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VIBRATIONS OF A COUPLED TWO-DEGREE-OF-FREEDOM SYSTEM

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In this paper the motion of a two-mass system with two degrees of freedom is discussed. The masses are connected with three springs. The motion of the system is described with a system of two coupled strong non-linear differential equations. For the case when the non-linearity is of a cubic type, the analytical solution of the system is obtained. It is a combination of a Jacobi elliptic function and a trigonometric function. An approximate analytical method based on the Krylov–Bogolubov procedure is developed for the system which contains small non-linearities. Two examples are considered: the case when all the three stiffnesses are non-linear and the case when small damping acts. The analytical solutions are compared with numerical ones. They show a good agreement.

1. INTRODUCTION

The dynamics of the two-mass system with two degrees of freedom has been widely discussed for a long time. Usually, it is assumed that the motion is described with a system of two coupled linear differential equations and the solution of the equations is given in the closed analytical form. Vakakis and Rand [1, 2] extended the investigations. They assumed that the two-degree-of-freedom system is non-linear and analyzed its global dynamics. Two cases are discussed: one, when it is assumed that the non-linearities in the system are small [1], and the second, when only the non-linearity of the stiffness connecting the two moving masses is small [2]. For these cases, an approximate analytical solution is developed, based on the linear solution of the system of differential equations. In paper [3], a special case of the two-mass system is considered. The masses are connected with a strong non-linear stiffness and have no connection with the fixed part. The motion is described with a system of two coupled ordinary second order differential equations with strong cubic non-linearity. The closed-form analytical solution is obtained by applying Jacobi elliptical functions.

In this paper an extension of the previous cases is undertaken. It is assumed that the two masses are connected to each other with a strong non-linear stiffness and to the fixed parts with linear or weak non-linear elastic elements. This physical model corresponds to many machine–stand–foundation systems. The mathematical model of such a two-degree-of-freedom system is a system of two coupled non-linear differential equations. Using the combination of Jacobi elliptical functions [4–6] and harmonic functions the exact general analytical solution of the system is obtained. For the case when the system contains not only strong but also some small non-linearities, the approximate analytical solution is developed based on the well-known Krylov–Bogolubov method for elliptical and trigonometrical functions [7, 8]. Two examples are considered: (1) a system which is under the influence of

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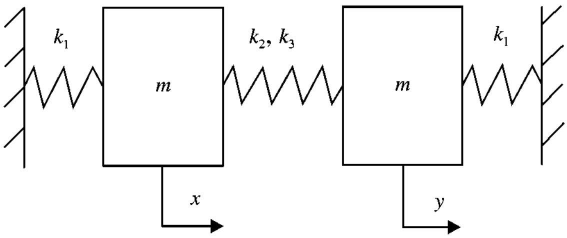

Figure 1. The model of the two-mass system.

non-linear stiffness; (2) a small non-linear damping force acts. The analytical solutions are compared with numerical ones, obtained by applying the Runge–Kutta method.

2. THE MODEL OF THE SYSTEM

The model of the two-mass system is shown in Figure 1. Two equal masses m are connected with the fixed bodies with stiffness k_1. The connection between the two bodies is a spring with non-linear properties. The linear coefficient of elasticity of the spring is k_2 and of the cubic non-linearity is k_3. The system has two degrees of freedom. The generalized co-ordinates are x and y. The mathematical model of the system is

$$m\ddot{x} + k_1 x + k_2(x - y) + k_3(x - y)^3 = \varepsilon f_1(x, \dot{x}, y, \dot{y}),$$

$$m\ddot{y} + k_1 y + k_2(y - x) + k_3(y - x)^3 = \varepsilon f_2(x, \dot{x}, y, \dot{y}), \tag{1}$$

where εf_i are small non-linearities ($i = 1, 2$). Introducing the new variables

$$X = x - y, \quad Y = x + y \tag{2}$$

and the notation suggested by Coppola and Rand [7], equations (1) are transformed into a system of two differential equations

$$\ddot{X} + \alpha X + \beta X^3 = (\varepsilon/m)(f_1 - f_2), \qquad \ddot{Y} + \Omega^2 Y = (\varepsilon/m)(f_1 + f_2), \tag{3}$$

where

$$\alpha = (k_1 + 2k_2)/m, \qquad \beta = 2k_3/m, \qquad \Omega^2 = k_1/m \tag{4}$$

and

$$f_i \equiv f_i((X + Y)/2, \quad (\dot{X} + \dot{Y})/2, \quad (Y - X)/2, \quad (\dot{Y} - \dot{X})/2), \quad i = 1, 2. \tag{5}$$

Differential equations (3) are coupled only with the small non-linear terms, i.e., with the terms with parameter ε.

