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DOMINANT DYNAMIC CHARACTERISTICS OF BUILT-UP STRUCTURES

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An approach for evaluating the dominant dynamic characteristics of built-up structures is presented. Based on a division of substructures and couplings into two groups, primary and subordinate, the approach is established by treating the members of the former group as precisely as possible, whereas the descriptions of those in the latter group are simplified substantially. Two generic built-up structures are discussed to illustrate how the approach promotes the understanding of the dynamic behaviour of built-up structures and also to illustrate its application. A corroboration of the applicability is provided from comparisons with the measured response of a shell–beam–plate structure.

1. INTRODUCTION

Structural dynamic analyses of complex structures today tend to follow two routes; finite element analysis, FEA, e.g., reference [1] and statistical energy analysis, SEA, e.g. reference [2]. Although both the techniques have prominent advantages regarding their applicability, reliability and stability for a broad range of structures, there still remain some aspects with respect to practical needs. For example, the behaviour of a complex structure revealed by FEA can be too detailed to interpret, whereas that revealed by SEA too general for certain purposes. The key issue with respect to engineering practice is to extract the dominant dynamic characteristics and thence enable correct design strategies or modifications.

To tackle this, a different route, offering a compromise between detail and manageability, is worthy of investigation. Ideally, the compromise should be such that only the dominant behaviour of the structure is revealed, which is less detailed than that produced by FEA but more detailed than that established by SEA. In recent years, reports of such investigations have been presented. An example is fuzzy structural analysis [3–5]. The key feature of that analysis technique is to treat simple (or master) substructures deterministically or precisely but to treat complicated (or fuzzy) substructures (or a group of substructures) probabilistically or statistically. Thereby, a substructure is said to be simple (or master) if it can be precisely described by currently available methods, otherwise it is categorized as complicated (or fuzzy). Another slant of fuzzy structural analysis is given in reference [6] whereby combined wave fields in a structure are considered. Long wavelength fields are considered simple (or master) and treated deterministically whereas short wavelength fields are said to be complicated (or fuzzy) and handled statistically. Technically, fuzzy structural

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analysis aims at providing the optimal description available and thus would appear particularly suitable for solving specific problems. Philosophically, however, it can be argued that fuzzy structural analysis (with its original definition, see reference [3]) does not directly give an insight into the behaviour of complex structures. This is so since parts of the structure are given simplified descriptions on grounds that they are complicated, rather than them contributing little to the system characteristics. Therefore, such a fuzzy structural analysis is likely to be successful when the simple (or master) substructures coincide with those influential for the system behaviour. Otherwise, it essentially provides information similar to that of SEA. Nevertheless, fuzzy structural analysis has promoted yet a different way of thinking for investigating the behaviour of complex structures, in line with the arguments given in reference [7].

To gain insight into the dynamic behaviour of complex structures, the basis for employing a simplified description should be that the subsystem only in a subordinate manner influences the dynamics of the complete system. This subordinate influence could then either manifest itself as a mainly local effect or in the very details of the system dynamics or be altogether negligible. Such a basis would ensure that the resulting description reveals only the dominant or governing dynamic behaviour. Proposed in this paper is an approach to identify the primary and subordinate subsystems and a methodology for establishing associated simplified descriptions for the latter subsystems. The hypothesis underlying the approach is that for built-up structures, some substructures play major roles in the system behaviour whilst others have a peripheral influence. Moreover, the classifications of primary or subordinate are extended to encompass also the joints by which the substructures are connected. Finally, it is demonstrated that the approach furnishes the information sought for a class of built-up structures made up by elementary substructures, such as finite rods, beams, plates and cylindrical shells.

2. DESCRIPTION OF THE APPROACH

Figure 1 illustrates a structure consisting of several substructures. Each substructure can be properly represented by its uncoupled dynamic characteristics, denoted by e_i, $i = 1, 2, \ldots, N$ in the figure. Depending upon the primary aim of the analysis, e_i can be in the form of a frequency response function, conventionally the mobility, as energy stored in the substructure, as stiffness and mass matrices resulting from discretizations of the substructures or as a set of data obtained from measurements. The representations of the couplings between the substructures, denoted by cp_i, $i = 1, 2, \ldots, M$, can be derived from the conditions of continuity of motion or force equilibrium along interfaces, be given by ratios of contact impedance/mobility [8], or coupling loss factors [2].

