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DETERMINATION OF THE FREQUENCY RESPONSE FUNCTION OF A CANTILEVERED BEAM SIMPLY SUPPORTED IN-SPAN

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1. INTRODUCTION

The first author recently established a formula for the receptance matrix of viscously damped discrete systems subject to several constraint equations [1]. The reliability of the formula derived was tested on an academic example of a spring–mass system with three degrees of freedom, the co-ordinates of which were assumed to be subject to a constraint equation. The aim of this note is to put forward the applicability of the method better on a more complex but practical system.

2. THEORY

The problem can be stated referring to the cantilever beam in Figure 1. The Bernoulli–Euler beam is assumed to be simply supported at a distance $s^* = \eta L$ from the fixed end. At the distance $x = l$, a harmonically varying vertical force $F(t)$ is acting on the beam. It is desired to determine the amplitude distribution of the beam due to this force. This problem can also be posed to find the frequency response function of the beam. In order to simplify the calculations in the following, damping will be omitted.

2.1. APPLICATION OF THE FORMULA IN REFERENCE [1]

Begin with the system in Figure 1, where it is first assumed that the support does not exist. The equation of the motion of the beam is

$$EIw^{\mathrm{IV}}(x, t) + m\ddot{w}(x, t) = F(t)\,\delta(x - l), \tag{1}$$

the exciting force being

$$F(t) = F_0 \mathrm{e}^{\mathrm{i}\Omega t}, \tag{2}$$

where the primes and overdots denote partial derivatives with respect to x and time t respectively. EI is the bending rigidity and m is the mass per unit length of the beam. $\delta(x)$ denotes the Dirac function.

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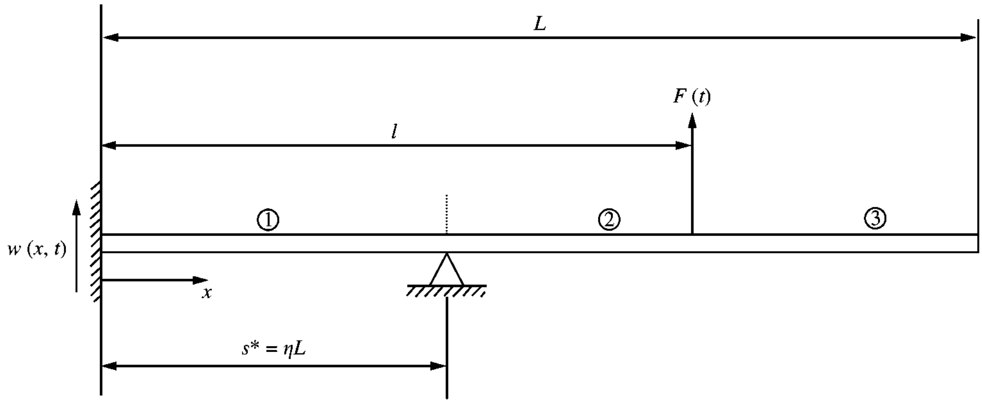

Figure 1. Cantilevered beam simply supported in-span, subject to a harmonically varying force.

The corresponding boundary conditions are

$$w(0,t) = w'(0,t) = w''(L,t) = w'''(L,t) = 0 \tag{3}$$

An approximate series solution of the differential equation (1) can be taken in the form

$$w(x,t) \approx \sum_{r=1}^{n} w_r(x)\,\eta_r(t), \tag{4}$$

where the $w_r(x)$ are the orthogonal eigenfunctions of the bare clamped-free beam, normalized with respect to the mass density. After substitution of expression (4) into the differential equation (1), both sides of the equation are multiplied by the sth eigenfunction $w_s(x)$ and integrated over the beam length. By using the orthogonality property of the eigenfunctions, the system of modal equations, i.e., the system of differential equations for $\eta_i(t)$, is obtained:

$$\ddot{\eta}_i(t) + \omega_i^2\eta_i(t) = N_i(t) \quad (i = 1, \ldots, n), \tag{5}$$

where

$$\omega_i^2 = (\beta_i L)^4 \frac{EI}{mL^4}, \quad \bar{\beta}_1 = \beta_1 L = 1{\cdot}875104068712, \quad \bar{\beta}_2 = \beta_2 L = 4{\cdot}694091132974,$$

$$N_i(t) = F(t)\,w_i(l). \tag{6}$$

The system of differential equations (5) can be written in matrix notation as

$$\ddot{\boldsymbol{\eta}}(t) + \boldsymbol{\omega}^2\boldsymbol{\eta}(t) = \mathbf{N}(t), \tag{7}$$

where

$$\boldsymbol{\eta}(t) = [\eta_1(t) \;\cdots\; \eta_n(t)]^{\mathrm{T}}, \qquad \boldsymbol{\omega}^2 = \mathbf{diag}(\omega_i^2), \qquad \mathbf{N}(t) = \bar{\mathbf{N}}\,\mathrm{e}^{\mathrm{i}\Omega t},$$

$$\bar{\mathbf{N}} = F_0\,\mathbf{w}(l), \qquad \mathbf{w}(x) = [w_1(x) \cdots w_n(x)]^{\mathrm{T}}. \tag{8}$$

