = 0{\cdot}55$ ms, and a half-power bandwidth of the lower Gaussian peak is $B_h = 8$ Hz. (a) $N = 1024$, $T = 0{\cdot}56$ s; (b) $N = 2048$, $T = 1{\cdot}13$ s; (c) $N = 4096$, $T = 2{\cdot}26$ s: – – –, theoretical; ——, proposed.

frequency resolution roughly given by the inverse of data length T^{-1}. As a result of difference approximation of the second order derivatives, moreover, slopes of the bias index curves take smaller values than the theoretically expected. On the other hand, those of the variance are kept constant according to ratios of the window energy E_w's to data length T's as given by equation (6), but exhibit slightly smaller values than the theoretical as the smoothing bandwidth becomes broad. The latter fact may be due to a violation of the pre-requisite that the variance evaluation given by equation (4) is meaningful only when the smoothing bandwidth is negligibly narrow in comparison to that of the narrowest peak in the signal spectrum.

However, as the smoothing bandwidth approaches the optimum point, each of the bias discrepancy decreases, and all the optimum values of the smoothing bandwidth may be judged to coincide with the theoretically expected. These illustrate the effectiveness of the proposed method.

Examples of power spectra estimated from a sample time series are shown in Figures 4, 5, and 6, where the half-power bandwidth of the lower frequency peak B_h is taken as 8, 24, and 48 Hz, respectively, with a data length parameter N being fixed as 1024. In these figures, dotted lines denote main portions of the theoretically assumed spectra up to 300 Hz, a solid line in (a) represents the periodogram, and those in (b), (c), and (d) are power spectra estimated by very narrow, optimum, and very broad smoothing bandwidths respectively.

From comparison of the results in these figures (a)–(d), the next features are clearly seen. Although all the periodograms in (a) suffer from heavy sampling variation, it comes to be

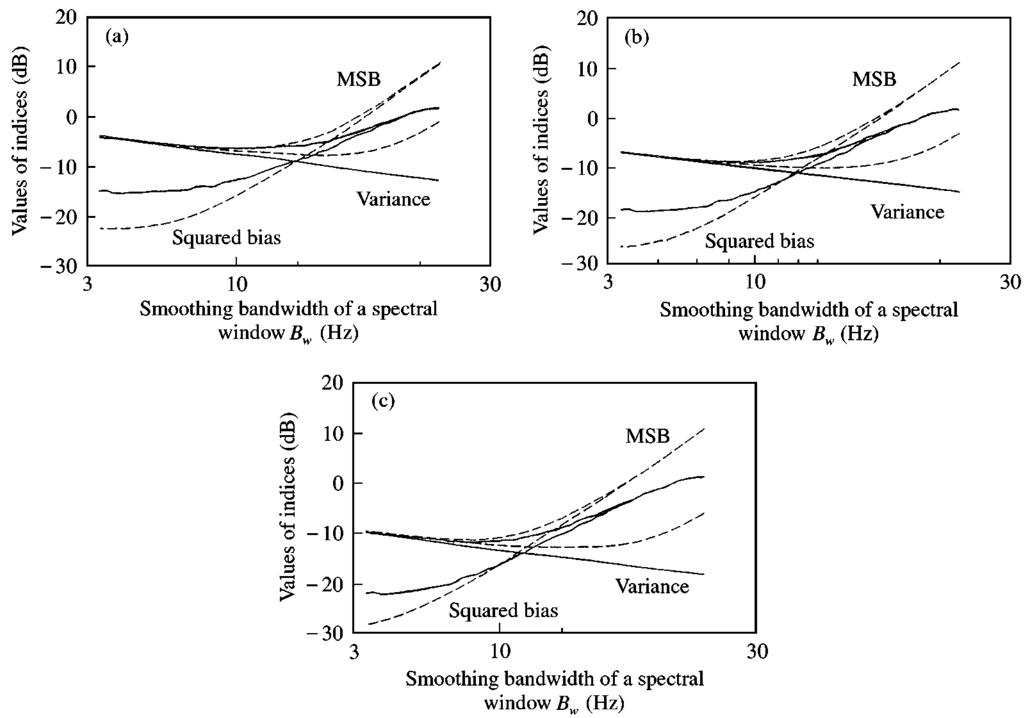

Figure 2. Characteristics of the proposed and theoretically expected indices estimated through power spectral estimation by 100 times, where sampling interval is $\Delta t = 0{\cdot}50$ ms, and a half-power bandwidth of the lower Gaussian peak is $B_h = 24$ Hz. (a) $N = 1024$, $T = 0{\cdot}56$ s; (b) $N = 2048$, $T = 1{\cdot}13$ s; (c) $N = 4096$, $T = 2{\cdot}26$ s: – – –, theoretical; ——, proposed.