3. GENERAL SOLUTION OF THE SYSTEM OF TWO STRONG NON-LINEAR DIFFERENTIAL EQUATIONS

For $\varepsilon = 0$, the differential equations of motion are

$$m\ddot{x} + k_1 x + k_2(x - y) + k_3(x - y)^3 = 0,$$
$$m\ddot{y} + k_1 y + k_2(y - x) + k_3(y - x)^3 = 0 \quad (6)$$

or using variables (2)

$$m\ddot{X} + (k_1 + 2k_2)X + 2k_3X^3 = 0, \qquad m\ddot{Y} + k_1 Y = 0. \quad (7)$$

Equations (7) represent a system of two separated differential equations which can be solved independently. The first equation is with strong cubic non-linearity and the second is linear. The general solution of the first equation is after reference [8],

$$X = A\,\mathrm{cn}(\omega t + \theta, k^2), \quad (8)$$

where cn is the Jacobi elliptical function [4] with the frequency ω and modulus k

$$\omega = \sqrt{\alpha + \beta A^2} = \sqrt{(k_1 + 2k_2 + 2k_3A^2)/m}, \quad (9)$$

$$k^2 = \beta A^2/2(\alpha + \beta A^2) = k_3A^2/(k_1 + 2k_2 + 2k_3A^2) \quad (10)$$

A and θ are arbitrary parameters. The frequency ω and the modulus of the Jacobi function k depend on the initial amplitude A. The modulus of the function is independent of the mass of the bodies.

The general solution of linear equation (7) is

$$Y = B\cos(\Omega t + \phi), \quad (11)$$

where B and ϕ are unknown parameters and the frequency of the linear oscillator is

$$\Omega = \sqrt{k_1/m}. \quad (12)$$

Substituting equations (8) and (11) into equation (2) the general solution of the system of equations (6) is

$$x = \tfrac{1}{2}A\,\mathrm{cn}(\omega t + \theta, k^2) + \tfrac{1}{2}B\cos(\Omega t + \phi),$$
$$y = \tfrac{1}{2}B\cos(\Omega t + \phi) - \tfrac{1}{2}A\,\mathrm{cn}(\omega t + \theta, k^2). \quad (13)$$

The values of A, B, θ and ϕ in equation (13) have to be determined according to the initial conditions.

3.1. THE INFLUENCE OF THE INITIAL CONDITIONS

Substituting the initial conditions

$$x(0) = x_0, \quad y(0) = y_0, \quad \dot{x}(0) = \dot{x}_0, \quad \dot{y}(0) = \dot{y}_0 \quad (14)$$

into equation (13), the following system of equations is obtained:

$$x_0 = \tfrac{1}{2} A\,\mathrm{cn}(\theta, k^2) + \tfrac{1}{2} B \cos\phi, \qquad y_0 = \tfrac{1}{2} B\cos\phi - \tfrac{1}{2} A\,\mathrm{cn}(\theta, k^2),$$

$$\dot{x}_0 = (\omega/2)\, A\,\mathrm{sn}(\theta, k^2)\,\mathrm{dn}(\theta, k^2) - (\Omega/2)\, B \sin\phi,$$

$$\dot{y}_0 = (\omega/2)\, A\,\mathrm{sn}(\theta, k^2)\,\mathrm{dn}(\theta, k^2) - (\Omega/2)\, B \sin\phi, \tag{15}$$

where sn and dn are also Jacobi elliptical functions [4]. Solving the system of equations (15)

$$B = \sqrt{(x_0 + y_0)^2 + (\dot{x}_0 + \dot{y}_0)^2/\Omega^2},$$

$$\phi = \arctan[-(\dot{x}_0 + \dot{y}_0)/\Omega(x_0 + y_0)],$$

$$A = \sqrt{(-\alpha + \sqrt{\alpha^2 + 2\alpha\beta(x_0 - y_0)^2 + \beta^2(x_0 - y_0)^4 + \beta(\dot{y}_0 - \dot{x}_0)^2})/\beta}$$

and the value of α is obtained from the equation

$$\mathrm{sc}(\alpha, k^2)\,\mathrm{dc}(\theta, k^2) = \frac{\dot{y}_0 - \dot{x}_0}{(x_0 - y_0)\omega} = \frac{(\dot{y}_0 - \dot{x}_0)\sqrt{m}}{(x_0 - y_0)\sqrt{k_1 + 2k_2 + 2k_3 A^2}}. \tag{16}$$

Assume some special cases of initial conditions.

