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AN INTEGRAL FORMULATION WITH RANDOM PARAMETERS ADAPTED TO THE STUDY OF THE VIBRATIONAL BEHAVIOUR OF STRUCTURES IN THE MIDDLE- AND HIGH-FREQUENCY FIELD

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The fundamentals of a new boundary element formulation applied to the study of structural dynamics on a wide frequency range is proposed in this paper. Random geometrical parameters are introduced to the integral representations in order to model the increasing sensitivity of the harmonic vibrational responses to any parameter perturbation when the frequency increases. It is shown that the significant unknowns of the formulation are the expectations of the square boundary kinematic unknowns. Indeed, these variables are able to describe accurately the responses of the structures in the low-frequency field and deliver a smooth response in the middle- and high-frequency field, corresponding to the global trend of the structural dynamic response in this frequency domain. The formulation is established for isolated and assembled one-, two- and three-dimensional structures. Numerical applications illustrate the effectiveness of the method.

1. INTRODUCTION

The harmonic dynamic behaviour of complex mechanical systems such as assembled beams, rods, membranes and plates, generally cannot be approached by means of analytical solutions. The finite element method (FEM) [1] and classical direct and indirect boundary element methods (BEM) [2] yield effective numerical solutions for such problems as long as the frequency range of the studied phenomenon does not reach the "high-frequency domain". High-frequency simulations involve a huge number of degrees of freedom which require a great amount of computing time. Therefore, 30 years ago new solutions were developed to investigate the high-frequency dynamic behaviour of mechanical structures. The most famous theory is the statistical energy analysis (SEA) [3]. The aim of the SEA is to evaluate the spread of the energy through complex systems divided into coupled subsystems. Since the early beginnings of the SEA, numerous extensions of the method have arisen. Among those, one can mention the wave intensity analysis [4] and the simplified energy method ([5–11]).

The energy methods previously mentioned brought about a large progress in structural high-frequency predictions in the last 20 years. Nevertheless, these formulations involve some drawbacks. According to Fahy [12], one of the most penalizing facts is the lack of confidence in the predictions given by these methods. Moreover, these different formulations are restricted to the high-frequency range and there is no way of utilizing them in the low-frequency domain.

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In this paper, the basis of a novel formulation is proposed, reliable on the whole frequency domain. The fundamental idea of the work relies on the fact that the dynamic behaviour of any mechanical system is intrinsically random, and the influence of this randomness grows with the frequency [13, 14].

The analytical formulation is based on the dynamic boundary integral equations. These integral equations are multiplied by well-chosen variables, in order to obtain a formulation governing the cross product of the kinematic variables. Gaussian randomness is then introduced to the geometrical parameters of the structures and the stochastic expectations of the terms in the new integral equations are taken into account. Some assumptions are introduced to limit the number of high order moment unknowns. Finally, the unknowns of the new formulation are the second order stochastic moments of the boundary kinematic variables.

The theory is valid for one-, two- and three-dimensional structures subjected to harmonic external sources, and is a generalization of the formulation previously developed by Viktorovitch *et al.* [15], for the case of one-dimensional structures.

Numerical applications for isolated and assembled one- and two-dimensional structures illustrate the effectiveness of the theory.

2. THE CLASSICAL BOUNDARY INTEGRAL EQUATIONS FOR SECOND AND FOURTH ORDER DIFFERENTIAL GOVERNING EQUATIONS

The boundary integral representations are formulated in this section. Different ways are proposed in the literature to obtain these integral equations. One can use the dynamic reciprocal theorem [16], considering two distinct elastic equilibrium states: the first system is the actual state of displacements, slopes, body and boundary forces and moments, and the second one corresponds to a unit force system in an infinite solid [17]. The following boundary integral formulations are derived for the particular cases of the isolated membrane and the plate. From these examples one can easily extend the formulations to one-dimensional structures as well as three-dimensional acoustics.

2.1. THE CASE OF THE MEMBRANE

The governing equation for a homogeneous, isotropic and linear elastic membrane of domain Ω and smooth boundary $\partial\Omega$, subjected to a transverse harmonic loading per unit area f, with the assumption of small deflections under in-plane tensile force T, is written [18] as

$$T\nabla^2 u + Tk^2 u = -f(\mathbf{x}), \tag{1}$$

where $u(\mathbf{x})$ (bold fonts denote vectors) is the amplitude of the transverse motion, k is the complex wave number

$$k^2 = \omega^2 \rho / T(1 + \mathrm{i}\eta) \approx k_0^2(1 - \mathrm{i}\eta), \tag{2}$$

k_0 is the real part of the complex wave number, ω denotes the circular frequency of vibration, η is the damping loss factor and ρ is the mass density per unit area. The fundamental solution for the unit force system on the infinite membrane is the Green kernel G, whose expression is

$$\mathrm{G}(\mathbf{x}, \boldsymbol{\xi}) = (1/4\mathrm{i})\,\mathrm{H}_0^{(2)}(kr), \quad r = |\mathbf{x} - \boldsymbol{\xi}|. \tag{3}$$

$H_0^{(2)}$ denotes the Hankel function of zero order and second type. The variable $\boldsymbol{\xi}$ is the loading location while $\mathbf{x}$ denotes the spatial position. The boundary integral equation may finally be written for $\boldsymbol{\xi}$ belonging to the smooth boundary of the membrane, $\partial\Omega$, as [19]

$$\tfrac{1}{2}u(\boldsymbol{\xi}) = \int_{\Omega_f} f(\mathbf{x})G(\mathbf{x}, \boldsymbol{\xi})\,\mathrm{d}\Omega + \int_{\partial\Omega} [u(\mathbf{x})T_n(G(\mathbf{x}, \boldsymbol{\xi})) - T_n(u(\mathbf{x}))G(\mathbf{x}, \boldsymbol{\xi})]\,\mathrm{d}\partial\Omega. \tag{4}$$

Ω_f denotes the domain of the membrane on which the loading is applied. T_n represents the boundary shear force. The coefficient $\frac{1}{2}$ appearing at the left-hand side of equation (4) comes from the calculus of the singular boundary integral. Finally, one must give boundary conditions in order to be able to solve the boundary integral equations. The boundary $\partial\Omega$ is defined by means of two subdomains $\partial\Omega_u$ and $\partial\Omega_T$ such that

$$\partial\Omega_u \cup \partial\Omega_T = \partial\Omega, \qquad \partial\Omega_u \cap \partial\Omega_T = \emptyset \tag{5}$$

The boundary conditions are then defined as

$$\begin{aligned} u(\mathbf{x}) &= \hat{u}(\mathbf{x}) && \forall \mathbf{x} \in \partial\Omega_u, \\ T_n(u(\mathbf{x})) &= \hat{T}_n(u(\mathbf{x})) && \forall \mathbf{x} \in \partial\Omega_T. \end{aligned} \tag{6}$$

2.2. THE CASE OF THE PLATE

In an analogous way, as for the membrane, one can write the governing equation of a homogeneous, isotropic, thin, linear, elastic plate in flexural motion. The assumption of small deflections is adopted, the surface of the plate is called Ω and its perimeter $\partial\Omega$. If the plate is subjected to a harmonic loading f and the deflection amplitude is labelled u, the governing equation can be written [18] as

$$D\nabla^4 u - Dk^4 u = f(\mathbf{x}), \tag{7}$$

where $k = \sqrt{\omega\sqrt{\rho h/D}}$ is the wave number. ρ the mass density, h and $D = h^3/12(1 - \nu^2)$, respectively, denote the thickness and the rigidity of the plate and ν is the Poisson coefficient. According to Butterfield [17] one can write the following boundary integral equation (written here for the point at the boundary) using the Green function of the infinite plate $G(kr) = (\mathrm{i}/8k^2)(H_0^2(\mathrm{i}kr) - H_0^2(kr))$ for $\forall \boldsymbol{\xi} \in \partial\Omega$,

$$\begin{aligned} \tfrac{1}{2}u(\boldsymbol{\xi}) = \int_{\Omega} G(\mathbf{x}, \boldsymbol{\xi}) f(\mathbf{x})\,\mathrm{d}\Omega + \int_{\partial\Omega} [u(\mathbf{x})V_n(G(\mathbf{x}, \boldsymbol{\xi})) \\ - u_{,n}(\mathbf{x})M_n(G(\mathbf{x}, \boldsymbol{\xi})) + M_n(u(\mathbf{x}))G_{,n}(\mathbf{x}, \boldsymbol{\xi}) - V_n(u(\mathbf{x}))G(\mathbf{x}, \boldsymbol{\xi})]\,\mathrm{d}\partial\Omega, \end{aligned} \tag{8}$$

where V_n, M_n, $(\,)_{,n} = \partial(\,)/\partial\mathbf{n}$ and $\mathbf{n}$ denote, respectively, the shear force, the bending moment, the normal slope and outward normal vector. In order to solve for the two boundary unknowns of the plate, a second integral equation (directly written at a boundary point) is required:

$$\begin{aligned} \frac{1}{2}\frac{\partial u(\boldsymbol{\xi})}{\partial \mathbf{n}_\xi} = \int_{\Omega} \frac{\partial G(\mathbf{x}, \boldsymbol{\xi})}{\partial \mathbf{n}_\xi} f(\mathbf{x})\,\mathrm{d}\Omega + \int_{\partial\Omega} \Bigg[u(\mathbf{x})V_n\left(\frac{\partial G(\mathbf{x}, \boldsymbol{\xi})}{\partial \mathbf{n}_\xi}\right) - u_{,n}(\mathbf{x})M_n\left(\frac{\partial G(\mathbf{x}, \boldsymbol{\xi})}{\partial \mathbf{n}_\xi}\right) \\ + M_n(u(\mathbf{x}))\left(\frac{\partial G(\mathbf{x}, \boldsymbol{\xi})}{\partial \mathbf{n}_\xi}\right)_{,n} - V_n(u(\mathbf{x}))\frac{\partial G(\mathbf{x}, \boldsymbol{\xi})}{\partial \mathbf{n}_\xi} \Bigg]\,\mathrm{d}\partial\Omega, \end{aligned} \tag{9}$$

where $\mathbf{n}_\xi$ represents the outward normal vector at the boundary point of co-ordinate ξ. Boundary conditions are required to solve the integral equations. Two distint partitions of the boundary $\partial\Omega$ are defined. They are built considering for each one two subdomains such that

$$\partial\Omega_u \cup \partial\Omega_V = \partial\Omega, \qquad \partial\Omega_u \cap \partial\Omega_V = \emptyset \tag{10}$$

and

$$S_u \cup S_V = \partial\Omega, \qquad S_u \cap S_V = \emptyset. \tag{11}$$

Boundary conditions are expressed in terms of displacement or shear force on the first partition. Boundary conditions in terms of slope or bending moment are defined by means of the second partition. Finally, one obtains

$$\begin{aligned} &u(\mathbf{x}) = \hat{u}(\mathbf{x}) && \text{for } \mathbf{x} \in \partial\Omega_u, \\ &V_n(u(\mathbf{x})) = \hat{V}_n(u(\mathbf{x})), && \forall \mathbf{x} \in \partial\Omega_V, \\ &u_{,n}(\mathbf{x}) = \hat{u}_{,n}(\mathbf{x}), && \forall \mathbf{x} \in S_u, \\ &M_n(u(\mathbf{x})) = \hat{M}_n(u(\mathbf{x}))) && \forall \mathbf{x} \in S_V. \end{aligned} \tag{12}$$

3. THE RANDOM FORMULATION

3.1. A GENERAL OVERVIEW OF THE METHOD

Trying to solve the previous boundary equations over a wide frequency range, and especially in the high-frequency field is unrealistic. Indeed, the computational cost increases with the frequency and rapidly becomes hard to handle. Moreover, it has been shown by Manohar and Keane [13, 14], that the deterministic response of any mechanical system is more and more sensitive to small perturbations of the geometrical and material parameters of the structures, when the frequency increases.

Within this context, the authors justified, in a previous paper by Viktorovitch *et al.* [15] dealing with one-dimensional structures. The introduction of randomness into the geometrical description of the structure makes it possible to obtain reliable information on the behaviour of a mechanical system, in a wide frequency range. It is shown in the present paper, that these previous theoretical developments can be extended to two-dimensional systems.

The random parameters are introduced to the boundary equations, and each of these equations is multiplied by a well-chosen boundary unknown. The expectations with respect to the different random variables, of the terms in the equations are considered. In order to obtain a consistent number of equations, the assumptions developed by the authors in reference [15] concerning the statistical independence of the different force and displacement variables are utilized in the present work.

3.2. THE RANDOM FORMULATION FOR ISOLATED STRUCTURES

The random formulation is developed for the boundary integral equations of the isolated membrane. The initial stage is the boundary formulation given by equation (4) and its boundary conditions. Randomness is then introduced to the definitions of the location of the loading and of the boundary of the structure. These two new random parameters are,

respectively, denoted by $\tilde{\Omega}_f$ and $\partial\tilde{\Omega}$. Moreover, the partition of the boundary defined in section 2.1 is naturally replaced by $\partial\tilde{\Omega} = \partial\tilde{\Omega}_u \cup \partial\tilde{\Omega}_T$. The symbol $(\tilde{\ })$ is also utilized to denote the random variable $\tilde{\xi}$, which is the random co-ordinate of a point belonging to $\partial\tilde{\Omega}$. Thus, equation (4) is rewritten considering the random description:

$$\frac{1}{2}u(\tilde{\xi}) = \int_{\tilde{\Omega}_f} f(\mathbf{x})\mathrm{G}(\mathbf{x},\tilde{\xi})\,\mathrm{d}\Omega + \int_{\partial\tilde{\Omega}_T} [u(\mathbf{x})\,T_n(\mathrm{G}(\mathbf{x},\tilde{\xi})) - \hat{T}_n(u(\mathbf{x}))\mathrm{G}(\mathbf{x},\tilde{\xi})]\,\mathrm{d}\partial\Omega_T$$

$$+ \int_{\partial\tilde{\Omega}_u} [\hat{u}(\mathbf{x})\,T_n(\mathrm{G}(\mathbf{x},\tilde{\xi})) - T_n(u(\mathbf{x}))\mathrm{G}(\mathbf{x},\tilde{\xi})]\,\mathrm{d}\partial\Omega. \tag{13}$$

The aim of this work is to derive an equation on the expectations of the cross product of the force and displacement unknowns. Therefore, for any point $\tilde{\xi} \in \partial\tilde{\Omega}$, the right- and left-hand sides of equation (13) are multiplied by the conjugate of the random boundary unknown at the same spatial position $\tilde{\xi}$. If $\tilde{\xi} \in \partial\tilde{\Omega}_T$, the boundary unknown is $u(\tilde{\xi})$, otherwise the boundary unknown is $T_n(u(\tilde{\xi}))$. The expectations of all the terms in the relationships are then expressed with respect to all the random variables introduced in the formulation. The symbol $\langle\ \rangle$ denotes the expectation operator.

Finally, the boundary $\partial\tilde{\Omega}_u \cup \partial\tilde{\Omega}_T$ is discretized into $N_u + N_T$ independent random boundary elements: $\partial\tilde{\Omega}_u = \bigcup_{i=1}^{N_u} \partial\tilde{\Omega}_{n_u}$ and $\partial\tilde{\Omega}_T = \bigcup_{i=1}^{N_T} \partial\tilde{\Omega}_{n_T}$, and the boundary unknowns are assumed to have a constant value on each of these boundary elements.

One finally obtains for $\tilde{\xi} \in \partial\tilde{\Omega}_T$

$$\begin{aligned}\tfrac{1}{2}\langle |u(\tilde{\xi})|^2 \rangle = {} & \left\langle u^*(\tilde{\xi}) \int_{\tilde{\Omega}_f} f(\mathbf{x})\mathrm{G}(\mathbf{x},\tilde{\xi})\,\mathrm{d}\Omega \right\rangle \\ & + \left\langle u^*(\tilde{\xi}) \int_{\partial\tilde{\Omega}_T} u(\mathbf{x})\,T_n(\mathrm{G}(\mathbf{x},\tilde{\xi}))\,\mathrm{d}\partial\Omega \right\rangle - \left\langle u^*(\tilde{\xi}) \int_{\partial\tilde{\Omega}_T} \hat{T}_n(u(\mathbf{x}))\mathrm{G}(\mathbf{x},\tilde{\xi})\,\mathrm{d}\partial\Omega \right\rangle \\ & + \left\langle u^*(\tilde{\xi}) \int_{\partial\tilde{\Omega}_u} \hat{u}(\mathbf{x})\,T_n(\mathrm{G}(\mathbf{x},\tilde{\xi}))\,\mathrm{d}\partial\Omega \right\rangle - \left\langle u^*(\tilde{\xi}) \int_{\partial\tilde{\Omega}_u} T_n(u(\mathbf{x}))\mathrm{G}(\mathbf{x},\tilde{\xi})\,\mathrm{d}\partial\Omega \right\rangle. \end{aligned} \tag{14}$$

For $\tilde{\xi} \in \partial\tilde{\Omega}_u$.

$$\begin{aligned}\tfrac{1}{2}\langle T_n^*(u(\tilde{\xi}))\hat{u}(\tilde{\xi}) \rangle = {} & \left\langle T_n^*(u(\tilde{\xi})) \int_{\tilde{\Omega}_f} f(\mathbf{x})\mathrm{G}(\mathbf{x},\tilde{\xi})\,\mathrm{d}\Omega \right\rangle \\ & + \left\langle T_n^*(u(\tilde{\xi})) \int_{\partial\tilde{\Omega}_T} u(\mathbf{x})\,T_n(\mathrm{G}(\mathbf{x},\tilde{\xi}))\,\mathrm{d}\partial\Omega \right\rangle \\ & - \sum_{n_T=1}^{N_T} \left\langle T_n^*(u(\tilde{\xi})) \int_{\partial\tilde{\Omega}n_T} \hat{T}_n(u(\mathbf{x}))\mathrm{G}(\mathbf{x},\tilde{\xi})\,\mathrm{d}\partial\Omega \right\rangle \\ & + \left\langle T_n^*(u(\tilde{\xi})) \int_{\partial\tilde{\Omega}_u} \hat{u}(\mathbf{x})\,T_n(\mathrm{G}(\mathbf{x},\tilde{\xi}))\,\mathrm{d}\partial\Omega \right\rangle \\ & - \left\langle T_n^*(u(\tilde{\xi})) \int_{\partial\tilde{\Omega}_u} T_n(u(\mathbf{x}))\mathrm{G}(\mathbf{x},\tilde{\xi})\,\mathrm{d}\partial\Omega \right\rangle. \end{aligned} \tag{15}$$

