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STABILITY AND BREATHING MOTIONS OF PRESSURIZED COMPRESSIBLE HYPERELASTIC SPHERICAL SHELLS

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The stability of homogeneous, isotropic, compressible, hyperelastic, thick spherical shells subjected to external dead-load traction are investigated within the context of the finite elasticity theory. The stability of the finitely deformed state and small, free, radial vibrations about this state are investigated using the theory of small deformations superposed on large elastic deformations. The frequencies of small free vibrations about the pre-stressed state are obtained numerically. The loss of stability occurs when the motions cease to be periodic. The critical values of stress and deformation are given for a foam rubber, slightly compressible rubber and a nearly incompressible rubber.

1. INTRODUCTION

The earlier research on compressible, hyperelastic solids were mostly limited to determining a suitable model representing the behavior [1–3], and to analyzing deformations and stresses in bodies of different geometries which undergo finite elastic deformations for various boundary conditions [4–6].

Parallel to the works mentioned above, in 1982, Ball [7] investigated discontinuous equilibrium solutions and cavitation in non-linear elasticity by modelling the appearance of a cavity in the interior of a solid, homogeneous, isotropic, hyperelastic body once a critical load is reached. Following Ball's work, a class of problems concerning the void nucleation and growth in such bodies has been the subject of extensive research [8–10]. The problem of bifurcation of a solid sphere made of compressible Varga material subjected to a uniform radial tension on its outer surface was investigated by Horgan [11]. In reference [11], an analytical solution was obtained for the critical stretch of the outer surface causing the formation of a void at the center.

Recently, Akyuz and Ertepinar have studied the breathing motions [12] and the asymmetric vibrations [13] of cylindrical shells of arbitrary wall thickness about a finitely deformed state caused by uniform external dead-load traction. For breathing motion [12], it was observed that the shells display a hardening behavior under an increasing inward pressure and a softening behavior under inflation, while the opposite was observed for asymmetric vibration [13]. As a result, in breathing motions, failure occurs due to the inflation without bound at a critical external outward pressure, while in asymmetric vibrations, the loss of stability occurs under a critical external inward pressure and the critical circumferential mode shape is independent of system parameters.

The present work deals with the breathing motions and the loss of stability of thick spherical shells subjected to finite radial deformations. The material of the body is assumed to be a polynomial material [3] which is homogeneous, isotropic, compressible, hyperelastic and reduces to Blatz–Ko material when some material constants are specialized. The shell is first subjected to a finite, uniform, radial extension. The stress and the displacement fields of this initial state are expressed using the theory of finite elasticity [14]. The resulting highly non-linear differential system of this state is solved numerically by using the multiple shooting method [15]. The spherical shells are then exposed to a secondary radial, dynamical displacement field. The formulation of this state is based on the theory of small deformations superposed on large elastic deformations which is due to Green *et al.* [16]. The boundary conditions of this state are obtained from the requirement that the secondary surface tractions vanish. The homogeneous, linear differential system governing the secondary state is solved numerically using the method of complementary functions. For a non-trivial solution of the problem, it is required that the characteristic determinant of the system vanishes. This determinant contains parameters pertaining to the finitely deformed state, the frequencies of small, free vibrations about this state, material constants and the initial geometry of the shell. Here, the solution of this equation yields the frequencies numerically. The loss of stability occurs when the motions about the finitely deformed state cease to be periodic, i.e., when the frequency of vibrations equals zero.

Numerical results are obtained to investigate the effects of several geometric and material properties on the frequencies and the critical stretch of the outer surface when instability occurs. The results corresponding to the limiting cases (such as solid shell, thin shell, foam rubber, nearly incompressible rubber) are obtained and compared with those existing in the literature.

2. FORMULATION

Consider a spherical shell made of a homogeneous, isotropic, compressible, and hyperelastic material. Let r_1 and r_2, respectively, denote the inner and outer radii of the undeformed shell. The shell is subjected to a uniform radial traction q on its exterior surface. The co-ordinates of a material point in the undeformed and the deformed states are, respectively, given by (the details of the formulation can be found in the text by Green and Zerna [14])

$$x_1 = r\sin\theta\cos\phi, \quad x_2 = r\sin\theta\sin\phi, \quad x_3 = r\cos\theta \tag{1}$$

and

$$X_1 = R(r)\sin\theta\cos\phi, \quad X_2 = R(r)\sin\theta\sin\phi, \quad X_3 = R(r)\cos\theta. \tag{2}$$