1. If the motion starts without initial velocity, i.e., $\dot{x}_0 = 0$, $\dot{y}_0 = 0$, coefficients (16) which depend on the initial conditions are

$$A = x_0 - y_0, \qquad B = x_0 + y_0, \quad \theta = 0, \quad \phi = 0. \tag{17}$$

Substituting equation (16) into equation (13) the general solution of system (6) is

$$x = \frac{x_0 - y_0}{2}\,\mathrm{cn}\left(t\sqrt{\alpha + \beta(x_0 - y_0)^2},\ \frac{\beta(x_0 - y_0)^2}{2[\alpha + \beta(x_0 - y_0)^2]}\right) + \frac{x_0 + y_0}{2}\cos(\Omega t),$$

$$y = \frac{x_0 + y_0}{2}\cos(\Omega t) - \frac{x_0 - y_0}{2}\,\mathrm{cn}\left(t\sqrt{\alpha + \beta(x_0 - y_0)^2},\ \frac{\beta(x_0 - y_0)^2}{2[\alpha + \beta(x_0 - y_0)^2]}\right), \tag{18}$$

i.e., using parameters (4) it can be expressed as

$$x = \frac{x_0 - y_0}{2}\,\mathrm{cn}\left(t\sqrt{\frac{k_1 + 2k_2 + 2k_3(x_0 - y_0)^2}{m}},\right.$$

$$\left.\frac{k_3(x_0 - y_0)^2}{k_1 + 2k_2 + 2k_3(x_0 - y_0)^2}\right) + \frac{x_0 + y_0}{2}\cos\left(t\sqrt{\frac{k_1}{m}}\right),$$

$$y = \frac{x_0 + y_0}{2}\cos\left(t\sqrt{\frac{k_1}{m}}\right)$$

$$- \frac{x_0 - y_0}{2}\,\mathrm{cn}\left(t\sqrt{\frac{k_1 + 2k_2 + 2k_3(x_0 - y_0)^2}{m}},\ \frac{k_3(x_0 - y_0)^2}{k_1 + 2k_2 + 2k_3(x_0 - y_0)^2}\right).$$

2. For the case when the initial position of the system is zero, i.e., $x_0 = 0$ and $y_0 = 0$, the initial coefficients are $\phi = \pi/2$ and $\theta = K(k^2)$ where $K(k^2)$ is the total elliptical integral of the first kind [5].

By assuming the case when the parameters of the system are

$$m = 1, \quad k_1 = k_2 = k_3 = 1$$

with initial conditions

$$x_0 = 1, \qquad y_0 = 0{\cdot}5, \qquad \dot{x}_0 = \dot{y}_0 = 0. \tag{19}$$

The system of equations

$$\ddot{x} + x + (x - y) + (x - y)^3 = 0, \qquad \ddot{y} + y + (y - x) + (y - x)^3 = 0 \tag{20}$$

with initial conditions (19) produces the closed-form analytical solution

$$x = \tfrac{1}{4}\,\mathrm{cn}(t\sqrt{7/2}, 1/14) + \tfrac{3}{4}\cos t, \qquad y = \tfrac{3}{4}\cos t - \tfrac{1}{4}\,\mathrm{cn}(t\sqrt{7/2}, 1/14). \tag{21}$$

In Figure 2(a) and 2(b), the $x - t$ and $y - t$ diagrams are plotted. The motion of both masses is periodical.

3.2. LINEAR CASE

For the case when the system is linear, i.e.,

$$k_3 = 0, \tag{22}$$

the differential equations of motion are

$$m\ddot{x} + k_1 x + k_2(x - y) = 0, \qquad m\ddot{y} + k_1 y + k_2(y - x) = 0. \tag{23}$$

Substituting equation (22) into equation (10), the modulus of the elliptical function cn is zero ($k^2 = 0$). The elliptical function cn with the modulus zero transforms into the harmonic function cosines, and the general solution of the system is after equation (13)

$$x = (A/2)\cos(\omega t + \theta) + (B/2)\cos(\Omega t + \phi),$$

$$y = (B/2)\cos(\Omega t + \phi) - (A/2)\cos(\omega t + \theta). \tag{24}$$

This solution is well known in the theory of linear vibrations.

3.3. SYSTEM WITH SOFT NON-LINEARITY

For the case when the non-linearity is soft and k_3 is negative ($k_3 < 0$), the differential equations of motion are