Finally, the collocation method is employed which enables the transformation of equations (14, 15) into a discrete set of equations. The discrete variables appearing in the formulation are denoted as follows: $u(\tilde{\mathbf{x}}_{n_T})$ (respectively, $T_n(u(\tilde{\mathbf{x}}_{n_u}))$ is the boundary unknown on the boundary element $\partial\tilde{\Omega}_{n_T}$ (respectively $\partial\tilde{\Omega}n_u$). One obtains $N_u + N_T$ equations for $\tilde{\mathbf{x}}_{k_T} \in \partial\tilde{\Omega}_{k_T}$, $k_T \in [1, N_T]$

$$\begin{aligned}
\tfrac{1}{2}\langle |u(\tilde{\mathbf{x}}_{k_T})|^2 \rangle = &\left\langle u^*(\tilde{\mathbf{x}}_{k_T}) \int_{\tilde{\Omega}_f} f(\mathbf{x}) \mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{k_T})\, \mathrm{d}\Omega \right\rangle \\
&+ \left\langle u^*(\tilde{\mathbf{x}}_{k_T}) \sum_{n_T=1}^{N_T} u(\tilde{\mathbf{x}}_{n_T}) \int_{\partial\tilde{\Omega}_{n_T}} T_n(\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{k_T}))\, \mathrm{d}\partial\Omega \right\rangle \\
&- \left\langle u^*(\tilde{\mathbf{x}}_{k_T}) \sum_{n_T=1}^{N_T} \int_{\partial\tilde{\Omega}_{n_T}} \hat{T}_n(u(\mathbf{x})) \mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{k_T})\, \mathrm{d}\partial\Omega \right\rangle \\
&+ \left\langle u^*(\tilde{\mathbf{x}}_{k_T}) \sum_{n_u=1}^{N_u} \int_{\partial\tilde{\Omega}_{n_u}} \hat{u}(\mathbf{x})\, T_n(\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{k_T}))\, \mathrm{d}\partial\Omega \right\rangle \\
&- \left\langle u^*(\tilde{\mathbf{x}}_{k_T}) \sum_{n_u=1}^{N_u} T_n(u(\tilde{\mathbf{x}}_{n_u})) \int_{\partial\tilde{\Omega}_{n_u}} \mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{k_T})\, \mathrm{d}\partial\Omega \right\rangle
\end{aligned} \tag{16}$$

and for $\tilde{\mathbf{x}}_{k_u} \in \partial\tilde{\Omega}_{k_u}$, $k_u \in [1, N_u]$

$$\begin{aligned}
\tfrac{1}{2}\langle T_n^*(u(\tilde{\mathbf{x}}_{k_u})) \rangle \hat{u}(\tilde{\mathbf{x}}_{k_u}) = &\left\langle T_n^*(u(\tilde{\mathbf{x}}_{k_u}) \int_{\tilde{\Omega}_f} f(\mathbf{x}) \mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{k_u})\, \mathrm{d}\Omega \right\rangle \\
&+ \left\langle T_n^*(u(\tilde{\mathbf{x}}_{k_u})) \sum_{n_T=1}^{N_T} u(\tilde{\mathbf{x}}_{n_T}) \int_{\partial\tilde{\Omega}_{n_T}} T_n(\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{k_u}))\, \mathrm{d}\partial\Omega \right\rangle \\
&- \left\langle T_n^*(u(\tilde{\mathbf{x}}_{k_u})) \sum_{n_T=1}^{N_T} \int_{\partial\tilde{\Omega}_{n_T}} \hat{T}_n(u(\mathbf{x})) \mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{k_u})\, \mathrm{d}\partial\Omega \right\rangle \\
&+ \left\langle T_n^*(u(\tilde{\mathbf{x}}_{k_u})) \sum_{n_u=1}^{N_u} \int_{\partial\tilde{\Omega}_{n_u}} \hat{u}(\mathbf{x})\, T_n(\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{k_u}))\, \mathrm{d}\partial\Omega \right\rangle \\
&- \left\langle T_n^*(u(\tilde{\mathbf{x}}_{k_u})) \sum_{n_u=1}^{N_u} T_n(u(\tilde{\mathbf{x}}_{n_u})) \int_{\partial\tilde{\Omega}_{n_u}} \mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{k_u})\, \mathrm{d}\partial\Omega \right\rangle.
\end{aligned} \tag{17}$$

Relationships (16) and (17) contain high order statistical moments. In order to obtain the fewest number of unknowns, some simplifications must be adopted and this is the aim of the next section.

3.2.1. *Limiting the number of high order moments*

To solve equations (16) and (17), some assumptions are necessary. Indeed, solving these equations requires a reduction in the number of statistical moments appearing in these equations. This reduction will be based on these assumptions. The procedure presented in what follows is similar to a "closure problem", frequently used for the study of turbulent flows in fluid mechanics. In that case, the closure hypotheses are generally applied to the Reynolds equations [20]. Therefore, the purpose of the assumptions is to define the correlation levels between the random terms of the integral equations.

At first, one must remark that the statistical properties of the different terms appearing in the integral equations are a consequence of the statistical properties of the geometry explicitly defined in the previous sections.

The statistical assumptions introduced in the following are based on a physical interpretation of the integral representations. Indeed, let one consider the following boundary element equation:

$$\tfrac{1}{2}\,u(\tilde{\boldsymbol{\xi}}) = \int_{\tilde{\Omega}_f} f(\mathbf{x})\mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}})\,\mathrm{d}\Omega + \sum_{n_T=1}^{N_T} \int_{\partial\tilde{\Omega}_{n_T}} [u(\mathbf{x})\,T_n(\mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}})) - \hat{T}_n(u(\mathbf{x}))\mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}})]\,\mathrm{d}\partial\Omega$$

$$+ \sum_{n_u=1}^{N_u} \int_{\partial\tilde{\Omega}_{n_u}} [\hat{u}(\mathbf{x})\,T_n(\mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}})) - T_n(u(\mathbf{x}))\mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}})]\,\mathrm{d}\partial\Omega. \tag{18}$$

The right-hand side of equation (18) can be interpreted as the sum of the contributions of extended sources located on the boundary elements of the structure. The amplitude of the sources are the boundary unknowns, as well as the contributions of the external loadings $f(\mathbf{x})$. This physical interpretation is analogous to the diffraction theory. One can remark at this stage that the positions of the different sources are independent random parameters.

In order to solve the problem, two random hypotheses are considered. These assumptions are based on the previous physical interpretation of the integral equations in terms of sources: external loadings and boundary sources.

The first assumption deals with the statistical behaviour of the different sources.

Assumption 1. *The contributions of two sources are statistically independent when the position of the source or the target points of the contributions are distinct.*

Assumption 1 may be simply justified by considering that the statistical independence of the different random variables introduced at the positions of the sources or the target points induces the decorrelation of the source contributions propagating along distinct paths.

At this stage, two types of sources are distinguished; the external loadings which are called primary sources and the boundary sources (on which no loading is applied) which are called secondary sources. The latter are constituted by the multiple wave reflexions of the waves stemming from the loadings.

The second assumption governs the random behaviour of the force and displacement variables.

Assumption 2. *It is considered that a force or a displacement variable expressed at any point* $\tilde{\boldsymbol{\xi}}$ *of the structure, is only correlated with the contributions of the primary sources at point* $\tilde{\boldsymbol{\xi}}$.

A justification of this last assumption is proposed. At first, it is necessary to consider that any force or displacement variable is correlated with the contribution of the external loading. Indeed, the causality relation between the contribution of the external loading and the force or displacement unknown is obvious. On the other hand, the different contributions of the secondary sources, are considered of second order; the contributions are supposed to have no decisive influence on the statistical behaviour of the unknowns. In conclusion, Assumption 2 is considered equivalent, taking into account the minimal correlation.

According to this last assumption, one may make the following simplifications of the equations for $-\forall \tilde{\mathbf{x}} \in \partial\tilde{\Omega}_{x_n}$ and $\tilde{\boldsymbol{\xi}} \in \tilde{\Omega} \backslash \partial\tilde{\Omega}_{x_n}$

$$\left\langle u^*(\tilde{\boldsymbol{\xi}}) u(\tilde{\mathbf{x}}_n) \int_{\partial\tilde{\Omega}_{x_n}} T_n(\mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}}))\, \mathrm{d}\partial\Omega \right\rangle \approx \langle u^*(\tilde{\boldsymbol{\xi}}) \rangle \left\langle u(\tilde{\mathbf{x}}_n) \int_{\partial\tilde{\Omega}_{x_n}} T_n(\mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}}))\, \mathrm{d}\partial\Omega \right\rangle,$$

$$\left\langle u^*(\tilde{\boldsymbol{\xi}})\, T_n(u(\tilde{\mathbf{x}}_n)) \int_{\partial\tilde{\Omega}_{x_n}} \mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}})\, \mathrm{d}\partial\Omega \right\rangle \approx \langle u^*(\tilde{\boldsymbol{\xi}}) \rangle \left\langle T_n(u(\tilde{\mathbf{x}}_n)) \int_{\partial\tilde{\Omega}_{x_n}} \mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}})\, \mathrm{d}\partial\Omega \right\rangle,$$

$$\left\langle T_n(u^*(\tilde{\boldsymbol{\xi}})) u(\tilde{\mathbf{x}}_n) \int_{\partial\tilde{\Omega}_{x_n}} T_n(\mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}}))\, \mathrm{d}\partial\Omega \right\rangle \approx \langle T_n(u^*(\tilde{\boldsymbol{\xi}})) \rangle \left\langle u(\tilde{\mathbf{x}}_n) \int_{\partial\tilde{\Omega}_{x_n}} T_n(\mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}}))\, \mathrm{d}\partial\Omega \right\rangle,$$

$$\left\langle T_n(u^*(\tilde{\boldsymbol{\xi}}))\, T_n(u(\tilde{\mathbf{x}}_n)) \int_{\partial\tilde{\Omega}_{x_n}} \mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}})\, \mathrm{d}\partial\Omega \right\rangle \approx \langle T_n(u^*(\tilde{\boldsymbol{\xi}})) \rangle \left\langle T_n(u(\tilde{\mathbf{x}}_n)) \int_{\partial\tilde{\Omega}_{x_n}} \mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}})\, \mathrm{d}\partial\Omega \right\rangle. \tag{19}$$