suppressed in the estimated spectra as shown from (b) to (d) as the smoothing bandwidth B_W of spectral windows becomes broad. But, the spectra in (d) show rather reasonable biases due to the excess smoothing. As a result, the spectra estimated with the optimum smoothing bandwidth in (c) provide the most precise estimate for all the signal spectra of $B_h = 8–48$ Hz.

These results illustrate well the fundamental characteristics of the observed data oriented stationary power spectral estimation by the proposed automatic determination of the smoothing bandwidth of spectral windows.

Figure 7(a) and 7(b) collectively show the finally obtained relations of the optimum half-valued smoothing bandwidth of spectral windows B_w and the minimum MSE of normalized power spectral estimates against the half-power bandwidth of the lower frequency peak B_h, respectively.

Figure 7(a) roughly reveals the facts that the optimum smoothing bandwidth of spectral windows B_W is in a linear relation to the half-power bandwidth of the lower frequency peak B_h, i.e., the narrowest peak in the signal spectrum, except for a data length parameter N being as small as 512, and that the proportional coefficient slightly decreases as the data length increases. The latter fact reflects the special features that due to difference approximation of the second order derivative, the slope of the proposed bias index against the smoothing bandwidth becomes large with the increase of data length, so that the optimum smoothing bandwidth of spectral windows B_W decreases with increase of frequency resolution, as mentioned previously.

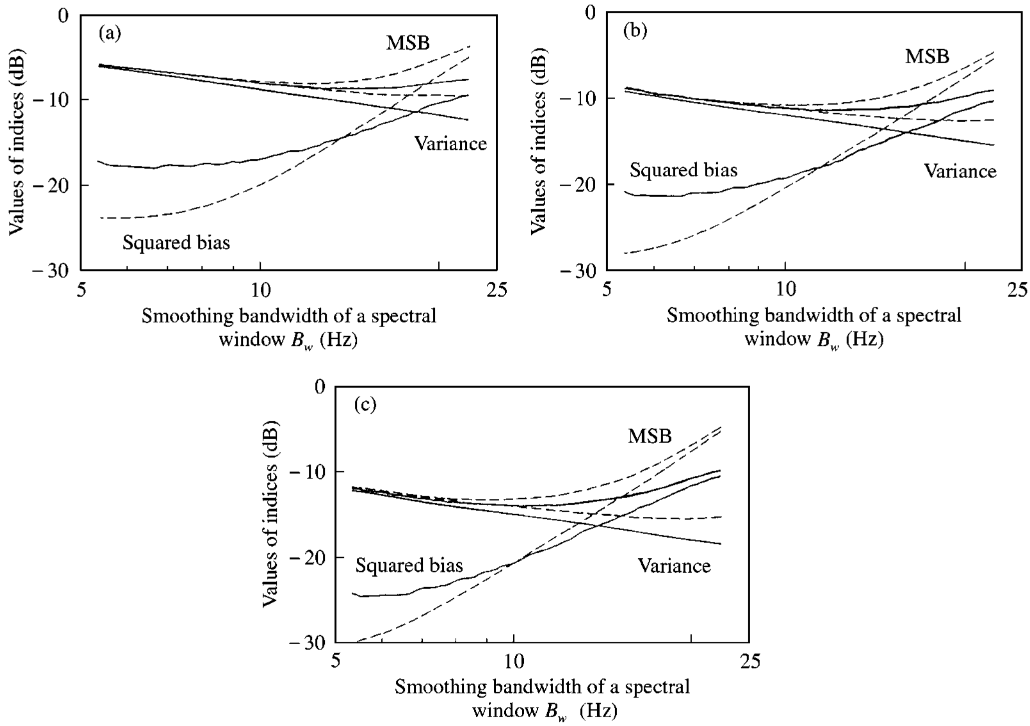

Figure 3. Characteristics of the proposed and theoretically expected indices estimated through power spectral estimation by 100 times, where sampling interval is $\Delta t = 0{\cdot}55$ ms, and a half-power bandwidth of the lower Gaussian peak is $B_h = 48$ Hz. (a) $N = 1024$, $T = 0{\cdot}56$ s; (b) $N = 2048$, $T = 1{\cdot}13$ s; (c) $N = 4096$, $T = 2{\cdot}26$ s: ‐ ‐ ‐, theoretical; ——, proposed.