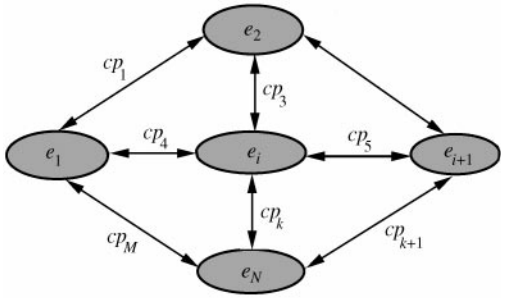

Figure 1. Built-up system with subsystems and couplings.

Similarly, the symbol S is used to represent the whole system characteristics sought, which can be the mobility, velocity/energy distribution or vibrational power transmission, etc. Once the system is assembled by imposing the chosen method from the representation of the couplings, S is expressed as a function of e_i, and cp_i (a list of symbols is given in Appendix C):

$$S = \varphi(e_1, e_2, \ldots, e_N, cp_1, cp_2, \ldots, cp_M). \tag{1}$$

The explicit form of the function $\varphi(\cdot)$ thus depends on the method by which the system is assembled. For instance, if the system is assembled by means of power balance between substructures, $\varphi(\cdot)$ is a linear algebraic operator. Equation (1) can be rewritten in compact form, as

$$S = \varphi(\mathbf{E}, \mathbf{CP}), \tag{2}$$

where $\mathbf{E} = \{e_i \,|\, i = 1, 2, \ldots, N\}$ and $\mathbf{CP} = \{cp_i \,|\, i = 1, 2, \ldots, M\}$ are the sets of substructure and coupling descriptions respectively.

By the hypothesis, the substructures and couplings are divided into primary and subordinate groups such that

$$\mathbf{E} = \mathbf{E}_{primary} \cup \mathbf{E}_{subordinate}, \quad \mathbf{CP} = \mathbf{CP}_{primary} \cup \mathbf{CP}_{subordinate}. \tag{3}$$

The issue of how to decide to which group a substructure or coupling belong will be dealt with in the next two sections, commenced with the couplings because the subsystem description is dependent on the type of coupling.

Whilst keeping precise representations of the primary substructures and couplings, those of subordinate substructures and couplings are simplified such that structural details contributing only to the fine details of the system behaviour are removed. Hence, an approach to the dominant dynamic characteristics of the system is expressed as

$$S^{Dominant} = \varphi(\mathbf{E}_{primary}, \mathbf{E}^{Simplified}_{subordinate}, \mathbf{CP}_{primary}, \mathbf{CP}^{Simplified}_{subordinate}). \tag{4}$$

In equation (4), the superscript "simplified" indicates the use of the simplified description for the subordinate substructures and couplings.

3. PRIMARY AND SUBORDINATE COUPLING

As stated, a coupling is classified as subordinate if its characteristics do not markedly influence the system response. To illustrate this it is appropriate to consider a single point and single component of motion connection, J, between two coupled substructures, as depicted in Figure 2.

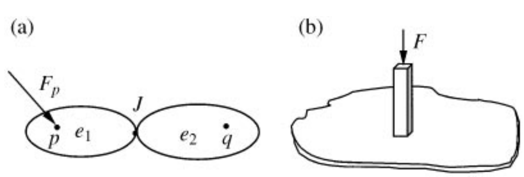

Figure 2. Two coupled substructures (a). A column-recipient seating structure (b).

To make the discussion explicit, the ratio of the uncoupled point mobilities of the two substructures at the contact point J is set to represent the coupling,

$$cp_{21} = Y^{(1)}_{JJ}/Y^{(2)}_{JJ}. \tag{5}$$

By means of this ratio, the coupling can be defined to be strong if $|cp_{21}| \approx 1$ and weak if either $|cp_{21}| \ll 1$ or $|cp^{-1}_{21}| \ll 1$.