$$m\ddot{x} + k_1 x + k_2(x - y) - k_3(x - y)^3 = 0,$$

$$m\ddot{y} + k_1 y + k_2(y - x) - k_3(y - x)^3 = 0. \tag{25}$$

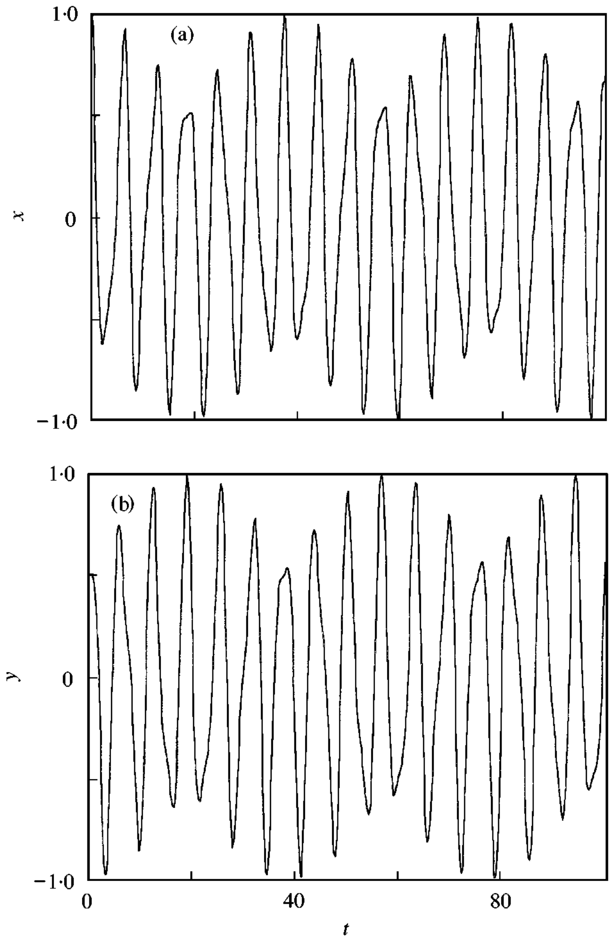

Figure 2. The x–t diagrams (a) and y–t diagram (b) for the system with strong cubic non-linearity.

The general solution has form (13) where modulus (10) and frequency (9) of the Jacobi elliptical function are

$$k^{*2} = k_3A^2/(k_1 + 2k_2 - 2k_3A^2) < 0, \tag{26}$$

$$\omega^{*2} = (k_1 + 2k_2 - 2k_3A^2)/m > 0. \tag{27}$$

The connection between the elliptical function cn with negative modulus and cd with positive modulus is according to reference [5]

$$\mathrm{cn}\left(\omega^* t + \alpha, -\frac{k_3A^2}{k_1 + 2k_2 - 2k_3A^2}\right) \equiv \mathrm{cd}\left[(\omega^* t + \alpha)\sqrt{\frac{k_1 + 2k_2 - k_3A^2}{k_1 + 2k_2 - 2k_3A^2}}, \frac{k_3A^2}{k_1 + 2k_2 - k_3A^2}\right].$$

The general solution of system (25) is

$$x = \tfrac{1}{2}A\,\mathrm{cd}(u, k^{**2}) + \tfrac{1}{2}B\cos(\Omega t + \phi), \qquad y = \tfrac{1}{2}B\cos(\Omega t + \phi) - \tfrac{1}{2}A\,\mathrm{cd}(u, k^{**2}), \tag{28}$$

where

$$u = (\omega^* t + \theta)\sqrt{(k_1 + 2k_2 - k_3 A^2)/(k_1 + 2k_2 - 2k_3 A^2)} \tag{29}$$

$$k^{**2} = k_3 A^2/(k_1 + 2k_2 - k_3 A^2). \tag{30}$$

4. APPROXIMATE SOLUTION OF THE SYSTEM

In this section, the approximate analytical solution of the system of differential equations (1) is obtained. It is based on the perturbation of solutions (13) of the system of two strong non-linear differential equations (6). The method of variable amplitude and phase (Krylov–Bogolubov) is extended for the solution which is a combination of elliptical and trigonometrical function. In the previous methods the Krylov–Bogolubov method was applied for the solutions which contain only elliptical functions or only trigonometrical functions. Now, it is the combination of both.

The trial solutions of equations (2) are assumed in the form of the generating solutions (8) and (11), as it is the usual procedure in the Krylov–Bogolubov method. The trial solution has the form

$$X = A(t)\,\mathrm{cn}[\psi_1(t), k^2(t)] \equiv A(t)\,\mathrm{cn}, \qquad Y = B(t)\cos\psi_2(t), \tag{31, 32}$$

where A, B, ω, ϕ, θ and k are time dependent,

$$\psi_2(t) = \Omega t + \phi(t) \tag{33}$$

and

$$\psi_1(t) = \int_t \omega(t)\,\mathrm{d}t + \theta(t) \tag{34}$$

as suggested by Yuste and Bejarano [8], or

$$\psi_1(t) = 4K(k^2)\varphi(t) \tag{35}$$

as suggested by Coppola and Rand [7]. $4K(k^2)\varphi(t)$ is the argument of the elliptical function which leads to the periodic variational equations that can be averaged. The phase equation $\theta(t)$ is not periodical one, and the averaging procedure cannot be applied for the non-periodical function. In this paper, the procedure suggested by Coppola and Rand [7] is applied.

The assumption in the Krylov–Bogolubov method is that the modulus–amplitude and frequency–amplitude relationships must be the same for the trial solution as for the generating solution and they are

$$\omega^2(t) = \alpha + \beta A^2(t), \qquad k^2(t) = \beta A^2(t)/(\alpha + \beta A^2(t)). \tag{36}$$