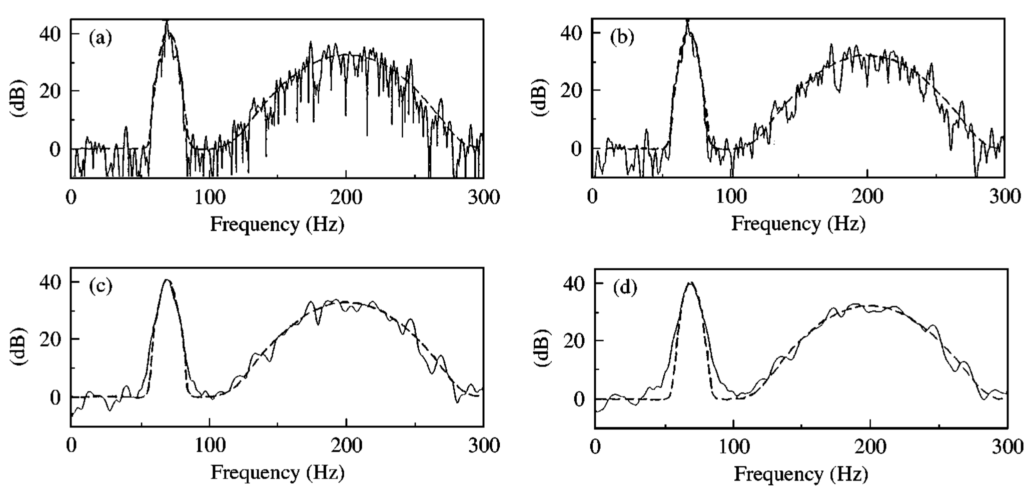

Figure 4. Typical examples of power spectra estimated from a sample time series when $B_h = 8$ Hz and $N = 1024$, where the dotted lines denote the theoretically assumed power spectrum, a solid line in (a) represents the sample periodogram, and those in (b), (c), and (d) are the power spectra estimated by very narrow, optimum, and very broad smoothing bandwidths respectively. (b) $B_w = 1{\cdot}26$ Hz; (c) $B_w = 4{\cdot}96$ Hz; (d) $B_w = 7{\cdot}49$ Hz.

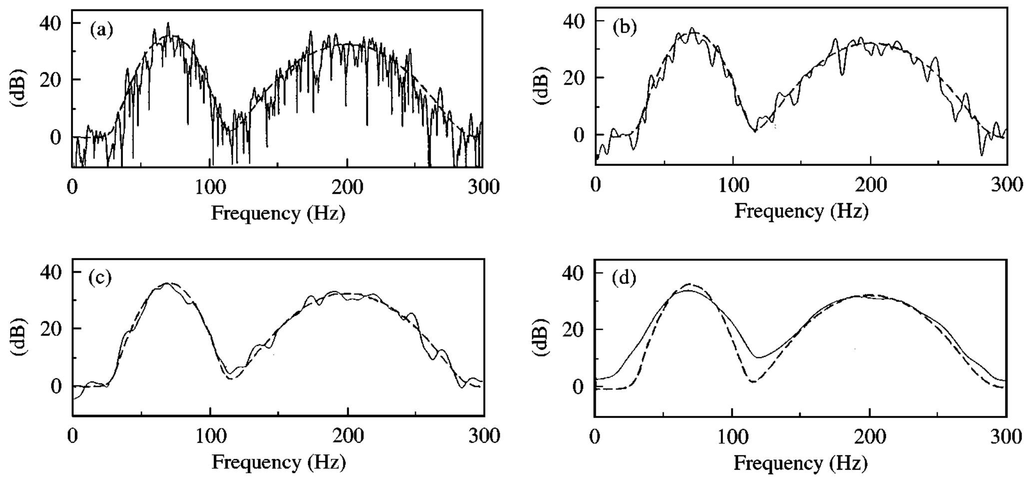

Figure 5. Typical examples of power spectra estimated from a sample time series when $B_h = 24$ Hz and $N = 1024$, where the dotted lines denote the theoretically assumed power spectrum, a solid line in (a) represents the sample periodogram, and those in (b), (c), and (d) are the power spectra estimated by very narrow, optimum, and very broad smoothing bandwidths respectively. (b) $B_w = 3{\cdot}22$ Hz; (c) $B_w = 7{\cdot}50$ Hz; (d) $B_w = 22{\cdot}4$ Hz.