According to the second assumption, the boundary unknowns are only correlated with the contributions of the primary sources. Consequently, they are not correlated with the propagative part of the boundary sources (which represent the contribution of a secondary source of amplitude equal to unity and located on the boundary). Therefore, one can write

$$\left\langle u(\tilde{\mathbf{x}}_n) \int_{\partial\tilde{\Omega}_{x_n}} \mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}})\, \mathrm{d}\partial\Omega \right\rangle \approx \langle u(\tilde{\mathbf{x}}_n) \rangle \left\langle \int_{\partial\tilde{\Omega}_{x_n}} \mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}})\, \mathrm{d}\partial\Omega \right\rangle,$$

$$\left\langle u(\tilde{\mathbf{x}}_n) \int_{\partial\tilde{\Omega}_{x_n}} T_n(\mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}}))\, \mathrm{d}\partial\Omega \right\rangle \approx \langle u(\tilde{\mathbf{x}}_n) \rangle \left\langle \int_{\partial\tilde{\Omega}_{x_n}} T_n(\mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}}))\, \mathrm{d}\partial\Omega \right\rangle,$$

$$\left\langle |u(\tilde{\mathbf{x}}_n)|^2 \int_{\partial\tilde{\Omega}_{x_n}} T_n(\mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}}))\, \mathrm{d}\partial\Omega \right\rangle \approx \langle |u(\tilde{\mathbf{x}}_n)|^2 \rangle \left\langle \int_{\partial\tilde{\Omega}_{x_n}} T_n(\mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}}))\, \mathrm{d}\partial\Omega \right\rangle,$$

$$\left\langle T_n(u(\tilde{\mathbf{x}}_n)) \int_{\partial\tilde{\Omega}_{x_n}} \mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}})\, \mathrm{d}\partial\Omega \right\rangle \approx \langle T_n(u(\tilde{\mathbf{x}}_n)) \rangle \left\langle \int_{\partial\tilde{\Omega}_{x_n}} \mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}})\, \mathrm{d}\partial\Omega \right\rangle,$$

$$\left\langle T_n(u(\tilde{\mathbf{x}}_n)) \int_{\partial\tilde{\Omega}_{x_n}} T_n(\mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}}))\, \mathrm{d}\partial\Omega \right\rangle \approx \langle T_n(u(\tilde{\mathbf{x}}_n)) \rangle \left\langle \int_{\partial\tilde{\Omega}_{x_n}} T_n(\mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}}))\, \mathrm{d}\partial\Omega \right\rangle,$$

$$\left\langle |T_n(u(\tilde{\mathbf{x}}_n))|^2 \int_{\partial\tilde{\Omega}_{x_n}} T_n(\mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}}))\, \mathrm{d}\partial\Omega \right\rangle \approx \langle |T_n(u(\tilde{\mathbf{x}}_n))|^2 \rangle \left\langle \int_{\partial\tilde{\Omega}_{x_n}} T_n(\mathrm{G}(\mathbf{x}, \tilde{\boldsymbol{\xi}}))\, \mathrm{d}\partial\Omega \right\rangle. \tag{20}$$

3.2.2. *Application of the random hypotheses to the stochastic integral equations*

Equations (16, 17) can now be rewritten using relationships (19, 20). One finally obtains for $\tilde{\mathbf{x}}_{k_T} \in \partial\tilde{\Omega}_{k_T}$, $k_T \in [1, N_T]$

$$\tfrac{1}{2} \langle |u(\tilde{\mathbf{x}}_{k_T})|^2 \rangle = \left\langle u^*(\tilde{\mathbf{x}}_{k_T}) \int_{\tilde{\Omega}_f} f(\mathbf{x}) \mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{k_T})\, \mathrm{d}\Omega \right\rangle$$

$$+ \langle u^*(\tilde{\mathbf{x}}_{k_T}) \rangle \sum_{\substack{n_T = 1 \\ n_T \neq k_T}}^{N_T} \langle u(\tilde{\mathbf{x}}_{n_T}) \rangle \left\langle \int_{\partial\tilde{\Omega}_{n_T}} T_n(\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{k_T}))\, \mathrm{d}\partial\Omega \right\rangle$$

$$
\begin{aligned}
&- \langle u^*(\tilde{\mathbf{x}}_{k_T})\rangle \sum_{n_T=1}^{N_T} \left\langle \int_{\partial\tilde{\Omega}_{n_T}} \hat{T}_n(u(\mathbf{x}))\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{k_T})\,\mathrm{d}\partial\Omega \right\rangle \\
&+ \langle u^*(\tilde{\mathbf{x}}_{k_T})\rangle \sum_{n_u=1}^{N_u} \left\langle \int_{\partial\tilde{\Omega}_{n_u}} \hat{u}(\mathbf{x})\, T_n(\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{k_T}))\,\mathrm{d}\partial\Omega \right\rangle \\
&- \langle u^*(\tilde{\mathbf{x}}_{k_T})\rangle \sum_{n_u=1}^{N_u} \langle T_n(u(\tilde{\mathbf{x}}_{n_u}))\rangle \left\langle \int_{\partial\tilde{\Omega}_{n_u}} \mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{k_T})\,\mathrm{d}\partial\Omega \right\rangle \\
&+ \langle |u^*(\tilde{\mathbf{x}}_{k_T})|^2\rangle \left\langle \int_{\partial\tilde{\Omega}_{k_T}} T_n(\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{k_T}))\,\mathrm{d}\partial\Omega \right\rangle. \qquad (21)
\end{aligned}
$$

For $\tilde{\mathbf{x}}_{k_u} \in \partial\tilde{\Omega}_{k_u}$, $k_u \in [1, N_u]$

$$
\begin{aligned}
\tfrac{1}{2}\langle T_n^*(u(\tilde{\mathbf{x}}_{k_u}))\rangle \hat{u}(\tilde{\mathbf{x}}_{k_u}) = &\left\langle T_n^*(u(\tilde{\mathbf{x}}_{k_u})) \int_{\tilde{\Omega}_f} f(\mathbf{x})\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{k_u})\,\mathrm{d}\Omega \right\rangle \\
&+ \langle T_n^*(u(\tilde{\mathbf{x}}_{k_u}))\rangle \sum_{n_T=1}^{N_T} \langle u(\tilde{\mathbf{x}}_{n_T})\rangle \left\langle \int_{\partial\tilde{\Omega}_{n_T}} T_n(\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{k_u}))\,\mathrm{d}\partial\Omega \right\rangle \\
&- \langle T_n^*(u(\tilde{\mathbf{x}}_{k_u}))\rangle \sum_{n_T=1}^{N_T} \left\langle \int_{\partial\tilde{\Omega}_{n_T}} \hat{T}_n(u(\mathbf{x}))\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{k_u})\,\mathrm{d}\partial\Omega \right\rangle \\
&+ \langle T_n^*(u(\tilde{\mathbf{x}}_{k_u}))\rangle \sum_{n_u=1}^{N_u} \left\langle \int_{\partial\tilde{\Omega}_{n_u}} \hat{u}(\mathbf{x})\, T_n(\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{k_u}))\,\mathrm{d}\partial\Omega \right\rangle \\
&- \langle T_n^*(u(\tilde{\mathbf{x}}_{k_u}))\rangle \sum_{\substack{n_u=1 \\ n_u \neq k_u}}^{N_u} \langle T_n(u(\tilde{\mathbf{x}}_{n_u}))\rangle \left\langle \int_{\partial\tilde{\Omega}_{n_u}} \mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{k_u})\,\mathrm{d}\partial\Omega \right\rangle \\
&- \langle |T_n^*(u(\tilde{\mathbf{x}}_{k_u}))|^2\rangle \left\langle \int_{\partial\tilde{\Omega}_{k_u}} \mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{k_u})\,\mathrm{d}\partial\Omega \right\rangle. \qquad (22)
\end{aligned}
$$

Equations (21, 22) are the fundamental relationships of this new stochastic integral formulation. A careful observation of these $N_u + N_T$ discretized equations enables one to point out the presence in the formulation of different order statistical moments of the unknowns. One can also distinguish the expectations of the boundary unknowns multiplied by the contribution of the external loading. In conclusion, the number of unknowns of the formulation is equal to $3(N_u + N_T)$. In order to clarify the previous statements, the unknowns are given. It is recalled that $\tilde{\mathbf{x}}_{n_T} \in \partial\tilde{\Omega}_{n_T}$, $n_T = 1, 2, \ldots, N_T$, and $\tilde{\mathbf{x}}_{n_u} \in \partial\tilde{\Omega}_{n_u}$ with $n_u = 1, 2, \ldots, N_u$.