ω_i $(i = 1, \ldots, n)$ are the eigenfrequencies of the bare cantilever beam.
Substitution of

$$\boldsymbol{\eta}(t) = \bar{\boldsymbol{\eta}}\,\mathrm{e}^{\mathrm{i}\Omega t} \tag{9}$$

into the matrix differential equation (7) yields

$$\bar{\boldsymbol{\eta}} = \mathbf{H}(\Omega)\,\bar{\mathbf{N}}, \tag{10}$$

where the receptance matrix is in the form

$$\mathbf{H}(\Omega) = (-\Omega^2\,\mathbf{I} + \boldsymbol{\omega}^2)^{-1} = \operatorname{diag}\left(\frac{1}{\omega_i^2 - \Omega^2}\right). \tag{11}$$

This expression could be written directly from equation (5) as in reference [1].

Now return to the actual system with the support at $x = \eta L$. The introduction of the support leads to the constraint equation

$$\sum_{r=1}^{n} w_r(s^*)\,\eta_r(t) = 0, \tag{12}$$

which can be written compactly as

$$\mathbf{a}_1^{\mathrm{T}}\boldsymbol{\eta} = 0, \tag{13}$$

where

$$\mathbf{a}_1^{\mathrm{T}} = \mathbf{w}^{\mathrm{T}}(s^*) = [w_1(s^*) \cdots w_n(s^*)]^{\mathrm{T}}, \qquad s^* = \eta L. \tag{14}$$

The amplitude vector $\bar{\boldsymbol{\eta}}$ in the constrained case can be written from equation (10) analogously as

$$\bar{\boldsymbol{\eta}} = \mathbf{H}_{cons}(\Omega)\,\bar{\mathbf{N}}, \tag{15}$$

where from reference [1] the receptance matrix of the constrained system reads as

$$\mathbf{H}_{cons}(\Omega) = \mathbf{H}(\Omega)\left[\mathbf{I} - \frac{\mathbf{w}(s^*)\,\mathbf{w}^{\mathrm{T}}(s^*)\,\mathbf{H}(\Omega)}{\mathbf{w}^{\mathrm{T}}(s^*)\,\mathbf{H}(\Omega)\,\mathbf{w}(s^*)}\right] \tag{16}$$

with $\mathbf{I}$ being the $(n \times n)$ unit matrix.

Therefore, the displacements of the constrained (i.e., supported) beam can be written by using equation (9) as

$$w_{cons}(x, t) = \bar{w}_{cons}(x)\,\mathrm{e}^{\mathrm{i}\Omega t}, \tag{17}$$

where

$$\bar{w}_{cons}(x) = \sum_{r=1}^{n} w_r(x)\bar{\eta}_r. \tag{18}$$

It is easy to show that the above expression can be reformulated as

$$\bar{w}_{cons}(x) = (\mathbf{w}^{\mathrm{T}}(x)\,\mathbf{H}_{cons}(\Omega)\mathbf{w}(l))\,F_0, \tag{19}$$

which in turn, after some rearrangements, leads to

$$\bar{w}_{cons}(x) = \mathbf{a}^{\mathrm{T}}(x)\,\mathbf{diag}\left(\frac{1}{\bar{\beta}_i^4 - \Omega^{*2}}\right)\left[\mathbf{I} - \frac{\mathbf{a}(s^*)\,\mathbf{a}^{\mathrm{T}}(s^*)\,\mathbf{diag}\,(1/(\bar{\beta}_i^4 - \Omega^{*2}))}{\mathbf{a}^{\mathrm{T}}(s^*)\,\mathbf{diag}\,(1/(\bar{\beta}_i^4 - \Omega^{*2}))\,\mathbf{a}(s^*)}\right]\mathbf{a}(l)\,\frac{F_0}{(EI/L^3)}, \tag{20}$$