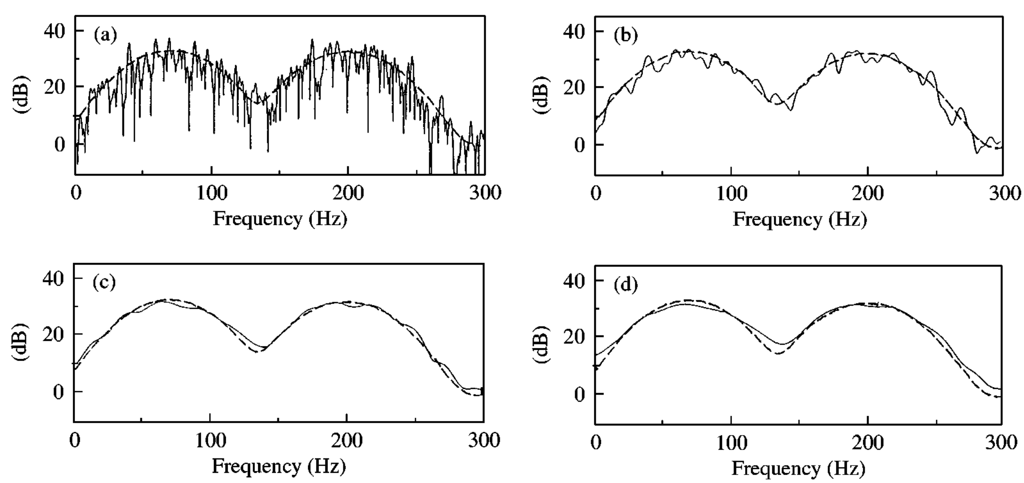

Figure 6. Typical examples of power spectra estimated from a sample time series when $B_h = 48$ Hz and $N = 1024$, where the dotted lines denote the theoretically assumed power spectrum, a solid line in (a) represents the sample periodogram, and those in (b), (c), and (d) are the power spectra estimated by very narrow, optimum, and very broad smoothing bandwidths respectively. (b) $B_w = 5{\cdot}32$ Hz; (c) $B_w = 12{\cdot}4$ Hz; (d) $B_w = 22{\cdot}4$ Hz.

On the other hand, the results in Figure 7(b) clearly show that the more the data length is available and the broader the narrowest peak of signal spectrum becomes, more precisely can the estimation be realized. That is, increase of data length by 8 times or bandwidth of the narrowest peak by 12 times improves the minimum MSE of the normalized estimate by about 8dB. This reflects the well-known special feature of the B–T method for power spectral estimation.