The constraint in the Krylov–Bogolubov method is that the time derivative of the trial solution must have the same form as the time derivative of the generating solution. For equations (31) and (32) it is

$$\dot{X} = A(t)\omega\,\mathrm{cn}_{\psi 1} = \dot{A}(\mathrm{cn} + 4\varphi A\,\mathrm{cn}_{\psi 1} K'k' + Ak'\,\mathrm{cn}_k) + 4KA\dot{\varphi}\,\mathrm{cn}_{\psi 1},$$

$$\dot{Y} = -B(t)\Omega\sin\psi_2, \tag{37}$$

where $(\cdot) \equiv \mathrm{d}/\mathrm{d}t$, $(') \equiv \mathrm{d}/\mathrm{d}A$, $\mathrm{cn}_{\psi 1}$ is the derivative of elliptical function with respect to the argument, and cn_k the derivative of elliptical function with respect to the modulus k. The task of finding the solution X and Y is transformed into finding six functions so that expressions (31) and (32) satisfy equations (2). Substituting the trial solutions into equations (2) and taking the aforementioned constraints the following four first order differential equations are obtained:

$$\dot{A} = (\varepsilon/m\omega)(f_1 - f_2)\,\mathrm{cn}_{\psi 1},$$

$$\dot{\varphi} = \frac{\omega}{4K} + \frac{\varepsilon}{m}(f_1 - f_2)\frac{1}{4KA\omega}\left[\mathrm{cn} - \frac{1-2k^2}{1-k^2}\left(Z\,\mathrm{cn}_{\psi 1} + k^2\,\mathrm{cn}(1-\mathrm{cn}^2)\right)\right],$$

$$\dot{B} = -(\varepsilon/m\Omega)(f_1 + f_2)\sin\psi_2, \qquad \dot{\phi} = -(\varepsilon/m\Omega\beta)(f_1 + f_2)\cos\psi_2, \tag{38}$$

where K is the complete elliptical integral of the first kind and Z the Jacobian Zeta function [5]

$$Z = Z(4K\varphi, k) = E(4K\varphi, k) - 4\varphi E \tag{39}$$

and

$$f_i \equiv f_i\{\tfrac{1}{2}[A\,\mathrm{cn}(\psi_1, k^2) + B\cos\psi_2], \tfrac{1}{2}[A\omega\,\mathrm{cn}_{\psi 1} - B\Omega\sin\psi_2],$$
$$\tfrac{1}{2}[B\cos\psi_2 - A\,\mathrm{cn}(\psi_1, k^2)], \tfrac{1}{2}[-A\omega\,\mathrm{cn}_{\psi 1} - B\Omega\sin\psi_2]\}, \quad i = 1, 2. \tag{40}$$

Using reference [4] it is

$$\mathrm{cn}_{\psi 1} = -\,\mathrm{sn}\,\mathrm{dn},$$
$$\mathrm{cn}_k = -\,(\mathrm{sn}\,\mathrm{dn}/k(1-k^2))\,[(1-k^2)4K\varphi$$
$$- E(4K\varphi, k)] - (k/(1-k^2))\,\mathrm{cn}(1-\mathrm{cn}^2), \tag{41}$$

where $E(4K\varphi, k)$ is the Legendre's incomplete elliptical integral of the second kind and E the complete elliptical integral of the second kind [5]. Solving equations (38) one obtains the approximate solutions of equations (1)

$$x = \tfrac{1}{2}A(t)\,\mathrm{cn}[\psi_1(t), k^2(t)] + \tfrac{1}{2}B(t)\cos\psi_2(t),$$
$$y = \tfrac{1}{2}B(t)\cos\psi_2(t) - \tfrac{1}{2}A(t)\,\mathrm{cn}[\psi_1(t), k^2(t)]. \tag{42}$$

To find the solution of equations (38) is not an easy task. Usually, the averaging procedure is introduced to simplify the problem. The averaged equations (38) are

$$\dot{A} = -\frac{\varepsilon}{m\omega}\frac{1}{2\pi}\frac{1}{4K}\int_0^{4K}\left[\int_0^{2\pi}(f_1 - f_2)\,\mathrm{cn}_{\psi 1}\,\mathrm{d}\psi_1\right]\mathrm{d}\psi_2,$$

$$\dot{\varphi} = \frac{\omega}{4K} + \frac{\varepsilon}{m}\frac{1}{2\pi}\frac{1}{4K}\int_0^{4K}\left[\int_0^{2\pi}(f_1 - f_2)\frac{1}{4KA\omega}\right.$$
$$\left.\times\left[\mathrm{cn} - \frac{1-2k^2}{1-k^2}\left(Z\,\mathrm{cn}_{\psi 1} + k^2\,\mathrm{cn}(1-\mathrm{cn}^2)\right)\right]\mathrm{d}\psi_1\right]\mathrm{d}\psi_2,$$

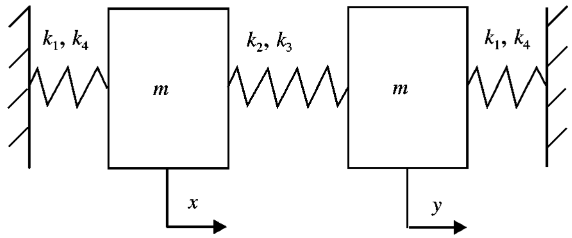

Figure 3. The model of the two-mass system with three non-linear springs.