The response of the system can now be formally written as $S = \varphi(e_1, e_2, cp_{21})$, and its Taylor series expansion for small $|cp_{21}|$ or $|cp^{-1}_{21}|$ is given by

$$S = (S)_{cp_{21}=0} + \left[\frac{\partial S}{\partial(cp_{21})}\right]_{cp_{21}=0} cp_{21} + \cdots \tag{6}$$

or

$$S = (S)_{cp^{-1}_{21}=0} + \left[\frac{\partial S}{\partial(cp^{-1}_{21})}\right]_{cp^{-1}_{21}=0} cp^{-1}_{21} + \cdots. \tag{7}$$

It should be noted that the above Taylor expansions must be carried out with respect to the variable of structure dynamics, in this case, the ratio of mobilities, and it should not be carried out with respect to a temporal or frequency variable, or spatial variables of co-ordinates, so as to make the expansion hold even for the case where S is presented in form of an integral or differential equation of time/frequency and spatial variables. The meaning of the first order approximations deserves some comments. Physically, $cp_{21} = 0$ occurs when substructure 1 acts as a constant velocity source to the receiving substructure 2. The approximation $(S)_{cp_{21}=0}$ can therefore be regarded as the description of the system under the, besides the actual boundary conditions, additional constraint that the velocity of the system at the contact point is given by that of the uncoupled driving substructure. The opposite case with $cp^{-1}_{21} = 0$, on the other hand, occurs when substructure 1 acts as a constant force source to the substructure 2. Therefore, $(S)_{cp^{-1}_{21}=0}$ should be regarded as the system description under the additional constraint of a blocked coupling point.

For $|cp_{21}|$ or $|cp^{-1}_{21}|$ substantially less than unity, it is generally valid to approximate the description S by the first term of the series provided this is not zero. Accordingly,

$$S \approx \begin{cases} (S)_{cp_{21}=0} & \text{for } |cp_{21}| \ll 1 \\ (S)_{cp^{-1}_{21}=0} & \text{for } |cp^{-1}_{21}| \ll 1. \end{cases} \tag{8}$$

Since this approximation is independent of cp_{21}, it is seen that the system characteristics are insensitive to the particulars of a weak coupling. In contrast, a valid approximation for S when the coupling is strong, i.e., $|cp_{21}| \approx 1$, must include several terms of the series expansion, which means that the system response is sensitive to the details of any strong coupling. Hence, it is established that any weak coupling can be treated as subordinate whereas a strong coupling should be considered as primary. This is analogous to the considerations for constant force and velocity source idealizations; see, e.g., reference [8].

Upon introducing this result in equation (4), the approach for evaluating the dominant dynamic characteristics formally becomes

$$S^{Dominant} = \varphi(\mathbf{E}_{primary}, \mathbf{E}^{Simplified}_{subordinate}, \mathbf{CP}_{primary}, \mathbf{CP}^{Simplified}_{subordinate} = 0) \tag{9a}$$

or

$$S^{Dominant} = \varphi(\mathbf{E}_{primary}, \mathbf{E}^{Simplified}_{subordinate}, \mathbf{CP}_{primary}, \mathbf{CP}^{Simplified}_{subordinate} = \infty) \tag{9b}$$

in the two situations respectively.

It should be noted, however, that when the system description, S, sought is the power transmission between the two substructures, the first order approximation of the response is always invalid. According to Mondot and Petersson [8], the complex power transmission $\underline{\Pi}$ between the two coupled substructures is expressed as

$$\underline{\Pi} = C_f \underline{S}_{rc}, \tag{10}$$

where $\underline{S}_{rc} = \frac{1}{2}|v_J^{(1)}|^2/Y_{JJ}^{(1)}$ is called source descriptor and depends only on the driving substructure data. $v_J^{(1)}$ is the free velocity of the driving substructure at the contact point J. The factor C_f is termed the coupling function and is, in the present notation, defined by

$$C_f = \begin{cases} cp_{21}^{*}/|1 + cp_{21}|^2, \\ cp_{21}^{-1}/|1 + cp_{21}^{-1}|^2. \end{cases} \tag{11}$$

It is readily seen that with the first order approximations introduced, the power transmission becomes $\Pi_{cp_{21}=0} = \Pi_{cp_{21}^{-1}=0} \equiv 0$, which obviously is not true for every case. Thus, even for weak coupling, a valid approximation of $\underline{\Pi}$ calls for more than one term of the series expansion. From this example, it is deduced that the suggested route for identifying a subordinate coupling is not applicable in SEA-like analyses.