The non-zero components of the contravariant metric tensors of the undeformed and deformed states, g^{ij} and G^{ij}, are

$$(g^{11}, g^{22}, g^{33}) = (R'^2, 1/r^2, 1/(r\sin\theta)^2), \quad (G^{11}, G^{22}, G^{33}) = (1, 1/R^2, 1/(R\sin\theta)^2), \tag{3}$$

where a prime denotes derivative with respect to r. Therefore, the three strain invariants are given by

$$I_1 = R'^2 + 2(R^2/r^2), \quad I_2 = (R^4/r^4) + 2(R^2R'^2/r^2), \quad I_3 = R^4R'^2/r^4. \tag{4}$$

The strain energy density function W of a homogeneous, isotropic, hyperelastic, compressible material can expressed in terms of three strain invariants

$$W = W(I_1, I_2, I_3). \tag{5}$$

The components of the stress tensor τ^{ij} are given by

$$\tau^{ij} = \Phi g^{ij} + \Psi B^{ij} + pG^{ij}, \tag{6}$$

where

$$\Phi = (2/\sqrt{I_3})\,\partial W/\partial I_1, \quad \Psi = (2/\sqrt{I_3})\,\partial W/\partial I_2, \quad p = 2\sqrt{I_3}(\partial W/\partial I_3),$$

$$B^{ij} = I_1 g^{ij} - g^{ir}g^{js}G_{rs}. \tag{7}$$

It is now further assumed that the strain energy density function W has the form

$$W = \frac{\mu}{2}\left[f(I_1 - 3) + (1-f)\left(\frac{I_2}{I_3} - 3\right) + 2(1-2f)(\sqrt{I_3} - 1) + (2f+\beta)(\sqrt{I_3} - 1)^2\right], \tag{8}$$

which has been proposed by Levinson and Burgess [3] and has been named "polynomial compressible material" by the authors. In equation (8), μ is the shear modulus of the material for vanishingly small strains, f is a material constant whose value lies between zero and unity, and β is expressed as

$$\beta = (4v - 1)/(1 - 2v), \tag{9}$$

where v is the Poisson ratio for the material as the deformations become vanishingly small. It is noted that, for highly elastic rubbers and rubber-like materials, $f = 0$ while for solid natural and synthetic rubbers, $f = 1$. When $v \to \frac{1}{2}$, the expression for the strain energy density function reduces to that of a neo-Hookean material. It is also noted that the Levinson–Burgess and Blatz–Ko models are identical for $v = 0{\cdot}25$ and $f = 0$.

The only non-zero equation of equilibrium for this finitely deformed state is the one in the radial direction and it is given by

$$\partial\tau^{11}/\partial R + (2\tau^{11} - R^2(\tau^{22} + \sin^2\theta\,\tau^{33}))/R = 0. \tag{10}$$

Substituting equations (3), (4), (6–8), into equation (10), the equation of equilibrium is obtained as

$$f(2rR'/R^2 - 2/R + r^2R''/R^2) + (1-f)(2r^4/R^5 - 2/R^2R'^3 + 3r^2R''/R^2R'^4)$$

$$+ (2f+\beta)(2RR'^2/r^2 - 2R^2R'/r^3 + R^2R''/r^2) = 0. \tag{11}$$

The shell is assumed to be free of tractions on its inner surface and subjected to a uniform tensile dead-load traction on its exterior surface with $q \geqslant 0$, therefore, the associated boundary conditions are

$$\tau^{11}(R_1) = 0 \quad \text{and} \quad \tau^{11}(R_2) = q(r_2/R_2)^2. \tag{12}$$

One now superposes a secondary dynamic displacement field onto the finitely deformed state which is described by

$$w_1 = u(R(r), t), \quad w_2 = w_3 = 0 \tag{13}$$

to investigate the existence of small, free, radial vibrations about the finitely deformed shell. In equations (13), w_1, w_2 and w_3 are the radial, the circumferential and the meridional components in the secondary dynamic displacement field.

The formulation of this state is based on the theory of small deformations superposed on large elastic deformations (a detailed discussion of the theory is given in reference [16]). The incremental metric tensors and the incremental stresses are given by

$$G^*_{ij} = w_{i,j} + w_{j,i} - 2\Gamma^r_{ij} w_r, \quad G^{*ij} = -\, G^{ir} G^{js} G^*_{rs} \tag{14}$$

$$\tau^{*ij} = g^{ij}\Phi^* + B^{ij}\Psi^* + B^{*ij}\Psi + G^{*ij}p + G^{ij}p^*, \tag{15}$$

where

$$\Phi^* = \frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_1^2} I_1^* + \frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_1 \partial I_2} I_2^* + \frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_1 \partial I_3} I_3^* - \frac{\Phi}{2I_3} I_3^*,$$

$$\Psi^* = \frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_1 \partial I_2} I_1^* + \frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_2^2} I_2^* + \frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_2 \partial I_3} I_3^* - \frac{\Psi}{2I_3} I_3^*,$$