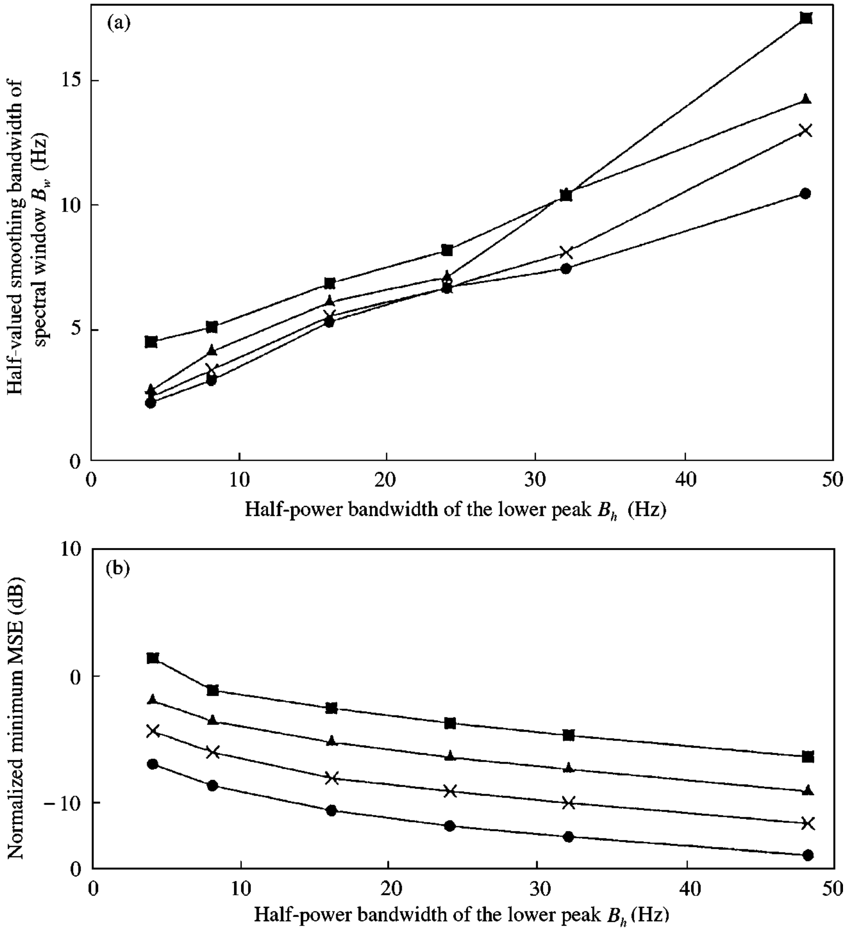

Figure 7. Finally obtained optimum half-valued bandwidth of spectral windows B_w and minimum MSE of the normalized spectral estimate against the half-power bandwidth of the lowest frequency peak B_h, where each value is evaluated through iterative spectral estimation by 100 times. (a) Optimal smoothing bandwidth versus half-power bandwidth of the lower peak; (b) normalized minimum MSE versus half-power bandwidth of the lower peak: ●, $N = 4096$; ×, $N = 2048$; ▲, $N = 1024$; ■, $N = 512$.

5. CONCLUSIONS

To develop an observed data-oriented precise stationary power spectral estimation, a practical and simple method for an automatic determination of smoothing bandwidth of spectral windows is proposed, on the basis of the fundamental reverse relation between bias and variance of power spectral estimate by the B–T method. The effectiveness and fundamental characteristics of the proposed method are experimentally examined through computer simulations, and the following main results are obtained:

(1) Proposed indices for evaluating the bias, variance, and MSE of the normalized spectral estimate from an observed time series work well, realizing the observed data-oriented precise power spectral estimation. But the slope of the bias index against the smoothing bandwidth gives a slightly small value, resulting in the necessity of improvement so as to make difference approximation of the second order derivative by a smaller interval than a half-valued bandwidth.

(2) The optimum smoothing bandwidth is in a linear relation to the bandwidth of the narrowest peak in a signal spectrum with a proportional coefficient of about one-fifth when Bohman's spectral window is used.

(3) The minimum MSE of the normalized spectral estimate is improved by about 8 dB as data length or bandwidth of the narrowest peak increase by 8 or 12 times, respectively, so far as the frequency resolution remains in the range of 1/100–1/2 of the latter.

Further high resolution and precise estimation may be possible by such a use of the proposed method to estimate the spectrum of residue series after prewhitening the original series by parametric methods such as AR [4] or ARMA model fitting. In these cases, however, the proposed method is expected to provide the basis for the precise automatic stationary power spectral estimation.

ACKNOWLEDGMENTS

The author would like to express his sincere gratitute to H. Matsura, who was a graduate student in Master Program in the University of Tsukuba, and is now with Japan Victor Co. Ltd., for preparation of a part of the figures.

REFERENCES

1. K. Sasaki 1989 *Transactions of Institute of Systems, Control, and Information Engineers* **2**, 97–107. An automatic determination method of smoothing bandwidth in power spectral estimation (in Japanese).
2. D. R. Brillinger and M. Rosenblatt 1967 in *Spectral Analysis of Time Series* (B. Harris, editor), 153–188. New York: John Wiley & Sons, Inc. Asymptotic theory of estimation of *k*-th order spectra.
3. M. B. Priestley 1981 *Spectral Analysis and Time Series*. Volume 1 (United States Version), 449–471. Orlando: Academic Press, Inc.
4. K. Sasaki and Y. Kaji 1993 *Transactions of Institute of Systems, Control, and Information Engineers* **6**, 328–339. An automatic precise estimation of stationary power spectra by multi-step fittings of AR model (in Japanese).
5. A. Papoulis 1973 *IEEE Transactions on Information Theory* **IT-19**, 9–12. Minimum bias windows for high-resolution spectral estimates.