1. First order moment: $\langle u(\tilde{\mathbf{x}}_{n_T})\rangle$ and $\langle T_n(u(\tilde{\mathbf{x}}_{n_u}))\rangle$.
2. Second order moment: $\langle |u(\tilde{\mathbf{x}}_{n_T})|^2\rangle$ and $\langle |T_n(u(\tilde{\mathbf{x}}_{n_u}))|^2\rangle$.
3. Expectation of the boundary unknowns multiplied by the contributions of the loading $\langle u^*(\tilde{\mathbf{x}}_{n_T}) \int_{\tilde{\Omega}_f} f(\mathbf{x})\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{n_T})\,\mathrm{d}\Omega\rangle$ and $\langle T_n^*(u(\tilde{\mathbf{x}}_{n_u})) \int_{\tilde{\Omega}_f} f(\mathbf{x})\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_{n_u})\,\mathrm{d}\Omega\rangle$.

What emerges from the previous assessment is that equations (21, 22) are not sufficient to solve the problem. It is consequently necessary to obtain a complementary number of equations.

At first, the expectation of the classical boundary integral equation (13) is considered. This new equation is discretized on the $N_u + N_T$ boundary elements previously defined. The collocation method is utilized and finally, Assumptions 1 and 2 are applied to this new

expression, and one obtains for $\tilde{\mathbf{x}}_k \in \partial\tilde{\Omega}_k$, $k = 1, \ldots, N_u + N_T$

$$\begin{aligned}\tfrac{1}{2}\langle u(\tilde{\mathbf{x}}_k)\rangle = &\left\langle \int_{\tilde{\Omega}_f} f(\mathbf{x})\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_k)\,\mathrm{d}\Omega \right\rangle + \sum_{n_T=1}^{N_T} \langle u(\tilde{\mathbf{x}}_{n_T})\rangle \left\langle \int_{\partial\tilde{\Omega}_{n_T}} T_n(\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_k))\,\mathrm{d}\partial\Omega \right\rangle \\ &- \sum_{n_T=1}^{N_T} \left\langle \int_{\partial\tilde{\Omega}_{n_T}} \hat{T}_n(u(\mathbf{x}))\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_k)\,\mathrm{d}\partial\Omega \right\rangle + \sum_{n_u=1}^{N_u} \left\langle \int_{\partial\tilde{\Omega}_{n_u}} \hat{u}(\mathbf{x})\, T_n(\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_k))\,\mathrm{d}\partial\Omega \right\rangle \\ &- \sum_{n_u=1}^{N_u} \langle T_n(u(\tilde{\mathbf{x}}_{n_u}))\rangle \left\langle \int_{\partial\tilde{\Omega}_{n_u}} \mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_k)\,\mathrm{d}\partial\Omega \right\rangle. \end{aligned} \tag{23}$$

Secondly, boundary equation (13) is multiplied by $\int_{\tilde{\Omega}_f} f^*(\mathbf{x})\mathrm{G}^*(\mathbf{x}, \tilde{\mathbf{x}}_n)\,\mathrm{d}\Omega$ ($n \in N_T + N_u$ and $\tilde{\mathbf{x}}_n$ is the collocation point), and the expectations of the terms of this new equation are taken into account. Utilizing Assumptions 1 and 2, it is possible to write the new equation as follows for $\tilde{\mathbf{x}}_k \in \partial\tilde{\Omega}_k$, $k = 1, \ldots, N_u + N_T$:

$$\begin{aligned}&\tfrac{1}{2}\left\langle u(\tilde{\mathbf{x}}_k) \int_{\tilde{\Omega}_f} f^*(\mathbf{x})\mathrm{G}^*(\mathbf{x}, \tilde{\mathbf{x}}_k)\,\mathrm{d}\Omega \right\rangle \\ &\quad = \left\langle \left| \int_{\tilde{\Omega}_f} f(\mathbf{x})\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_k)\,\mathrm{d}\Omega \right|^2 \right\rangle \\ &\qquad + \sum_{\substack{n_T=1 \\ n_T \neq k}}^{N_T} \left\langle \int_{\tilde{\Omega}_f} f^*(\mathbf{x})\mathrm{G}^*(\mathbf{x}, \tilde{\mathbf{x}}_k)\,\mathrm{d}\Omega \right\rangle \langle u(\tilde{\mathbf{x}}_{n_T})\rangle \left\langle \int_{\partial\tilde{\Omega}_{n_T}} T_n(\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_k))\,\mathrm{d}\partial\Omega \right\rangle \\ &\qquad - \sum_{n_T=1}^{N_T} \left\langle \int_{\tilde{\Omega}_f} f^*(\mathbf{x})\mathrm{G}^*(\mathbf{x}, \tilde{\mathbf{x}}_k)\,\mathrm{d}\Omega \right\rangle \left\langle \int_{\partial\tilde{\Omega}_{n_T}} \hat{T}_n(u(\mathbf{x}))\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_k)\,\mathrm{d}\partial\Omega \right\rangle \\ &\qquad + \sum_{n_u=1}^{N_u} \left\langle \int_{\tilde{\Omega}_f} f^*(\mathbf{x})\mathrm{G}^*(\mathbf{x}, \tilde{\mathbf{x}}_k)\,\mathrm{d}\Omega \right\rangle \left\langle \int_{\partial\tilde{\Omega}_{n_u}} \hat{u}(\mathbf{x})\, T_n(\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_k))\,\mathrm{d}\partial\Omega \right\rangle \\ &\qquad - \sum_{\substack{n_u=1 \\ n_u \neq k}}^{N_u} \left\langle \int_{\tilde{\Omega}_f} f^*(\mathbf{x})\mathrm{G}^*(\mathbf{x}, \tilde{\mathbf{x}}_k)\,\mathrm{d}\Omega \right\rangle \langle T_n(u(\tilde{\mathbf{x}}_{n_u}))\rangle \left\langle \int_{\partial\tilde{\Omega}_{n_u}} \mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_k)\rangle\,\mathrm{d}\partial\Omega \right\rangle \\ &\qquad + a_T \left\langle u(\tilde{\mathbf{x}}_k) \int_{\tilde{\Omega}_f} f^*(\mathbf{x})\mathrm{G}^*(\mathbf{x}, \tilde{\mathbf{x}}_k)\,\mathrm{d}\Omega \right\rangle \left\langle \int_{\partial\tilde{\Omega}_{k_T}} (T_n(\mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_k))\,\mathrm{d}\partial\Omega \right\rangle \\ &\qquad - a_u \left\langle T_n(u(\tilde{\mathbf{x}}_k)) \int_{\tilde{\Omega}_f} f^*(\mathbf{x})\mathrm{G}^*(\mathbf{x}, \tilde{\mathbf{x}}_k)\,\mathrm{d}\Omega \right\rangle \left\langle \int_{\partial\tilde{\Omega}_{k_u}} \mathrm{G}(\mathbf{x}, \tilde{\mathbf{x}}_k)\,\mathrm{d}\partial\Omega \right\rangle \end{aligned} \tag{24}$$

with

$$\begin{aligned} a_T &= 0 \quad \forall \tilde{\mathbf{x}}_k \in \partial\tilde{\Omega}_u \\ a_T &= 1 \quad \forall \tilde{\mathbf{x}}_k \in \partial\tilde{\Omega}_T \end{aligned} \quad \text{and} \quad \begin{aligned} a_u &= 0 \quad \forall \tilde{\mathbf{x}}_k \in \partial\tilde{\Omega}_T, \\ a_u &= 1 \quad \forall \tilde{\mathbf{x}}_k \in \partial\tilde{\Omega}_u. \end{aligned} \tag{25}$$

Equations (21–24) enable one to evaluate the entire set of unknowns of the stochastic boundary integral formulation.

In order to obtain the second-order moment of a force or displacement unknown at an inner point of the structure, the random description is introduced to the classical integral equation. The latter equation is then multiplied by itself, and the expectation of the new equation is considered. Finally, the statistical assumptions are applied to the stochastic

integral representation. One can easily obtain this relationship by following the procedure outlined above.

3.3. ANALYTICAL REPRESENTATIONS OF THE RANDOM BOUNDARIES

In order to solve the set of equations (24), the numerical evaluation of the expectations of boundary and domain integrals must be carried out. The integration paths are random, therefore it is not possible to commutate the expectation and the integral operators. To simplify this evaluation, it will be shown that the random variable can be judiciously chosen in order to get rid of the randomness in the integration path by means of a change of variable. The way to proceed is illustrated for two different types of boundaries: an arc of a circle and a straight segment.

3.3.1. *The random circular boundary*

A circle $\partial\Omega$, is divided into n elements. One of these elements $\partial\Omega_i$ is defined by the radius of the circle r, the current angle θ and the two bounds of the element, respectively, defined by their co-ordinates (θ_i, r) and (θ_{i+1}, r). Figure 1 illustrates this notation.