$$p^* = I_3 \left(\frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_1 \partial I_3} I_1^* + \frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_2 \partial I_3} I_2^* + \frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_3^2} I_3^* \right) + \frac{p}{2I_3} I_3^*, \tag{16}$$

$$B^{*ij} = (g^{ij} g^{rs} - g^{ir} g^{js}) G^*_{rs}$$

and Γ^r_{ij} are the Christoffel symbols of the second kind, and a comma denotes differentiation with respect to the following subscript. The incremental strain invariants I_1^*, I_2^*, and I_3^*, are given by

$$I_1^* = 2R'^2 w_{1,R} + 4(R/r^2) w_1, \qquad I_2^* = (4R^2 R'^2/r^2) w_{1,R} + (4RR'^2/r^2 + 4R^3/r^4) w_1,$$

$$I_3^* = (2R^4 R'^2/r^4) w_{1,R} + (4R^3 R'^2/r^4) w_1. \tag{17}$$

Hence, in the absence of body forces, the incremental equations of motion reduce to

$$\tau'^{ij}_{,j} + \Gamma^i_{jm}\tau'^{mj} + \Gamma^j_{jm}\tau'^{mi} + \Gamma'^i_{jm}\tau^{mj} + \Gamma'^j_{jm}\tau^{mi} = \rho \ddot{w}^i + \rho'(\partial\theta^i/\partial X^j)\ddot{X}^j, \tag{18}$$

where $\ddot{w}^i$ are the linear increment of the acceleration component in the i direction, and Γ'^i_{jm} are the incremental values of the Christoffel symbols of the second kind. In equation (18), ρ is the mass density of a finitely deformed body and ρ' is the increment of ρ.

Considering equation (13) and also noting that $\ddot{w}^1 = \ddot{w}_1$, the only non-vanishing equation of motion is

$$\frac{\partial \tau^{*11}}{\partial R} + \frac{1}{R}(2\tau^{*11} - 2R^2\tau^{*22}) + 2w_{1,RR}\tau^{11} + \frac{2}{R}\left(w_{1,R} - \frac{w_1}{R}\right)(\tau^{11} + R^2\tau^{22}) = \rho \ddot{w}_1, \tag{19}$$

which upon substitution of equations (6–8) and (13–17), equation (19) reduces to

$$u_{,rr}\left\{f\frac{r^2}{R^2R'}+(1-f)\frac{3r^2}{R^2R'^5}+(2f+\beta)\frac{R^2}{r^2R'}\right\}$$

$$+u_{,r}\left\{f\frac{2r}{R^2R'}+(1-f)\left(\frac{6r}{R^2R'^5}-\frac{12r^2R''}{R^2R'^6}\right)+(2f+\beta)\left(\frac{4R}{r^2}-\frac{2R^2}{r^2R'}\right)\right\}$$

$$+u\left\{f\left(\frac{6}{R^2R}-\frac{4r^2R''}{R^3R'}-\frac{8r}{R^3}\right)+(1-f)\left(\frac{8r}{R^3R'^4}-\frac{12r^2R''}{R^3R'^5}-\frac{14r^4}{R^6R'}\right)-(2f+\beta)\frac{2R'}{r^2}\right\}$$

$$=\frac{\rho}{\mu}\ddot{u}, \tag{20}$$

where a dot denotes differentiation with respect to time. The principle of conservation of mass states the relationship between the current mass density ρ and the mass density ρ_0 of the natural state as

$$\rho=(\sqrt{g}/\sqrt{G})\rho_0. \tag{21}$$

In equation (21), g and G denote, respectively, the determinants $[g_{ij}]$ and $[G_{ij}]$. For this secondary state, the boundary conditions, which are obtained from the requirement that the secondary surface tractions vanish, are

$$\tau^{*11}-G^{*11}\tau^{11}=0 \quad \text{at} \quad R=R_1 \text{ and } R=R_2. \tag{22}$$