4. PRIMARY AND SUBORDINATE SUBSYSTEMS

From linear theory, each subsystem can be completely represented dynamically by two parts of information: i.e., frequency response to excitation and spatial distribution of the response. This is for instance seen in the transfer mobility, which can be expressed as

$$Y_{qp} = T_{qp} Y_{pp}. \tag{12}$$

Herein, Y_{pp} is the point mobility at an arbitrary point p representing the frequency response of the structure to excitation and T_{qp} is the ratio of the velocity at another arbitrary point q to that at p, which represents the spatial distribution of the response. For convenience, T_{qp} is termed motion transmissibility. With the motion transmissibility involving only the zeroes of the subsystem characteristics but the point mobility in addition the poles, the former is markedly less sensitive to structural detail. Hence, if, in establishing the description of a complete assembled system, the motion transmissibility is sufficient for some subsystems, these subsystems are classified as subordinate.

In fact, a recent study [9] shows that the motion transmissibility for finite elementary structures, such as rods, beams, plates and cylindrical shells, can be estimated from the corresponding partial infinite structure. Hereby, one-dimensional systems are modelled as semi-infinite while two-dimensional ones rely upon quarter-infinite models. This means that for a system assembled from a number of elementary subsystems, the simplified description required for the subordinate ones can therefore readily be obtained from semi- or quarter-infinite models.

5. TWO ILLUSTRATIVE SYSTEMS

With the approach illustrated in section 2 and the methods for determining and simplifying the subordinate substructures and couplings in sections 3 and 4, two illustrative systems can be discussed. This also demonstrates the application of the approach. Unless specifically mentioned, the couplings in the systems to be discussed are assumed to be weak, point-like and involve only one component of motion.

5.1. TWO-ELEMENT SYSTEM

Consider a two-element system as shown in Figure 2(a). The column seating-recipient system shown in Figure 2(b) constitutes a practical counterpart. Moreover, the system description, S, considered in this example is the mobility.

For point-coupled subsystems, their uncoupled mobilities have convenient descriptions: i.e., e_1 is given by $Y_{ij}^{(1)}$ and e_2 by $Y_{ij}^{(2)}$. The coupling is assessed by the ratio $cp_{21} = Y_{JJ}^{(1)}/Y_{JJ}^{(2)}$. From this, the point mobility of the complete system at point p can be written as

$$Y_{pp} = Y_{pp}^{(1)}\left(1 - \frac{cp_{21}}{1 + cp_{21}}\, T_{Jp}^{(1)} T_{pJ}^{(1)}\right) \tag{13}$$

and the transfer mobility between points p and q as

$$Y_{qp} = Y_{Jp}^{(1)} T_{qJ}^{(2)} \frac{1}{1 + cp_{21}}. \tag{14}$$

Since $|cp_{21}| \ll 1$ or $|cp_{21}^{-1}| \ll 1$ as assumed, the first order approximation of the point mobility, Y_{pp}, is obtained from equation (13) as $Y_{pp} \approx Y_{pp}^{(1)}$, which involves only e_1. This means that for the evaluation of the point mobility, substructure 1 constitutes the primary substructure whereas substructure 2 is subordinate. By simplifying the description of substructure 2, in this case such that it is omitted, the dominant point mobility of the system is found to be given by

$$Y_{pp}^{Dominant} = Y_{pp}^{(1)}. \tag{15}$$

The first order approximation of the transfer mobility in the same situation is obtained from equation (14) as $Y_{qp} \approx Y_{Jp}^{(1)} T_{qJ}^{(2)}$. As seen, this approximation requires the mobility of substructure 1 and the motion transmissibility of substructure 2. Again, therefore, substructure 1 constitutes a primary subsystem and 2 is subordinate. By describing substructure 2 using the corresponding partial infinite structure, the dominant transfer mobility is obtained as