where

$$\Omega^* = \frac{\Omega}{\boldsymbol{\omega}_0}, \qquad \boldsymbol{\omega}_0^2 = \frac{EI}{mL^4},$$

$$\mathbf{w}^{\mathrm{T}}(x) = \frac{1}{\sqrt{mL}}\,\mathbf{a}^{\mathrm{T}}(x) = \frac{1}{\sqrt{mL}}\,[a_1(x) \cdots a_n(x)],$$

$$a_i(x) = \cosh\bar{\beta}_i\frac{x}{L} - \cos\bar{\beta}_i\frac{x}{L} - \bar{\eta}_i^*\left(\sinh\bar{\beta}_i\frac{x}{L} - \sin\bar{\beta}_i\frac{x}{L}\right),$$

$$\bar{\eta}_i^* = \frac{\cosh\bar{\beta}_i + \cos\bar{\beta}_i}{\sinh\bar{\beta}_i + \sin\bar{\beta}_i}. \tag{21}$$

Expression (20) is the amplitude distribution on the supported beam subject to the harmonic force, which was looked for.

In the case $F_0 = 1$, the right-hand side of equation (20) represents nothing else but the frequency response function of the beam in Figure 1.

2.2. SOLUTION THROUGH A BOUNDARY VALUE PROBLEM FORMULATION

In order to prove the validity of expression (20), the only way is to compare this with the results of a boundary value problem formulation.

The bending vibrations of the three beam portions shown in Figure 1 are governed by the partial differential equation

$$EIw_i^{\mathrm{IV}}(x,t) + m\ddot{w}_i(x,t) = 0 \quad (i = 1,2,3) \tag{22}$$

with the following boundary and transition conditions:

$$\begin{gathered} w_1(0,t) = w_1'(0,t) = 0, \qquad w_1(s^*,t) = w_2(s^*,t), \qquad w_1'(s^*,t) = w_2'(s^*,t), \\ w_1''(s^*,t) = w_2''(s^*,t), \qquad w_2(l,t) = w_3(l,t), \qquad w_2'(l,t) = w_3'(l,t), \\ w_2''(l,t) = w_3''(l,t), \qquad w_3''(\mathrm{L},t) = w_3'''(\mathrm{L},t) = 0, \\ EIw_2'''(l,t) - EI\,w_3'''(l,t) + F_0\,\mathrm{e}^{\mathrm{i}\Omega t} = 0. \end{gathered} \tag{23}$$

If harmonic solutions of the form

$$w_i(x,t) = W_i(x)\,\mathrm{e}^{\mathrm{i}\Omega t} \tag{24}$$

are substituted into equation (22), the following ordinary differential equations are obtained for the amplitude functions $W_i(x)$:

$$W_i^{\mathrm{IV}}(x) - \bar{\Lambda}^4 W_i(x) = 0 \quad (i = 1,\ 2,\ 3), \tag{25}$$

where

$$\bar{\Lambda}^4 = \frac{m\Omega^2}{EI}. \tag{26}$$

The corresponding boundary and matching conditions now read as

$$\begin{gathered} W_1(0) = W_1'(0) = 0, \qquad W_1(s^*) = W_2(s^*) = 0, \qquad W_1'(s^*) = W_2'(s^*), \\ W_1''(s^*) = W_2''(s^*), \qquad W_2(l) = W_3(l), \qquad W_2'(l) = W_3'(l), \\ W_2''(l) = W_3''(l), \qquad W_3''(L) = W_3'''(L) = 0, \\ W_2'''(l) - W_3'''(l) + \frac{F_0}{EI} = 0. \end{gathered} \tag{27}$$

The general solutions of the differential equations (25) are

$$\begin{gathered} W_1(x) = c_1 \sin\bar{\Lambda}x + c_2\cos\bar{\Lambda}x + c_3\sinh\bar{\Lambda}x + c_4\cosh\bar{\Lambda}x, \\ W_2(x) = c_5 \sin\bar{\Lambda}x + c_6\cos\bar{\Lambda}x + c_7\sinh\bar{\Lambda}x + c_8\cosh\bar{\Lambda}x, \\ W_3(x) = c_9 \sin\bar{\Lambda}x + c_{10}\cos\bar{\Lambda}x + c_{11}\sinh\bar{\Lambda}x + c_{12}\cosh\bar{\Lambda}x, \end{gathered} \tag{28}$$

where c_1–c_{12} are unknown integration constants to be determined. Substitution of expressions (28) into conditions (27) yields, after rearrangement, the following set of 10 inhomogeneous equations for the determination of the coefficients c_i:

$$\mathbf{Ac} = \mathbf{b}. \tag{29}$$

TABLE 1

Dimensionless vibration amplitudes at various sections of the beam due to the harmonic force $F_0\, e^{i\Omega t}$ at $l/L = 1$. $\Omega = 5\sqrt{EI/mL^4}$ is chosen