The solution of equation (20) may be assumed to be of the form

$$u=R^*(R)\mathrm{e}^{\mathrm{i}\omega t}, \tag{23}$$

where ω is the frequency of vibrations about the finitely deformed state and R^* is an unknown function of R which, in turn, is a function of r. Hence, equation (20) is reduced to

$$R^*_{,rr}\{f+(1-f)3/R'^4+(2f+\beta)R^4/r^4\}$$

$$+R^*_{,r}\{f(2/r)+(1-f)(6/rR'^4-12R''/R'^5)+(2f+\beta)(4R^3R'/r^4-2R^4/r^5)\}$$

$$+R^*\{f(6/r^2-4R''/R-8R'/rR)+(1-f)(8/rRR'^3-12R''/RR'^4-14r^2/R^4)$$

$$-(2f+\beta)2R^2R'^2/r^4\}=-(\rho_0/\mu)\omega^2R^*. \tag{24}$$

Using equations (15), (16), and (23), the boundary conditions expressed by equation (22) are given by

$$R^*_{,r}\left\{f\frac{r^2}{R^2}+(1-f)\frac{3r^2}{R^2R'^4}+(2f+\beta)\frac{R^2}{r^2}\right\}$$

$$+R^*\left\{-f\frac{2r^2R'}{R^3}+(1-f)\frac{2r^2}{R^3R'^3}+(2f+\beta)\frac{2RR'}{r^2}\right\}=0 \quad \text{at} \quad R=R_1 \quad \text{and} \quad R=R_2. \tag{25}$$

The system of equations, equations (11) and (12), governing the finitely deformed state and the system of equations, equations (33) and (34), governing the superposed dynamic state are solved by a numerical scheme in the next section.

3. THE ANALYSIS OF THE PROBLEM

For the analysis of the problem, it is desirable to express the variables in a non-dimensional form. For this purpose, the following quantities are introduced:

$$\bar{r} = r/r_2, \quad \bar{R} = R/r_2, \quad \bar{R}^* = R^*/r_2, \quad \bar{\tau}^{11} = \tau^{11}/\mu, \quad \bar{q} = q/\mu,$$
$$\bar{u} = u/r_2, \quad \bar{\omega} = \omega\sqrt{\rho_0 r_2^2/\mu}. \tag{26}$$

The non-dimensional form of equation (11) is

$$f(2\bar{r}\bar{R}'/\bar{R}^2 - 2/\bar{R} + \bar{r}^2\bar{R}''/\bar{R}^2) + (1-f)(2\bar{r}^4/\bar{R}^5 - 2\bar{r}/\bar{R}^2\bar{R}'^3 + 3\bar{r}^2\bar{R}''/\bar{R}^2\bar{R}'^4)$$
$$+ (2f+\beta)(2\bar{R}\bar{R}'^2/\bar{r}^2 - 2\bar{R}^2\bar{R}'/\bar{r}^3 + \bar{R}^2\bar{R}''/\bar{r}^2) = 0, \tag{27}$$

while the associated boundary conditions given by equations (12) reduce to

$$f\frac{\bar{r}_1^2\bar{R}_1'}{\bar{R}_1^2} - (1-f)\frac{\bar{r}_1^2}{\bar{R}_1^2\bar{R}_1'^3} + (2f+\beta)\frac{\bar{R}_1^2\bar{R}_1'}{\bar{r}_1^2} + (1-4f-\beta) = 0,$$
$$f\frac{\bar{R}_2'}{\bar{R}_2^2} - (1-f)\frac{1}{\bar{R}_2^2\bar{R}_1'^3} + (2f+\beta)\bar{R}_2^2\bar{R}_2' + (1-4f-\beta) = \bar{q}\left(\frac{1}{\bar{R}_2}\right)^2. \tag{28}$$

The non-dimensional governing equation for the secondary state is

$$A\bar{R}^*_{,rr} + B\bar{R}^*_{,r} + C\bar{R}^* = 0, \tag{29}$$

where

$$A = f + (1-f)3/\bar{R}'^4 + (2f+\beta)\bar{R}^4/\bar{r}^4,$$
$$B = f(2/\bar{r}) + (1-f)(6/\bar{r}\bar{R}'^4 - 12\bar{R}''/\bar{R}'^5) + (2f+\beta)(4\bar{R}^3\bar{R}'/\bar{r}^4 - 2\bar{R}^4/\bar{r}^5),$$
$$C = f(6/\bar{r}^2 - 4\bar{R}''/\bar{R} - 8\bar{R}'/\bar{r}\bar{R}) + (1-f)(8/\bar{r}\bar{R}\bar{R}'^3 - 12\bar{R}''/\bar{R}\bar{R}'^4 - 14\bar{r}^2/\bar{R}^4)$$
$$- (2f+\beta)(2\bar{R}^2\bar{R}'^2/\bar{r}^4) + \bar{\omega}^2. \tag{30}$$

The boundary conditions of this state, equations (25), reduce to

$$M\bar{R}^*_{,r} + N\bar{R}^* = 0 \quad \text{at} \quad R = R_1 \text{ and } R = R_2, \tag{31}$$

where

$$M = f(\bar{r}^2/\bar{R}^2) + (1-f)3\bar{r}^2/\bar{R}^2\bar{R}'^4 + (2f+\beta)(\bar{R}^2/\bar{r}^2),$$
$$N = -f(2\bar{r}^2\bar{R}'/\bar{R}^3) + (1-f)(2\bar{r}^2/\bar{R}^3\bar{R}'^3) + (2f+\beta)(2\bar{R}\bar{R}'/\bar{r}^2). \tag{32}$$