$$Y_{qp}^{Dominant} = Y_{Jp}^{(1)} T_{qJ}^{(2),\, Simplified}. \tag{16}$$

For the opposite situation whereby $|cp_{21}^{-1}| \ll 1$, the point and transfer mobilities can be expressed as

$$Y_{pp} = Y_{pp}^{(1)}\left(1 - \frac{1}{1 + cp_{21}^{-1}}\, T_{Jp}^{(1)} T_{pJ}^{(1)}\right) \tag{17}$$

and

$$Y_{qp} = T_{pJ}^{(1)} Y_{qJ}^{(2)} \frac{1}{1 + cp_{21}^{-1}} \tag{18}$$

respectively. Accordingly, with respect to the point mobility, substructure 1 is primary and 2 is subordinate. In contrast, however, substructure 2 is primary whilst 1 is subordinate with

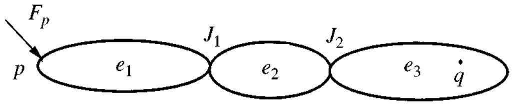

Figure 3. Three-element built-up structure.

respect to the transfer mobility. Thus, upon repeating the above analysis, the dominant point and transfer mobilities are found to be given by

$$Y_{pp}^{Dominant} = \tilde{Y}_{pp}^{(1)} \tag{19}$$

and

$$Y_{qp}^{Dominant} = T_{pJ}^{(1),Simplified} Y_{qJ}^{(2)}, \tag{20}$$

where $\tilde{Y}_{pp}^{(1)}$ is the point mobility of substructure 1 with the contact point blocked.

5.2. THREE-ELEMENT SYSTEMS

Figure 3 shows a three-element system. If F_{J_1} and F_{J_2} are the interface forces at joints J_1 and J_2, respectively, the point and transfer mobilities of the complete system are given by

$$Y_{pp} = Y_{pp}^{(1)} - Y_{pJ_1}^{(1)} F_{J_1}/F_p, \quad Y_{qp} = Y_{qJ_2}^{(3)} F_{J_2}/F_p. \tag{21a,b}$$

From continuity of velocity at the joints, the system of equations,

$$\begin{aligned} (Y_{J_1J_1}^{(1)} + Y_{J_1J_1}^{(2)})F_{J_1} - Y_{J_1J_2}^{(2)} F_{J_2} &= Y_{J_1p}^{(1)} F_p, \\ Y_{J_2J_1}^{(2)} F_{J_1} - (Y_{J_2J_2}^{(2)} + Y_{J_2J_2}^{(3)})F_{J_2} &= 0, \end{aligned} \tag{22}$$

can be solved and after some rearrangement the point mobility and transfer mobilities to some arbitrary point q on the third substructure are obtained for three different configurations as

(1)

$$Y_{pp} = Y_{pp}^{(1)} - \frac{cp_{21} Y_{pJ_1}^{(1)} T_{pJ_1}^{(1)}}{1 + cp_{21} - T_{J_2J_1}^{(2)} T_{J_1J_2}^{(2)} cp_{32}(cp_{32}+1)^{-1}}, \tag{23}$$

$$Y_{qp} = \frac{T_{qJ_2}^{(3)} T_{J_2J_1}^{(2)} Y_{J_1p}^{(1)}}{1 + cp_{21} + cp_{32} + cp_{21}cp_{32} - T_{J_2J_1}^{(2)} T_{J_1J_2}^{(2)} cp_{32}}, \tag{24}$$

for a configuration where $|cp_{21}| \ll 1$ and $|cp_{32}| \ll 1$,

(2)

$$Y_{pp} = Y_{pp}^{(1)} - \frac{Y_{pJ_1}^{(1)} T_{pJ_1}^{(1)}}{1 + cp_{21}^{-1}(1 - cp_{32}(1+cp_{32})^{-1} T_{J_2J_1}^{(2)} T_{J_1J_2}^{(2)})}, \tag{25}$$

$$Y_{qp} = \frac{T_{qJ_2}^{(3)} Y_{J_1J_2}^{(2)} T_{pJ_1}^{(1)}}{1 + cp_{21}^{-1} + cp_{32} + cp_{21}^{-1} cp_{32}(1 - T_{J_2J_1}^{(2)} T_{J_1J_2}^{(2)})}, \tag{26}$$