Then, a random variable is introduced in the definition of the radius r, such as $\partial\tilde{\Omega}_i(\theta) = \tilde{r}$. More precisely, the random boundary element is defined using the three conditions

$$\partial\tilde{\Omega}_i(\theta_i) = r, \qquad \partial\tilde{\Omega}_i(\theta_{i+1}) = r, \qquad \partial\tilde{\Omega}_i(\theta_{i+1/2}) = r + \tilde{\varepsilon}_i \tag{26}$$

with $\theta_{i+1/2} = (\theta_i + \theta_{i+1})/2$ and $\tilde{\varepsilon}_i$ is a zero mean random variable. Finally, $\partial\tilde{\Omega}_i$ is defined as a parabolic function of θ

$$\partial\tilde{\Omega}_i(\theta) = \frac{\tilde{\varepsilon}_i\theta^2 - (\theta_i + \theta_{i+1})\tilde{\varepsilon}_i\theta + \theta_i\theta_{i+1}\tilde{\varepsilon}_i - r(\theta_i\theta_{i+1/2} - \theta_i\theta_{i+1} - \theta_{i+1/2}^2 + \theta_{i+1/2}\theta_{i+1})}{-\theta_i\theta_{i+1/2} + \theta_i\theta_{i+1} + \theta_{i+1/2}^2 - \theta_{i+1/2}\theta_{i+1}}. \tag{27}$$

In order to evaluate the integral of any function $F(\theta)$ over the integration path $\partial\tilde{\Omega}_i$, one may write

$$\int_{\partial\tilde{\Omega}_i} F(\tilde{\mathbf{x}})\,\mathrm{d}\partial\tilde{\Omega}_i = \int_{\theta_i}^{\theta_{i+1}} F(\theta)\sqrt{\frac{\partial}{\partial\theta}\partial\tilde{\Omega}_i^2(\theta) + \partial\tilde{\Omega}_i^2(\theta)}\,\mathrm{d}\theta. \tag{28}$$

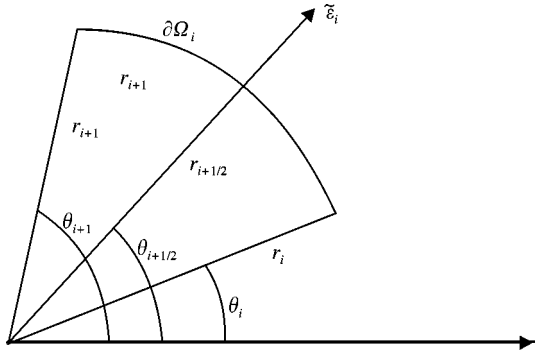

Figure 1. Notation for circular boundary element.

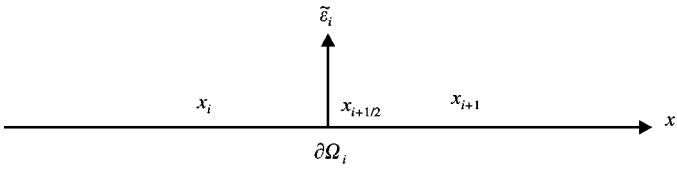

Figure 2. Notation for straight boundary element.

Observation of the right-hand term of the above expression allows one to say that the random variable has disappeared from the integration path. One can finally interchange the integration and the statistical operators. This way of introducing the random parameter in the definition of the boundary is particularly well suited for circular boundaries; however, this random definition could easily be extended to any other types of boundaries which can be defined in terms of polar co-ordinates.

3.3.2. *The random straight boundary*

Usual mechanical systems are made of assembled rectangular panels with straight boundaries. For this type of geometry, the random variable is introduced using a similar process as for the circular boundaries. Indeed, one may consider a straight boundary element $\partial\tilde{\Omega}_i$, bounded by two points (x_i, y_i) and (x_{i+1}, y_{i+1}). The local frame of reference is chosen so that $y_i = y_{i+1} = y$. The boundary element is represented by Figure 2.

The random variable is introduced in the definition of the boundary element. Finally, the random element $\partial\tilde{\Omega}_i$ is a parabolic function of the co-ordinate x, defined by three points:

$$\partial\tilde{\Omega}_i(x_i) = y, \qquad \partial\tilde{\Omega}_i(x_{i+1}) = y, \qquad \partial\tilde{\Omega}_i(x_{i+1/2}) = y + y_{\varepsilon_i}. \tag{29}$$

$\tilde{y}_{\varepsilon_i}$ is a zero mean random variable and $x_{i+1/2} = 1/2(x_i + x_{i+1})$. The analytical expression of $\partial\tilde{\Omega}_i$ may be written as

$$\partial\tilde{\Omega}_i(x) = \frac{\tilde{y}_{\varepsilon_i}x^2 - (x_i + x_{i+1})\tilde{y}_{\varepsilon_i}x + x_i x_{i+1}\tilde{y}_{\varepsilon_i} - y(x_i x_{i+1/2} - x_i x_{i+1} - x_{i+1/2}^2 + x_{i+1/2}x_{i+1})}{- x_i x_{i+1/2} + x_i x_{i+1} + x_{i+1/2}^2 - x_{i+1/2}x_{i+1}}. \tag{30}$$

The integrals may be evaluated over $\partial\tilde{\Omega}_i$ using a change of variables:

$$\int_{\partial\tilde{\Omega}_i} F(\tilde{x})\,\mathrm{d}\partial\tilde{\Omega}_i = \int_{x_i}^{x_{i+1}} F(x)\sqrt{\frac{\partial}{\partial x}\,\partial\tilde{\Omega}_i^2(x) + \partial\tilde{\Omega}_i^2(x)}\,\mathrm{d}x. \tag{31}$$

The integration path does not depend on the random variable anymore. Therefore, it is possible to switch the integration and the expectation operators.

3.4. THE RANDOM FORMULATION APPLIED TO ASSEMBLED STRUCTURES

In the previous sections, the basics of the random formulation have been explained for isolated structures. The aim of the authors is now to show that the present method has some advantages over the other high-frequency formulations, when applied to the prediction of the vibrational behaviour of assembled structures.

Classical high-frequency methods such as the SEA [3] are usually employed to treat the behaviour of complex mechanical systems in the high-frequency field. However, some

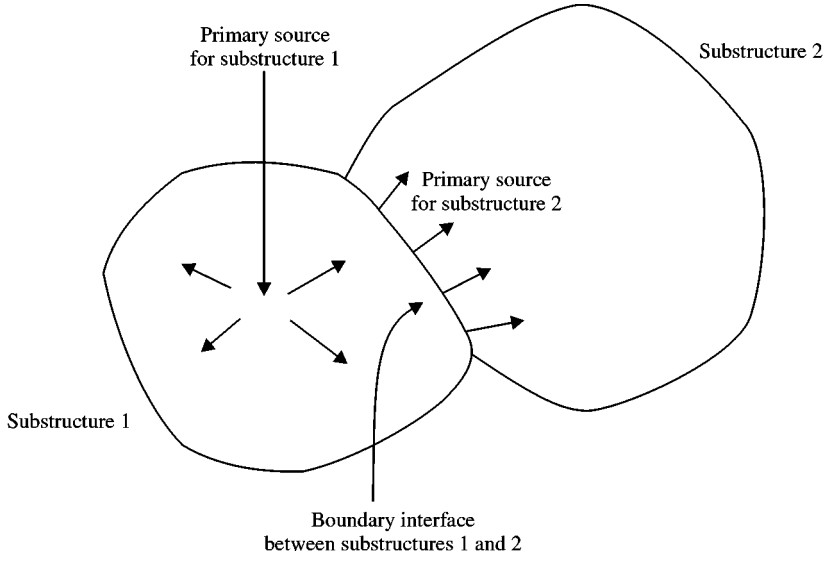

Figure 3. Illustration of Assumption 3, definition of the primary sources.

difficulties may arise when utilizing such methods in this context. The characterization of the interfaces between the different subsystems by means of coupling factors is still nowadays an open research field. Another big difficulty is encountered when the substructures of a complex mechanical system have very different properties. When this is the case, their respective modal densities in a given frequency interval differ, and the classical high-frequency assumptions used in the SEA for example, may not be valid for all the subsystems of the considered structure.

The interesting aspect of the random formulation proposed in this paper is that the boundary conditions utilized to solve the random integral boundary equations are expressed in terms of kinematic variables. Thus, the interface conditions between two subsystems are defined using the classical kinematic relationships. Moreover, the integral random formulation is valid in the low-, middle- and high-frequency domains. Therefore, the formulation is not affected by the differences of modal densities of the subsystems of the studied structure.

For isolated systems, two assumptions have been adopted which remain valid for the case of assembled structures. On the other hand, the way of defining the sources (primary or secondary sources) has to be up-dated for the case of assembled systems. Indeed, Viktorovitch *et al.* [15] proved that an additional correlation should be taken into account for assembled one-dimensional structures. Therefore, they proposed the following assumption which is still relevant for two-dimensional structures:

Assumption 3. *The boundaries connecting two substructures, of which one contains a primary source, become primary sources for the other substructure. This assumption enables one to express the correlation between an external loading located on one subsystem and a force displacement unknown on another substructure.*

This last assumption is illustrated in Figure 3. One can remark that the classification of the different sources (secondary and primary) enables distinction of the sources that directly

transmit the contribution of the loading from the sources reflecting the waves coming from the loading.

4. NUMERICAL APPLICATIONS OF THE RANDOM BOUNDARY INTEGRAL FORMULATION

In a previous article by Viktorovitch *et al.* [15], dealing with the random boundary element formulation applied to one-dimensional systems, a large number of examples of isolated and assembled rods and beams highlighted the effectiveness of the formulation in the low-, the middle- and the high-frequency ranges. Moreover, the observation of the frequency responses obtained by the random boundary formulation, showed the same behaviour for the different mechanical systems. That is to say, the low-frequency response is accurately obtained, the high-frequency behaviour of the structure is described by a smooth response and the middle-frequency range appears to be a transition zone between the low- and the high-frequency range.

In the following sections, it will be shown through different numerical applications that the conclusions drawn for one-dimensional systems can be extended to two-dimensional structures. Indeed, the boundary element formulation, which is the basis of the random formulation, is particularly well suited for structures bounded by one- or two-dimensional elements.

Therefore, some simulations have been carried out to observe the frequency responses obtained by the random boundary formulation. These simulations are presented in the following sections.