For the finitely deformed state, no closed-form solution of the highly non-linear system of equations, equations (27) and (28), seems possible. To solve these equations numerically, the boundary value problem is converted to an initial value problem by using the multiple shooting method [15]. For this purpose, equation (27) is first converted into a set of two first order equations of the form.

$$y_1' = y_2,$$

$$y_2' = \frac{f\left(\dfrac{2}{y_1} - \dfrac{2xy_2}{y_1^2}\right) + (1-f)\left(\dfrac{2x}{y_1^2 y_2^3} - \dfrac{2x^4}{y_1^5}\right) + (2f+\beta)\left(\dfrac{2y_1^2 y_2}{x^3} - \dfrac{2y_1 y_2^2}{x^2}\right)}{f\dfrac{x^2}{y_1^2} + (1-f)\dfrac{3x^2}{y_1^2 y_2^4} + (2f+\beta)\dfrac{y_1^2}{x^2}} \tag{33}$$

and the associated boundary conditions, equations (28), are now expressed as

$$f\frac{x^2 y_2}{y_1^2} - (1-f)\frac{x^2}{y_1^2 y_2^3} + (2f+\beta)\frac{y_1^2 y_2}{x^2} + (1-4f-\beta) = 0 \quad \text{at } R_1,$$

$$f\frac{y_2}{y_1^2} - (1-f)\frac{1}{y_1^2 y_2^3} + (2f+\beta)y_1^2 y_2 + (1-4f-\beta) = \bar{q}\left(\frac{1}{y_1}\right)^2 \quad \text{at } R_2, \tag{34}$$

where

$$x = \bar{r}, \quad y_1 = \bar{R}, \quad y_2 = \bar{R}'. \tag{35}$$

Note that the boundary conditions given by equation (34) can be rewritten in terms of the outer stretch ratio $\lambda(= R_2/r_2$ which is the value of y_1 at R_2) and wall thickness χ ($= r_1/r_2$ which is the value of x at R_1) as

$$f\frac{\chi^2 y_2}{y_1^2} - (1-f)\frac{\chi^2}{y_1^2 y_2^3} + (2f+\beta)\frac{y_1^2 y_2}{\chi^2} + (1-4f-\beta) = 0 \quad \text{at } R_1,$$

$$f\frac{y_2}{\lambda^2} - (1-f)\frac{1}{\lambda^2 y_2^3} + (2f+\beta)\lambda^2 y_2 + (1-4f-\beta) = \bar{q}\left(\frac{1}{\lambda}\right)^2 \quad \text{at } R_2. \tag{36}$$

The solution of equations (33) together with equations (34) is obtained by using a FORTRAN code called BVPSOL and developed by Deuflard and Bader [15]. This subroutine is a "(B)oundary (V)alue (P)roblem (So)lver for highly non-linear two-point boundary value problems using (L)ocal linear solver (condensing algorithm) for the solution of the arising linear subproblems by multiple shooting approach". For a nearly solid shell, 4001 equally spaced nodal points along the radial direction are used to attain the desired accuracy. The number of necessary nodal points decreases as the shell thickness increases.

Equation (29) which governs the secondary state is linear in $\bar{R}^*$ but, since no closed-form solution is available for the finitely deformed state, the coefficients A, B, and C in equation (29) contain constants known pointwise and the unknown frequency, and, hence, numerical approach must be used to obtain the solution of this state. The method of complementary functions is used to transform the linear boundary value problem to an initial value problem. For this purpose, the unknown function $\bar{R}^*$ is expressed as

$$\bar{R}^*(r) = c_1\bar{R}_1^* + c_2\bar{R}_2^*, \tag{37}$$

where c_1 and c_2 are constants, and $\bar{R}_1^*$ and $\bar{R}_2^*$ are the two homogenous solutions of equation (29). Next, equation (29) is converted into a set of two first order equations of the form

$$z_1' = z_2, \quad z_2' = -(Bz_2 + Cz_1)/A, \tag{38}$$

where

$$z_1 = \bar{R}^*, \quad z_2 = \bar{R}^*_{,r}. \tag{39}$$

To determine $\bar{R}_1^*$ and $\bar{R}_2^*$ at nodal points, equations (38) are integrated with initial conditions $z_1(R_1) = 1$ and $z_2(R_1) = 0$, and $z_1(R_1) = 0$ and $z_2(R_1) = 1$ respectively. The Runge–Kutta method of order four is used to integrate equation (38). Equation (31) now reduces to