$\bar{X}$ \ η	0·1		0·2		0·3		0·4		0·5	
0·1	0	0	0·030497	0·030734	−0·004108	−0·004113	−0·002091	−0·002092	−0·001380	−0·001380
0·2	−0·016266	−0·016230	0	0	−0·008214	−0·008213	−0·005572	−0·005574	−0·004136	−0·004138
0·3	−0·051199	−0·051222	−0·236753	−0·238491	0	0	−0·006264	−0·006264	−0·006197	−0·006198
0·4	−0·100962	−0·101143	−0·674390	−0·679561	0·030027	0·029987	0	0	−0·005501	−0·005503
0·5	−0·161851	−0·162253	−1·270533	−1·280610	0·080140	0·080055	0·016316	0·016311	0	0
0·6	−0·230419	−0·231072	−1·984560	−2·000426	0·145886	0·145758	0·041542	0·041530	0·011747	0·011742
0·7	−0·303628	−0·304553	−2·779070	−2·801531	0·223021	0·222848	0·073522	0·073503	0·028814	0·028804
0·8	−0·379017	−0·380235	−3·621659	−3·651081	0·307669	0·307447	0·110206	0·110186	0·049712	0·049702
0·9	−0·454885	−0·456419	−4·486890	−4·523432	0·396514	0·396234	0·149730	0·149701	0·073027	0·073011
1·0	−0·530480	−0·532332	−5·358391	−5·402254	0·487013	0·486659	0·190506	0·190472	0·097468	0·097442

$\bar{X}$ \ η	0·6		0·7		0·8		0·9	
0·1	−0·000985	−0·000985	−0·000707	−0·000707	−0·000473	−0·000473	−0·000245	−0·000245
0·2	−0·003147	−0·003148	−0·002351	−0·002351	−0·001615	−0·001616	−0·000855	−0·000856
0·3	−0·005301	−0·005302	−0·004220	−0·004219	−0·003018	−0·003017	−0·001641	−0·001640
0·4	−0·006269	−0·006271	−0·003018	−0·005621	−0·004273	−0·004275	−0·002417	−0·002418
0·5	−0·004886	−0·004886	−0·005820	−0·005821	−0·004981	−0·004982	−0·003000	−0·003000
0·6	0	0	−0·004174	−0·004174	−0·004754	−0·004756	−0·003216	−0·003215
0·7	0·009133	0·009130	0	0	−0·003216	−0·003219	−0·002894	−0·002895
0·8	0·021680	0·021679	0·007069	0·007070	0	0	−0·001873	−0·001871
0·9	0·036442	0·036435	0·016237	0·016233	0·005021	0·005015	0	0
1·0	0·052269	0·052256	0·026437	0·026429	0·011054	0·011040	0·002683	0·002678

The expression of the (10×10) coefficient matrix $\mathbf{A}$ is given in Appendix A. The vectors $\mathbf{c}$ and $\mathbf{b}$ are defined as

$$\mathbf{c}^{\mathrm{T}} = [c_1 \quad c_2 \quad c_5 \quad c_6 \quad c_7 \quad c_8 \quad c_9 \quad c_{10} \quad c_{11} \quad c_{12}],$$

$$\mathbf{b}^{\mathrm{T}} = \left[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -\frac{F_0}{EI\bar{\Lambda}^3} \quad 0 \quad 0\right]. \tag{30}$$

Lengthy expressions of the elements of vector $\mathbf{c}$, which were obtained using MATHEMATICA via symbolic computation, are not given here due to space limitations. However, it is important to note that the vector $\mathbf{c}$ and therefore the amplititude functions $W_1(x)$, $W_2(x)$ and $W_3(x)$ in equations (28) contain the common factor $F_0/(EI/L^3)$ which has the dimension of length. Having obtained $W_i(x)$ $(i = 1, 2, 3)$, it is possible to determine the steady state amplitude at any point x of the beam, due to the harmonic force at a point $x = l$.

3. NUMERICAL APPLICATIONS

This section is devoted to the numerical evaluations of the formulae established in the preceding sections. As an example, $l/L = 1$ and $\Omega^* = 5$ are chosen. This means that a harmonically varying vertical force of the radian frequency $5\sqrt{EI/mL^4}$ is acting at the tip of the beam, shown in Figure 1.