for a configuration where $|cp_{21}^{-1}| \ll 1$ and $|cp_{32}| \ll 1$ and

(3)

$$Y_{pp} = Y_{pp}^{(1)} - \frac{Y_{pJ_1}^{(1)} T_{pJ_1}^{(1)}}{1 + cp_{21}^{-1}[1 - T_{J_2J_1}^{(2)} T_{J_1J_2}^{(2)}(1 + cp_{32}^{-1})^{-1}]}, \tag{27}$$

$$Y_{qp} = \frac{Y_{qJ_1}^{(3)} T_{J_1J_2}^{(2)} T_{pJ_1}^{(1)}}{1 + cp_{21}^{-1} + cp_{32}^{-1} + cp_{21}^{-1} cp_{32}^{-1} - cp_{21}^{-1} T_{J_2J_1}^{(2)} T_{J_1J_2}^{(2)}}, \tag{28}$$

for a configuration in which $|cp_{21}^{-1}| \ll 1$ and $|cp_{32}^{-1}| \ll 1$.

By setting $cp_{21} = cp_{32} = 0$ for the first configuration, $cp_{21}^{-1} = cp_{32} = 0$ for the second and $cp_{21}^{-1} = cp_{32}^{-1} = 0$ for the third, the first order approximations can be obtained, from which the dominant point and transfer mobilities are established as

(1)

$$Y_{pp}^{Dominant} = Y_{pp}^{(1)}, \qquad Y_{qp}^{Dominant} = T_{qJ_2}^{(3),Simplified} T_{J_2J_1}^{(2),Simplified} Y_{J_1p}^{(1)}, \tag{29,30}$$

(ii)

$$Y_{pp}^{Dominant} = \tilde{Y}_{pp}^{(1)}, \qquad Y_{qp}^{Dominant} = T_{qJ_2}^{(3),Simplified} Y_{J_1J_2}^{(2)} T_{pJ_1}^{(1),Simplified}, \tag{31,32}$$

(iii)

$$Y_{pp}^{Dominant} = Y_{pp}^{(1)} - Y_{pJ_1}^{(1)} T_{pJ_1}^{(1)} = \tilde{Y}_{pp}^{(1)}, \tag{33}$$

$$Y_{qp}^{Dominant} = Y_{qJ_1}^{(3)} T_{J_1J_2}^{(2),Simplified} T_{pJ_1}^{(1),Simplified}. \tag{34}$$

This shows that for the first configuration, the first substructure plays a major role in determining the system behaviour. For the second configuration, the first substructure is more influential than the others when the point mobility is considered, whereas the second substructure takes the dominant role for the transfer mobility to any point on the third substructure. Finally, the first substructure is dominant for the point mobility, whereas the third substructure is primary for transfer mobility to any point on that substructure.

6. GENERALIZATION OF THE APPROACH

In order to generalize the approach, three important issues must be addressed. The first is the fact that the concept of weak coupling is frequency dependent. For instance, a coupling which is weak in one frequency band can be strong in another. This means that the nature of the coupling cannot be assessed for a wide frequency range unless the criterion for weak coupling, introduced in section 3, is modified such that it is less rapidly dependent upon frequency. The second issue is multi-point connections between subsystems since, for such, the coupling is no longer described by a single quantity. The third stems from the fact that the coupling usually involves multiple components of motion and excitation whereby, in addition to multiple coupling quantities, these are generally dimensionally dissimilar.