$$\begin{bmatrix} M(R_1)\bar{R}^*_{1,r}(R_1) + N(R_1)\bar{R}_1^*(R_1) & M(R_1)\bar{R}^*_{2,r}(R_1) + N(R_1)\bar{R}_2^*(R_1) \\ M(R_2)\bar{R}^*_{1,r}(R_2) + N(R_2)\bar{R}_1^*(R_2) & M(R_2)\bar{R}^*_{2,r}(R_2) + N(R_2)\bar{R}_2^*(R_2) \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{40}$$

where the characteristic determinant, which must vanish for a non-trivial solution, contains parameters pertaining to the finitely deformed state, material properties and initial geometry of the shell and the unknown frequencies of small, free vibration about the finitely deformed state. When the frequency of vibrations ceases to be real valued, the corresponding finitely deformed state becomes unstable.

4. DISCUSSION OF THE RESULTS

The approach used in this work enables one to locate the points corresponding to the critical stretch ratio for which the shell with a pre-existing void of arbitrary radius becomes unstable. Illustrative examples are worked out numerically to investigate the effects of material constants f and ν, the outer stretch ratio λ and the wall thickness χ on the frequency $\bar{\omega}$ and the critical outer stretch ratio, λ_{cr}, which is defined as the stretch ratio of the outer surface when the frequency approaches zero. In particular, the behaviors of the foam rubber, the nearly incompressible ($\nu = 0{\cdot}499$), and the slightly compressible hyperelastic materials represented by the Levinson–Burgess model are investigated. For the foam rubber, ($f = 0$ and $\nu = 0{\cdot}25$), Levinson–Burgess and Blatz–Ko materials are identical. When $f = 1$ and $\nu = 0{\cdot}5$, Levinson–Burgess material reduces to neo-Hookean material which is incompressible. In this limiting case, the results obtained reduce to those obtained by Wang and Ertepinar [17].

Figure 1 displays the variation of radial stress on the outer surface, $\bar{\tau}^{11}$ as a function of outer stretch ratio λ for a spherical shell with a thickness ratio $\chi = 0{\cdot}90$ for three different hyperelastic materials, namely the foam rubber ($f = 0$, $\nu = 0{\cdot}25$), the slightly compressible material ($f = 1$, $\nu = 0{\cdot}46$), and the nearly incompressible material ($f = 1$, $\nu = 0{\cdot}499$). In this figure, the points corresponding to the critical values are also indicated. It is observed that the critical outer radial stress decreases as the material compressibility increases. On the other hand, the outer critical stretch ratios are almost equal for foam rubber and slightly

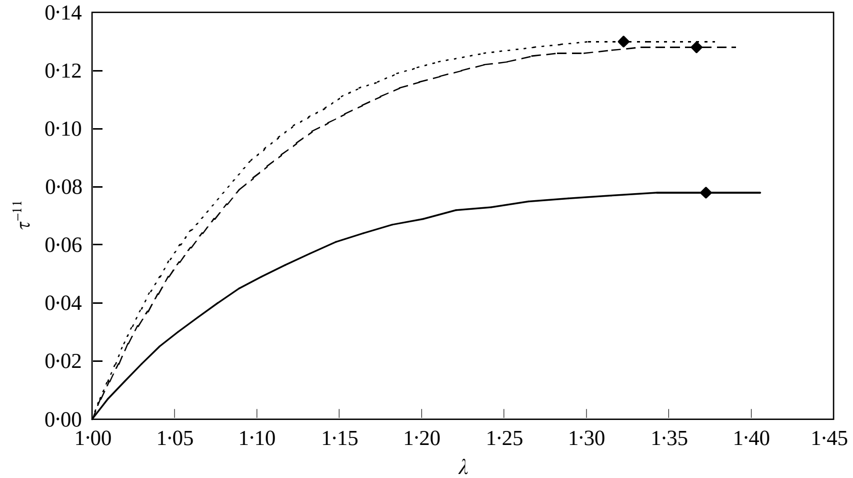

Figure 1. The outer radial stress versus outer stretch ratio for $\chi = 0{\cdot}90$: ◆, points corresponding to critical value; ——, foam rubber; – – –, slightly compressible rubber; ······, nearly incompressible material.