4.1. ISOLATED STRUCTURES

In this section, the random formulation is applied to isolated two-dimensional structures such as membranes and plates.

4.1.1. *Frequency and spatial variation of the random formulation boundary unknowns: circular membrane*

In this first example, a point loaded circular membrane is considered. The simulations are carried out considering a loading of 1 N located initially at the centre of the membrane (which is the origin of the frame and then at a point with co-ordinates ($x = 0{\cdot}5$; $y = 0{\cdot}5$). The geometric and material properties of the structure are summarized in Table 1.

In the different figures, the curves obtained by the random formulation are called SIF-0$\alpha\beta$, SIF denotes the stochastic integral formulation and 0$\alpha\beta$ is the value of the standard deviation when a Gaussian law is considered (respectively, the width of the spectrum for a uniform distribution law).

TABLE 1

Geometric and material properties of the circular membrane

Radius (m)	T (N/m)	η (%)	ρ (kg/m^2)
2	1	3	1

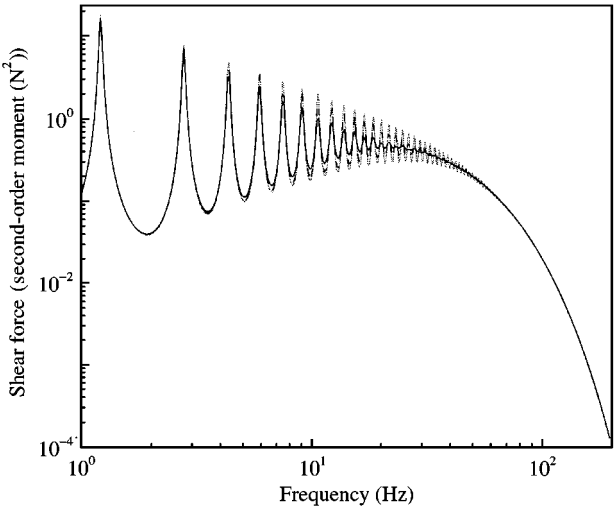

Figure 4. Frequency variation of the boundary force (loading located at the centre of the membrane) for the circular membrane. Key: ·····, deterministic; - - - -, SIF-005 ($a = 0{\cdot}05$); ——, SIF-010 ($a = 0{\cdot}1$).

Figure 4 represents the frequency variation of the second-order moment of the boundary shear force obtained for a uniform distribution law with a width a successively equal to 0·05 and 0·1. Observation of the curves SIF-005 and SIF-010 of Figure 4 highlights some important elements concerning the effectiveness of the random formulation. At first, one can globally state that the influence of the randomness increases with frequency. Indeed, the curves SIF-005 and SIF-010 give a precise representation of the modal behaviour in the low-frequency range. On the other hand, the high-frequency behaviour of the random formulation simulation is smooth and only delivers information on the general trend of the frequency variation of the boundary unknown. Moreover, comparison of the two curves SIF-005 and SIF-010 clearly shows the influence of the random variables. When the value of the standard deviation increases, the transition zone between the precise low-frequency representation and the high-frequency curve is translated towards the high-frequency field.

Consequently, the choice of the values of the standard deviation fixes the limits between the low and the high frequencies, and this limit is a function of the ratio between the standard deviation and the wavelength.

On the other hand, it must not be forgotten that the standard deviation is related to the geometrical indeterminations of the studied structure. Choosing too large a value for σ or a will lead to a non-realistic description of the structure. Therefore, relevant results cannot be expected in this situation.

4.1.2. *Frequency and spatial variation of the random formulation boundary unknowns: circular plate*

In order to illustrate the effectiveness of the random formulation, different types of structures have to be considered. In particular, the response of the random formulation when dealing with flexural motion of two-dimensional systems is of great interest. This is the aim of this section which is devoted to the comparison of the results obtained by the random formulation and the deterministic response for the case of plate bending. The material and geometric properties of the structure are summarized in Table 2.

TABLE 2

Mechanical and geometric properties of the circular plate

Radius (m)	D (N m)	η (%)	ρh (kg/m^2)
0·5	19·23	4	7·8

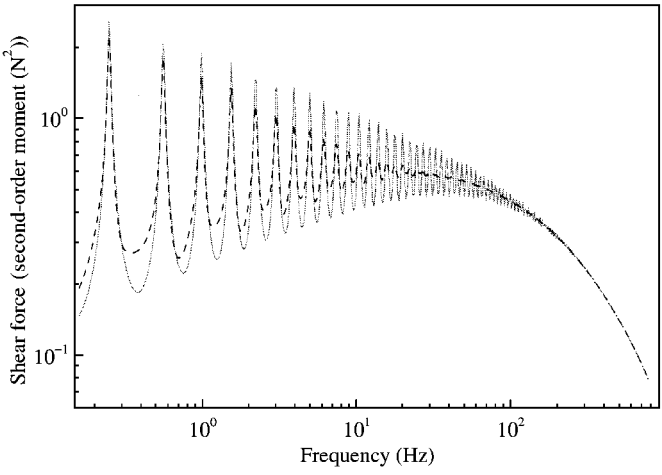

Figure 5. Frequency variation of the second order moment of the shear force at the boundary of the bending plate. Key: ----, SIF-007 ($a = 0{\cdot}07$); ·····, deterministic response.

The random law governing the behaviour of the random variables is a uniform distribution law, and the value of the width of the law is 0·07. Figure 5 represents the frequency variation of the second order moment of the shear force at any location on the boundary. In this first simulation, the loading is located at the centre of the plate, therefore the boundary unknowns are constant at any location of the boundary. The comparison between the frequency variation of the random simulations and the deterministic curves leads to the same remarks as for the membrane. That is to say, the curve SIF-007 shows good agreement with the deterministic in the low-frequency range. Moreover, the random response in the high-frequency field may be considered as a smooth trend of the deterministic frequency variation.

4.2. ASSEMBLED STRUCTURES

The random formulation is now applied to assembled structures. For engineering purposes, the validation of the formulation for this class of system is crucial since assembled systems constitute the most common industrial mechanical systems. In this section, it will be shown that the formulation offers some advantages over other middle- and high-frequency formulations. Especially, the SIF may be applied to a structure whose different subsystems have very different material properties. When this is the case, the usual high-frequency methods such as SEA are ineffective since the three frequency ranges (low, middle and high frequencies) defined for each separate subsystems do not match.

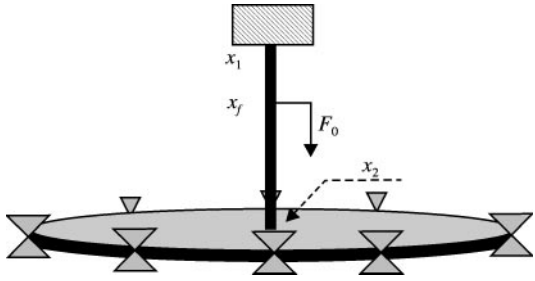

Figure 6. Coupling of a rod and a clamped circular membrane.

Two examples are proposed to illustrate the behaviour of the random formulation. The first example is concerned with a one-dimensional structure connected to a two-dimensional system. The last numerical application is a pure two-dimensional coupling.

4.2.1. *The example of a rod and a membrane coupling*

The simulation presented in this section is dealing with the coupling of a circular membrane and a rod submitted to longitudinal loading. Figure 6 illustrates the system. The rod as well as the membrane are clamped.

The rod is fixed to the membrane, the continuity of the longitudinal displacements and forces is guaranteed at the joint between the two substructures.

The boundaries of the rod and the loading location (respectively, $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_f$) are random variables and follow a Gaussian distribution law. Randomness is also introduced in the description of the boundary of the membrane as described in section 3.3. The random variables introduced in the description of the membrane are listed below. They follow a uniform distribution law of width denoted as a.

The first order moments: $\langle u(\tilde{\mathbf{x}}_1)\rangle$, $\langle u(\tilde{\mathbf{x}}_2)\rangle$, $\langle T_n(u(\tilde{\mathbf{x}}_2))\rangle$ and $\langle T_n(w(\tilde{\mathbf{x}}_k))\rangle$.

The second order moments: $\langle|u(\tilde{\mathbf{x}}_1)|^2\rangle$, $\langle|u(\tilde{\mathbf{x}}_2)|^2\rangle$, $\langle|T_n(u(\tilde{\mathbf{x}}_2))|^2\rangle$, $\langle|T_n(w(\tilde{\mathbf{x}}_k))|^2$, $\langle u^*(\tilde{\mathbf{x}}_2)T_n(u(\tilde{\mathbf{x}}_2))\rangle$ and $\langle T_n^*(u(\tilde{\mathbf{x}}_2))u(\tilde{\mathbf{x}}_2)\rangle$.

Expectations of the boundary unknowns of the rod multiplied by the contribution of the loading: $\langle \mathrm{G}^*(\tilde{\mathbf{x}}_f, \tilde{\mathbf{x}}_1)u(\tilde{\mathbf{x}}_1)\rangle$, $\langle \mathrm{G}^*(\tilde{\mathbf{x}}_f, \tilde{\mathbf{x}}_2)u(\tilde{\mathbf{x}}_2)\rangle$ and $\langle \mathrm{G}^*(\tilde{\mathbf{x}}_f, \tilde{\mathbf{x}}_2)T_n(u(\tilde{\mathbf{x}}_2))\rangle$.