The displacements at various sections of the beam, non-dimensionalized by dividing by $F_0/(EI/L^3)$ are given in Table 1. η represents the non-dimensional position of the support, whereas $\bar{x} = x/L$ denotes the non-dimensional position of the point, the displacement of which we are interested in. The values in the first columns are values obtained from formula (20), where $n = 15$ is taken in the series expansion (4) and $\bar{\beta}_1$–$\bar{\beta}_{15}$ in equation (21) taken from reference [2] are correct up to 12 decimal places. The values in the second columns are exact values obtained by the direct solution of the boundary value problem outlined in section 2.2.

The agreement between the values in both columns justifies expression (20), obtained on the basis of a formula established for the receptance matrix of viscously damped discrete systems subject to several constraint equations. It is worth noting that the agreement of the numbers in both columns becomes excellent if many more decimal places are considered in $\bar{\beta}_i$ values.

4. CONCLUSIONS

This note is concerned with the determination of the frequency response function of a cantilevered beam, which is simply supported in-span. The frequency response function is obtained through a formula, which was established for the receptance matrix of discrete systems subjected to linear constraint equations. The comparison of the numerical results obtained with those via a boundary value problem formulation justifies the approach used here.

REFERENCES

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APPENDIX A

The matrix $\mathbf{A}$ in equation (29) is

$$
\mathbf{A} = \begin{bmatrix}
\sin\bar{\Lambda}\eta L - \sinh\bar{\Lambda}\eta L & \cos\bar{\Lambda}\eta L - \cosh\bar{\Lambda}\eta L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sin\bar{\Lambda}\eta L & \cos\bar{\Lambda}\eta L & \sinh\bar{\Lambda}\eta L & \cosh\bar{\Lambda}\eta L & 0 & 0 & 0 & 0 \\
\cos\bar{\Lambda}\eta L - \cosh\bar{\Lambda}\eta L & -(\sin\bar{\Lambda}\eta L + \sinh\bar{\Lambda}\eta L) & -\cos\bar{\Lambda}\eta L & \sin\bar{\Lambda}\eta L & -\cosh\bar{\Lambda}\eta L & -\sinh\bar{\Lambda}\eta L & 0 & 0 & 0 & 0 \\
-(\sin\bar{\Lambda}\eta L + \sinh\bar{\Lambda}\eta L) & -(\cos\bar{\Lambda}\eta L + \cosh\bar{\Lambda}\eta L) & \sin\bar{\Lambda}\eta L & \cos\bar{\Lambda}\eta L & -\sinh\bar{\Lambda}\eta L & -\cosh\bar{\Lambda}\eta L & 0 & 0 & 0 & 0 \\
0 & 0 & \sin\bar{\Lambda}l & \cos\bar{\Lambda}l & \sinh\bar{\Lambda}l & \cosh\bar{\Lambda}l & -\sin\bar{\Lambda}l & -\cos\bar{\Lambda}l & -\sinh\bar{\Lambda}l & -\cosh\bar{\Lambda}l \\
0 & 0 & \cos\bar{\Lambda}l & -\sin\bar{\Lambda}l & \cosh\bar{\Lambda}l & \sinh\bar{\Lambda}l & -\cos\bar{\Lambda}l & \sin\bar{\Lambda}l & -\cosh\bar{\Lambda}l & -\sinh\bar{\Lambda}l \\
0 & 0 & -\sin\bar{\Lambda}l & -\cos\bar{\Lambda}l & \sinh\bar{\Lambda}l & \cosh\bar{\Lambda}l & \sin\bar{\Lambda}l & \cos\bar{\Lambda}l & -\sinh\bar{\Lambda}l & -\cosh\bar{\Lambda}l \\
0 & 0 & -\cos\bar{\Lambda}l & \sin\bar{\Lambda}l & \cosh\bar{\Lambda}l & \sinh\bar{\Lambda}l & \cos\bar{\Lambda}l & -\sin\bar{\Lambda}l & -\cosh\bar{\Lambda}l & -\sinh\bar{\Lambda}l \\
0 & 0 & 0 & 0 & 0 & 0 & -\sin\bar{\Lambda}L & -\cos\bar{\Lambda}L & \sinh\bar{\Lambda}L & \cosh\bar{\Lambda}L \\
0 & 0 & 0 & 0 & 0 & 0 & -\cos\bar{\Lambda}L & \sin\bar{\Lambda}L & \cosh\bar{\Lambda}L & \sinh\bar{\Lambda}L
\end{bmatrix}.
$$