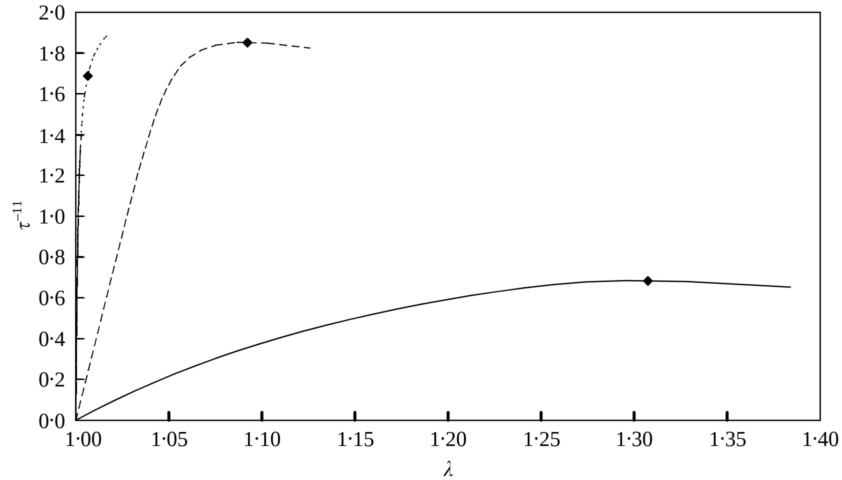

Figure 2. The outer radial stress versus outer stretch ratio for $\chi = 0{\cdot}10$: ◆, points corresponding to critical value; ——, foam rubber; – – –, slightly compressible rubber; ······, nearly incompressible material.

compressible material. It is also seen that for all material models, the pre-stressed spherical shell shows a softening behavior as λ increases. The softening behavior becomes more pronounced as λ approaches λ_{cr}.

The results of a similar investigation are shown in Figure 2 for a spherical shell with a shell thickness $\chi = 0{\cdot}10$. For this thick shell, the outer critical stretch ratio is significantly larger for foam rubber as compared to the other two models while the outer critical radial stress is larger for slightly compressible material. This figure indicates that foam rubber shows ductile behavior. By comparing Figures 1 and 2, it is seen that for the nearly incompressible and slightly compressible materials, the ductility decreases as the shell thickness increases.

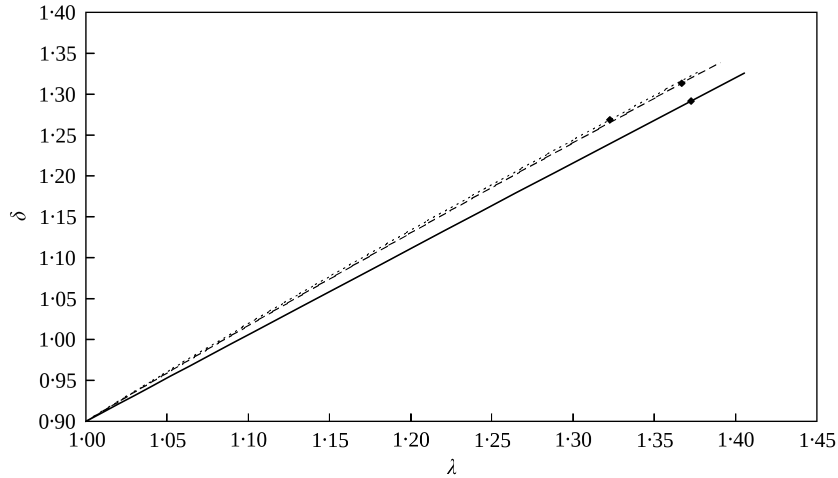

Figure 3. Void growth versus outer stretch ratio for $\chi = 0{\cdot}90$: ◆, points corresponding to critical value; ——, foam rubber; – – –, slightly compressible rubber; ······, nearly incompressible material.

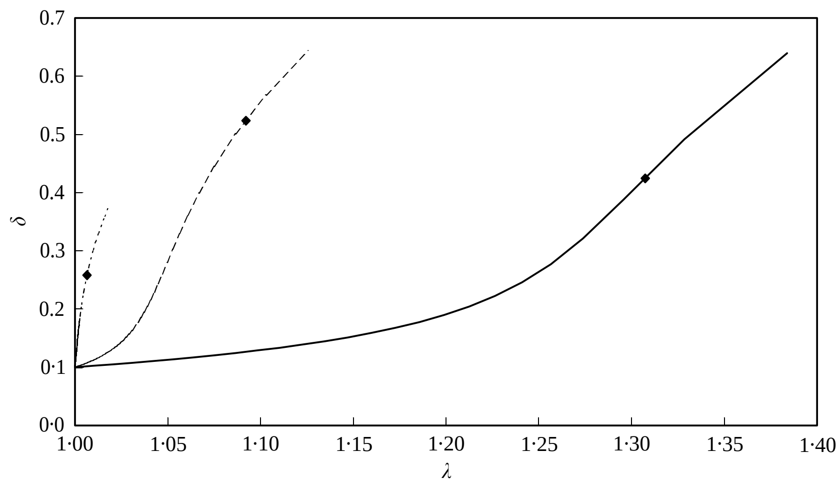

Figure 4. Void growth versus outer stretch ratio for $\chi = 0{\cdot}10$: ◆, points corresponding to critical value; ——, foam rubber; – – –, slightly compressible rubber; ······, nearly incompressible material.