$$\dot{B} = -\frac{\varepsilon}{m\Omega}\frac{1}{2\pi}\frac{1}{4K}\int_0^{4K}\left[\int_0^{2\pi}(f_1 + f_2)\sin\psi_2 \, \mathrm{d}\psi_1\right]\mathrm{d}\psi_2,$$

$$\dot{\phi} = -\frac{\varepsilon}{m\Omega B}\frac{1}{2\pi}\frac{1}{4K}\int_0^{4K}\left[\int_0^{2\pi}(f_1 + f_2)\cos\psi_2 \, \mathrm{d}\psi_1\right]\mathrm{d}\psi_2. \tag{43}$$

The procedure of averaging of the elliptical functions is widely analyzed in papers [7, 8]. Solving averaged equations (43) the $A(t)$, $\varphi(t)$, $B(t)$ and $\phi(t)$ are obtained. Substituting these functions into equation (42) the approximate solutions of equations (1) are obtained.

4.1. THE SYSTEM WITH NON-LINEAR STIFFNESS

Consider the case when the stiffnesses are with small non-linearities and the differential equation of motion is

$$m\ddot{x} + (k_1 x + \varepsilon k_4 x^3) + k_2(x - y) + k_3(x - y)^3 = 0,$$

$$m\ddot{y} + (k_1 y + \varepsilon k_4 y^3) + k_2(y - x) + k_3(y - x)^3 = 0, \tag{44}$$

i.e.,

$$\ddot{X} + \alpha X + \beta X^3 = -(\varepsilon k_4/4m) X (X^2 + 3Y^2),$$

$$\ddot{Y} + \Omega^2 Y = (\varepsilon k_4/4m) Y (Y^2 + 3X^2). \tag{45}$$

The model of the system is shown in Figure 3. The approximate solutions of equations (45) are according to equation (31), (32) and (43)

$$X = A_0 \,\mathrm{cn}[Q_1 t + \varphi_0, k^2(A_0)], \qquad Y = B_0 \cos[(\Omega + Q_2)t + \phi_0], \tag{46, 47}$$

where

$$A_0 = \mathrm{const}, \qquad B_0 = \mathrm{const},$$

$$\Omega = \sqrt{k_1/m}, \qquad Q_2 = (3\varepsilon k_4/8m\Omega)\,[A_0^2\langle \mathrm{cn}^2\rangle + B_0^2/4],$$

$$Q_1 = \omega - (\varepsilon k_4/4m\omega)(1/(1-k^2))$$

$$\times \{A_0^2[(1-2k^2+2k^4)\langle \mathrm{cn}^4\rangle + (1-2k^2)k^2\langle \mathrm{cn}^6\rangle + (1-2k^2)\langle Z\,\mathrm{sn}\,\mathrm{dn}\,\mathrm{cn}^3\rangle]$$

$$+ 3B_0^2\langle\cos^2\psi_2\rangle[(1-2k^2+2k^4)\langle \mathrm{cn}^2\rangle$$

$$+ (1-2k^2)k^2\langle \mathrm{cn}^4\rangle + (1-2k^2)\langle Z\,\mathrm{sn}\,\mathrm{dn}\,\mathrm{cn}\rangle]\},$$

$$\omega \equiv \omega(A_0) = \sqrt{\alpha + \beta A_0^2} = \sqrt{(k_1 + 2k_2 + 2k_3A_0^2)/m},$$

$$k^2 \equiv k^2(A_0) = \frac{\beta A_0^2}{2(\alpha + \beta A_0^2)} = \frac{k_3 A_0^2}{k_1 + 2k_2 + 2k_3 A_0^2}.$$

The averaged Jacobi elliptical functions are (see reference [5])

$$\langle\cos^2\psi_2\rangle = \frac{1}{2\pi}\int_0^{2\pi}\cos^2\psi_2\,\mathrm{d}\psi_2 = \tfrac{1}{2},$$

$$\langle\cos^4\psi_2\rangle = \frac{1}{2\pi}\int_0^{2\pi}\cos^4\psi_2\,\mathrm{d}\psi_2 = \tfrac{3}{8},$$

$$\langle \mathrm{cn}^2\rangle = \frac{1}{4K}\int_0^{4K}\mathrm{cn}^2[\psi_1(t), k^2]\,\mathrm{d}\psi_1 = \frac{1}{k^2}\left[\frac{E}{K} - (1-k^2)\right],$$

$$\langle \mathrm{cn}^4\rangle = \frac{1}{4K}\int_0^{4K}\mathrm{cn}^4[\psi_1(t), k^2]\,\mathrm{d}\psi_1 = \frac{1}{3k^4}\left[2(2k^2-1)\frac{E}{K} + (2-3k^2)(1-k^2)\right],$$

$$\langle \mathrm{cn}^6\rangle = \frac{1}{4K}\int_0^{4K}\mathrm{cn}^6[\psi_1(t), k^2]\,\mathrm{d}\psi_1$$