Expectations of the boundary unknowns of the membrane by the contribution of the coupling between the rod and the membrane: $\langle u^*(\tilde{\mathbf{x}}_2)T_n^*(\mathrm{G}(\tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_k))T_n(w(\tilde{\mathbf{x}}_k))\rangle$ and $\langle T_n^*(u(\tilde{\mathbf{x}}_2))\mathrm{G}^*(\tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_k)T_n(w(\tilde{\mathbf{x}}_k))\rangle$.

u denotes the displacement field of the rod whereas w is the displacement field of the membrane. The geometric and material properties of the two substructures are summarized in Tables 3 and 4.

Figure 7 represents the vibratory behaviour of the force at the joint between the rod and the membrane with $\sigma = 0{\cdot}15$ and $a = 0{\cdot}8$. The observation of the frequency variations obtained by the random formulation leads to similar conclusions as for the isolated structures studied previously. Indeed, the random formulation describes accurately the low-frequency deterministic response while the high-frequency trend of the deterministic response is smoothly illustrated.

Figure 8 represents the results obtained by the SIF formulation along the boundary of the membrane, when the width of the uniform distribution law is fixed to zero and highlights a very interesting property of the random formulation. Considering that the membrane or

TABLE 3

Geometric and material properties of the rod (*rod–membrane example*)

Length (m)	*ES* (N)	η (%)	ρS (kg/m)
10	$6{\cdot}6 \times 10^{-3}$	0·5	$2{\cdot}45 \times 10^{-2}$

TABLE 4

Geometric and material properties of the membrane (*rod–membrane example*)

Radius (m)	T (N/m)	η (%)	ρ (kg/m^2)
5	1	1	1

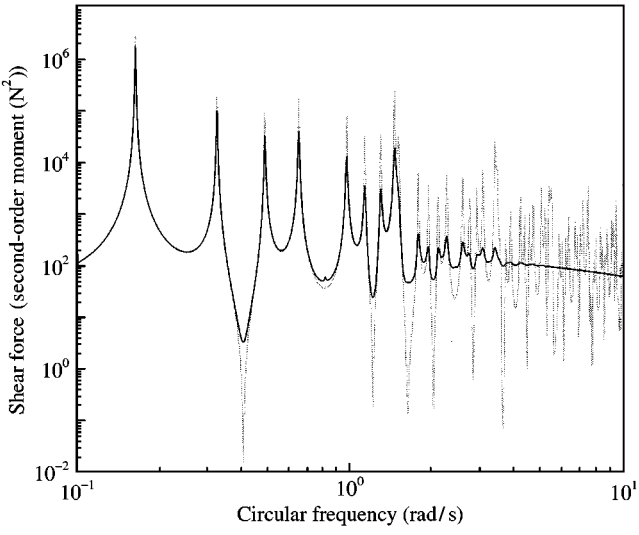

Figure 7. Frequency variation of the second order moment of the shear force at the joint in the rod and the membrane system. Deterministic result (·····) and SIF simulations; (——) for $\sigma = 0{\cdot}15$ and $a = 0{\cdot}8$.

the rod is deterministic enables one to obtain the modal behaviour of the considered deterministic substructure on the whole frequency domiain whilst the random formulation illustrates the trend of the behaviour of the other substructure, for which random variables are introduced.

4.2.2. *Two coupled membranes*

The last example deals with the coupling of two clamped membranes (shown in Figure 9). This example illustrates the effectiveness of the formulation as a vibratory predictive tool in

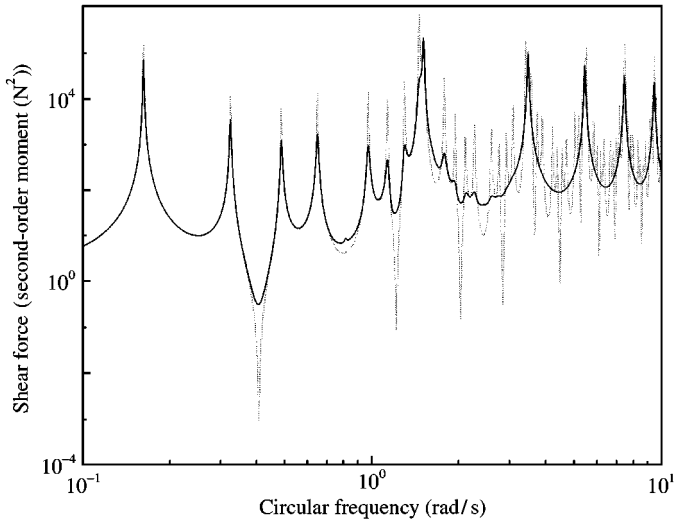

Figure 8. Frequency variation of the second order moment of the membrane boundary shear force in the rod–membrane system. Deterministic result (·····) and SIF simulations (——) for $\sigma = 0{\cdot}15$ and $a = 0$.

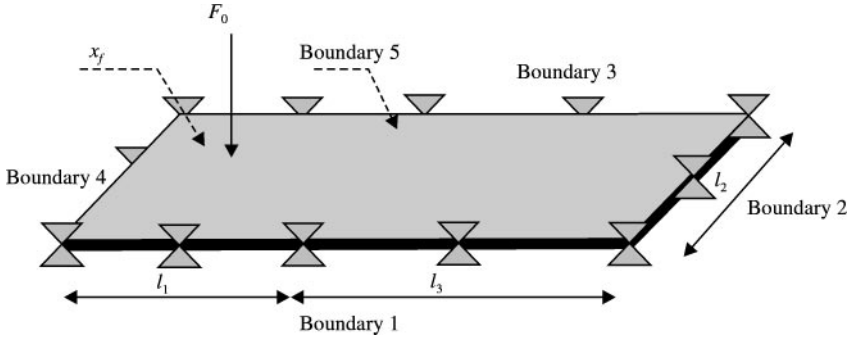

Figure 9. Two coupled membranes.

the middle- and high-frequency ranges. The coupling conditions are treated by means of the usual boundary conditions. This is not the case of the classical energy methods in which boundary conditions are formulated in terms of coupling factors whose derivation usually requires theoretical developments [11].

The whole structure is excited by a point loading located as illustrated in Figure 9. The two membranes share the same material properties which are summarized in Table 5. Table 6 contains the geometric properties of the structure.

The spatial variation of the boundary unknowns obtained by means of the random formulation are compared to the deterministic results on boundary 1. The frequency value is $\omega = 15$ rad/s.

Observation of Figure 10 allows the same conclusions as for the previous simulations. Indeed, the smoothing effect of the random formulation is once more illustrated for this particular example.

TABLE 5

Material properties of the two membranes

T (N/m)	η (%)	ρ (kg/m)
20	2	10

TABLE 6

Geometric properties of the two membranes

l_1 (m)	l_2 (m)	l_3 (m)	(x_f, y_f)
2	2	3	(1,1)

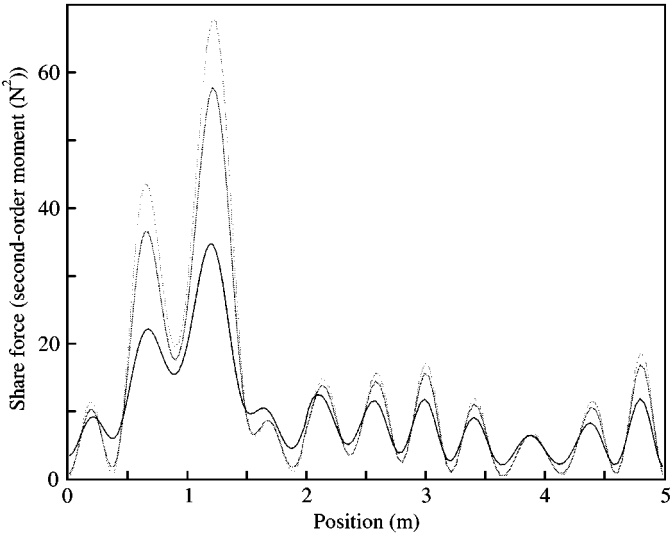

Figure 10. Two coupled membranes: Spatial variation of the second order moment of the force on boundary 1 for two coupled membranes. (·····) deterministic response versus SIF simulations, ——, $a = 0{\cdot}01$ (SIF-001) and $a = 0{\cdot}03$ (SIF-003).

5. CONCLUSION

The fundamentals of a novel formulation well adapted to wide frequency range investigation are defined in this paper, for one-, two- and three-dimensional systems. The SIF developed based on a stochastic description of the geometry of the structures. The output parameters are the second order statistical moments of the kinematic variables.

The major advantage of the SIF lies in the fact that this new formulation is valid for the whole frequency range. The SIF responses are able to describe the low-frequency modal behaviour of the structure with a precision close to the deterministic results, without being affected by the random description. In the high-frequency domain, the increasing complexity of the deterministic response is erased and the SIF only delivers the relevant information which is the global behaviour of the response of the structure. The middle

frequency field corresponds to a transition zone between the precise low-frequency results and the smooth and global high-frequency response. In conclusion, it is possible to state that the three major assumptions introduced in this paper give the possibility of building a unique formulation relevant on the whole frequency domain. This last point is one of the aspects of the theory. Until now, classical middle- and high-frequency formulations were based on hypotheses restricting the models to the high-frequency range [3, 11].

Another advantage of the formulation arises from the fact that the equations are based on the classical governing equations, which enables the writting of the usual boundary and assembly conditions in terms of displacements and forces. The derivation of coupling factors (such as for the SEA [3]) is not required for the random formulation.

Finally, the case of assembled structures made of substructures with very different modal densities can be handled with the SIF. Moreover, the differences between the substructures are detected by the formulation as it has been shown in the paper. This last point is very important since the usual high-frequency methods are unable to treat such problems.

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