In Figure 3, the void growth $\delta(= \bar{R}_1/\bar{r}_1)$ is plotted as a function of outer stretch ratio λ for shells with a shell thickness $\chi = 0{\cdot}90$. For all material models, the δ–λ relationship is almost linear. Among the three material models, the nearly incompressible material has the smallest critical void growth and critical outer stretch ratio.

Figure 4 displays the variation of the void growth as a function of outer stretch ratio λ for a spherical shell with a shell thickness $\chi = 0{\cdot}10$. Behaviour is similar to that shown in Figure 3, in that the nearly incompressible material has the smallest critical void growth and critical outer stretch ratio. It is also seen that the outer critical stretch ratio is larger for foam rubber while the critical void growth is larger for slightly compressible material.

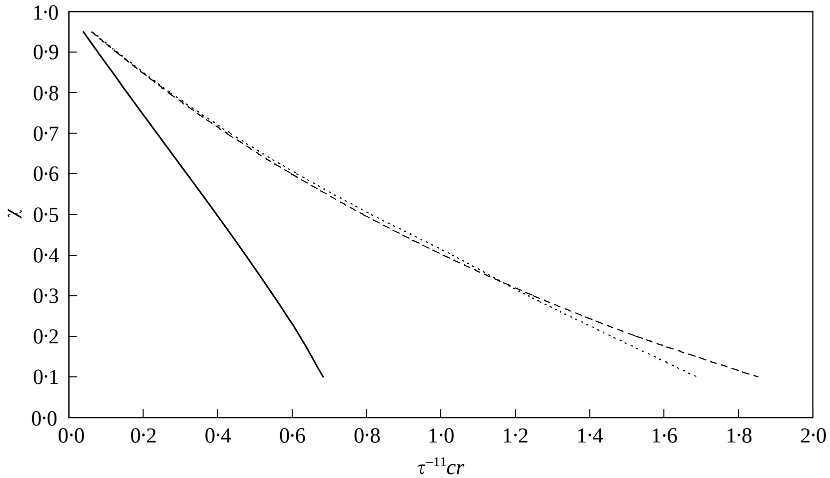

Figure 5. Critical radial stress on the outer surface versus thickness ratio curves: ——, foam rubber; – – –, slightly compressible rubber; ······, nearly incompressible material.

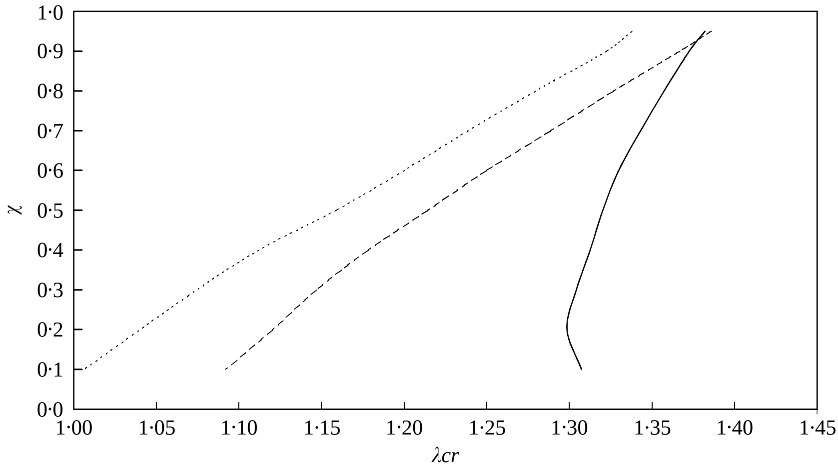

Figure 6. Critical stretch ratio on the outer surface versus thickness ratio curves: ——, foam rubber; – – –, slightly compressible rubber; ······, nearly incompressible material.

The change of critical radial stress at the outer surface of the shell as a function of the thickness ratio is investigated in Figure 5. The foam rubber has the smallest critical outer radial stress values for all thicknesses. For materials with $f = 1$ (the slightly compressible and nearly incompressible materials), it is observed that, for the thick shells, $\bar{\tau}_{cr}^{11}$ increases as the ν decreases, the effect of ν becomes insignificant for moderately thick and thin shells.

Figure 6 shows the change of critical stretch ratio of the shell as a function of the thickness ratio. It is seen that, for all shell thickness, $\bar{\lambda}_{cr}$ increases as the compressibility of the material increases.

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