$$= (1/15k^6)[(23k^4 - 23k^2 + 8)(E/K) + 15k^6 - 34k^4 + 27k^2 - 8].$$

Substituting equations (46) and (47) into equations (2) the general solutions are

$$x_a = \tfrac{1}{2}A_0\,\mathrm{cn}[Q_1 t + \varphi_0, k^2(A_0)] + \tfrac{1}{2}B_0\cos(Q_2 t + \phi_0),$$

$$y_a = \tfrac{1}{2}B_0\cos(Q_2 t + \phi_0) - \tfrac{1}{2}A_0\,\mathrm{cn}[Q_1 t + \varphi_0, k^2(A_0)], \qquad (48)$$

where $A_0, B_0, \phi_0, \varphi_0$ are constant values dependent on initial conditions. For the parameter values $k_1 = k_2 = k_3 = k_4 = m = 1$, $\varepsilon = 0{\cdot}1$ and initial conditions $A_0 = 0{\cdot}5$, $B_0 = 1{\cdot}5$, $\phi_0 = 0$, $\varphi_0 = 0$, i.e., $x_0 = 1$, $\dot{x}_0 = 0$, $y_0 = 0{\cdot}5$, $\dot{y}_0 = 0$ the history–time diagrams for both masses are plotted. In Figure 4(a), the x_a–t diagram obtained analytically (see equations (48)) and the diagram obtained by numerically solving equations (44) using the Runge–Kutta method, the x_n–t diagram, are plotted. In Figure 4(b), the y_a–t diagram obtained analytically and the numerical y_n–t diagram are plotted. The motions are periodical. The difference between analytical and numerical solutions is negligible.

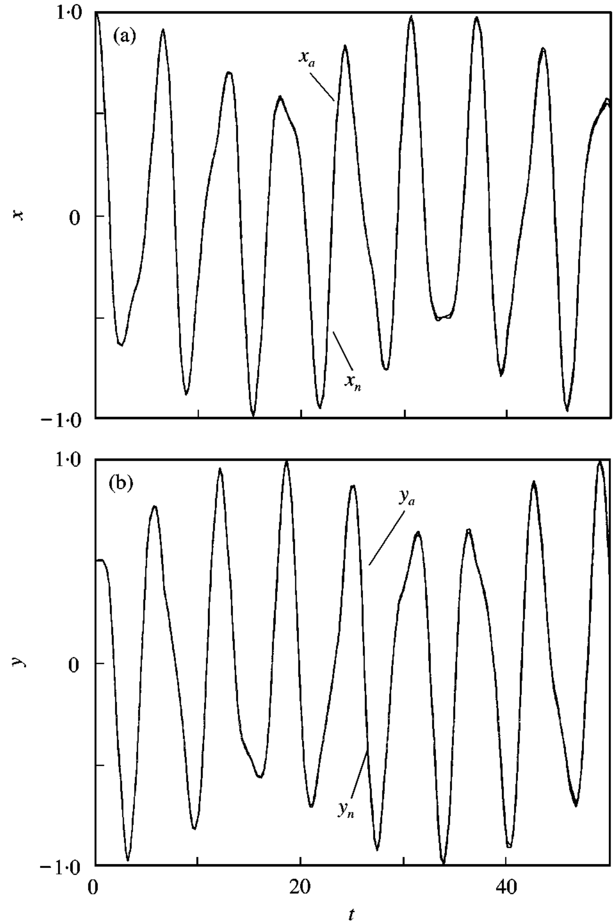

Figure 4. The x–t diagrams (a) and y–t diagram (b) for the system with three non-linear springs.

4.2. THE SYSTEM WITH SMALL DAMPING

Consider the case when the small damping exists. The model of the system is shown in Figure 5. The mathematical model of the system is

$$\ddot{x} + k_1 x + k_2(x - y) + k_3(x - y)^3 = -\varepsilon d\,\dot{x},$$

$$\ddot{y} + k_1 y + k_2(y - x) + k_3(y - x)^3 = -\varepsilon d\,\dot{y} \tag{49}$$

or introducing variables (2)

$$\ddot{X} + \alpha X + \beta X^3 = -(\varepsilon d/m)((\dot{X} + \dot{Y})/2),$$

$$\ddot{Y} + \Omega^2 Y = -(\varepsilon d/m)((\dot{Y} - \dot{X})/2). \tag{50}$$

Equations (49) and (50) are solved analytically and numerically. For the initial conditions $t = 0$, $A_0 = 0{\cdot}5$, $B_0 = 1{\cdot}5$, $\varphi_0 = 0$, $\phi_0 = 0$, the analytical solutions of equations (49) have

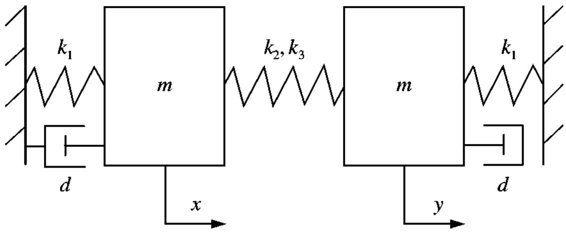

Figure 5. The model of the two-mass system with damping.

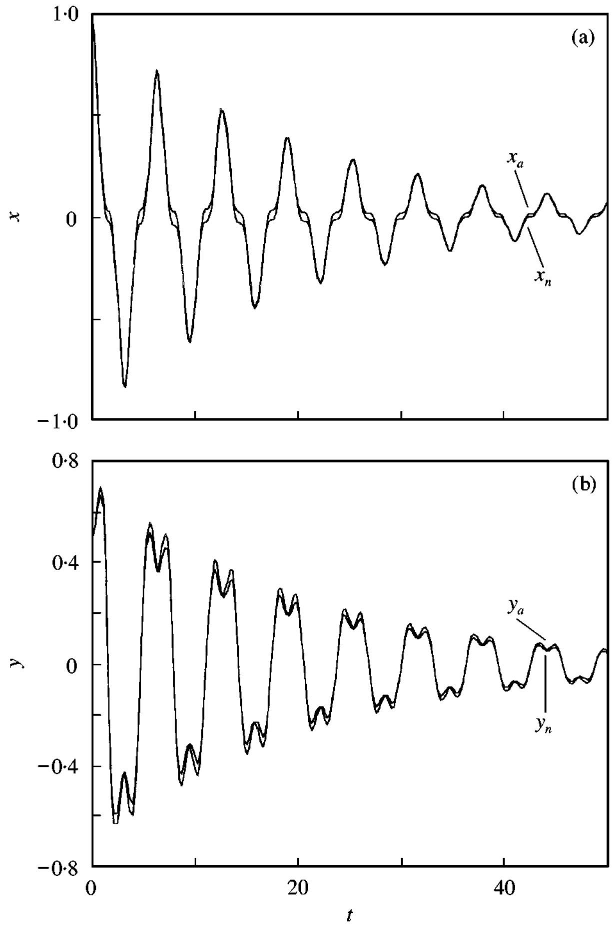

Figure 6. The x–t diagrams (a) and y–t diagrams (b) for the system with damping.

form (13) and they are

$$x_a = A(t)\,\mathrm{cn}[4\varphi(t)K, k^2] + B_0 \mathrm{e}^{-\varepsilon dt/2m},$$
$$y_a = B_0 \mathrm{e}^{-\varepsilon dt/2m} - A(t)\,\mathrm{cn}[4\varphi(t)K, k^2], \tag{51}$$

where $A(t)$ and $\varphi(t)$ are the solutions of the first order differential equations

$$\dot{A} = -\frac{\varepsilon d A}{m\omega}\frac{1}{4K}\int_0^{4K} \mathrm{cn}_{\psi 1}^2 \, \mathrm{d}\psi_1,$$

$$\dot{\varphi} = (\omega/4K) + (\varepsilon d/m)(1/4K)\int_0^{4K} \frac{\mathrm{cn}_{\psi 1}}{4K\omega}\left[\mathrm{cn} - \frac{1-2k^2}{1-k^2}(Z\mathrm{cn}_{\psi 1} + k^2 \mathrm{cn}(1-\mathrm{cn}^2))\right]\mathrm{d}\psi_1. \quad (52)$$

For the parameter values $k_1 = k_2 = k_3 = 1$, $m = 1$, $\varepsilon d = 0{\cdot}1$, analytical solutions (51) are compared with numerical one (x_n, y_n) which are obtained by using the Runge–Kutta procedure. In Figure 6(a), the x_a–t and x_n–t diagrams are plotted. In Figure 6(b), the time–history diagrams y_a–t and y_n–t are compared. The initial conditions are $t = 0$, $x_0 = 1$, $\dot{x}_0 = 0$, $y_0 = 0{\cdot}5$, $\dot{y}_0 = 0$. The motion has a tendency for decrease. Comparing the analytical and numerical solutions it can be seen that the analytical solution improves upon the numerical one. The difference between the solutions increases in time.

5. CONCLUSION

It can be concluded that

1. The motion of a two-mass system connected with a strong non-linear cubic stiffness which is described with a system of two strong non-linear cubic differential equations can be obtained in closed analytical form. The motions of the masses are periodical. The motions are described as the combination of a Jacobi elliptical function and a trigonometrical function.
2. An approximate method for solving coupled strong non-linear differential equations with small non-linearities has been developed based on the well-known Krylov–Bogolubov procedure.
3. The motion of the two-mass system with three non-linear stiffnesses differs from the system where only the stiffness which connects the masses is non-linear. Comparing analytical results (13) and (48) and also Figures 2 and 4, it can be seen that the motions are periodic but the period of vibrations differs for these cases: it is longer for the first case.
4. The approximate analytical results are in good agreement with exact numerical solutions. It proves the correctness of the analytical procedure.

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