6.1. WEAK COUPLING FOR BROADBAND FREQUENCY RANGE

To generalize the criterion of weak coupling, equation (6) is revisited. The deviation of the first order approximation from the complete response can be written as

$$\Delta S = S - S_{cp_{21}=0} = \left[\frac{\partial S}{\partial(cp_{21})}\right]_{cp_{21}=0} cp_{21} + \cdots. \tag{35}$$

As cp_{21} varies in magnitude with frequency between its upper and lower limits, there is a corresponding variation in the deviation ΔS. This variation, due to the system resonances and antiresonances, can be expressed by local variations superimposed on a global variation, such that $\Delta S = (\Delta S)_G(\Delta S)_L$. The local variation $(\Delta S)_L$, encompasses the details of the response of the system and is disregarded in the present approach. The global variation, $(\Delta S)_G$, on the other hand, comprises the fundamental features of interest herein and should therefore be thoroughly considered. When $(\Delta S)_G$ is small, it can be argued that the first order approximation $(S)_{cp_{21}=0}$ is valid in a global sense and can be used to evaluate the dominant response of the system. $(\Delta S)_G$ generally varies slowly with frequency and it is reasonable to assume that sufficiently broad regions of weak coupling can be identified.

A suitable measure of the global variation can be found from the geometric average of the upper and lower variation limits. The feature of the geometric average is that the peaks and troughs in the variation are equally accounted for. By invoking Skudrzyk's result that the geometric average of adjacent maxima and minima of the point mobility is equal to the characteristic mobility (17), i.e., $\sqrt{Y_{max}Y_{min}} \approx Y_\infty$, the global coupling strength parameter $(cp_{21})_G$, can be given by

$$(cp_{21})_G = \sqrt{|cp_{21}|_{max}|cp_{21}|_{min}} = \sqrt{\left|\frac{Y^{(1)}_{max}}{Y^{(2)}_{min}}\right| \cdot \left|\frac{Y^{(1)}_{min}}{Y^{(2)}_{max}}\right|} = \frac{Y^{(1)}_\infty}{Y^{(2)}_\infty} = cp^\infty_{21}. \tag{36}$$

For $cp^\infty_{21} \ll 1$, the higher order terms on the right-hand side of equation (35) can be neglected, so that the global variation is approximated by

$$(\Delta S)_G \approx \left(\left[\frac{\partial S}{\partial(cp_{21})}\right]_{cp_{21}=0}\right)_G (cp_{21})_G. \tag{37}$$

As is seen, $(\Delta S)_G$ is negligible when $(cp_{21})_G$ is sufficiently small. Therefore, a criterion for determining weak coupling for a wide frequency range can be formulated as $(cp_{21})_G < \varepsilon$, where ε is a small number.

Repeating the above analysis with equation (7) yields a similar criterion for the complementary situation in which the coupling is weak whereby $(cp_{21})_G^{-1} < \varepsilon$.

6.2. MULTIPLE POINTS AND SINGLE COMPONENT OF MOTION AND EXCITATION

Figure 4 illustrates a configuration with a multi-point connection between the subsystems. In such a case, the coupling strength parameter, cp_{21}, becomes a matrix $\mathbf{cp}_{21} = [Y^{(2)}_{ij}]^{-1}[Y^{(1)}_{ij}]$, where $Y^{(1)}_{ij}$ and $Y^{(2)}_{ij}$, respectively, are the uncoupled mobilities of the substructures 1 and 2, along the interface. When the transmission is confined to one component of motion and excitation, the geometric averages of the maximum and minimum envelopes to the point mobilities for any point along the interface are identical: i.e., $(Y^{(1)}_{ii})_G = Y^{(1)}_\infty$ and $(Y^{(2)}_{ii})_G = Y^{(2)}_\infty$, $i = 1, 2, 3, \ldots$. If κ is used to denote the spatial variation of the point mobility, one may write $Y^{(1)}_{ii} = Y^{(1)}_\infty \kappa^{(1)}_{ii}$ and $Y^{(2)}_{ii} = Y^{(2)}_\infty \kappa^{(2)}_{ii}$. Then, the

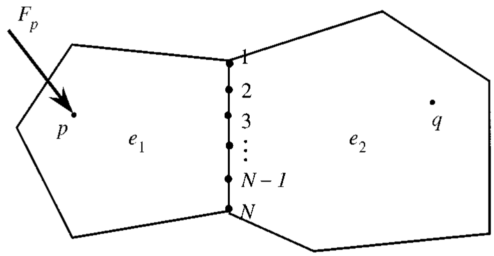

Figure 4. A two-element structure with the internal forces at a junction indicated.

coupling matrix $\mathbf{cp}_{21} = cp_{21}^{\infty}[T_{ij}^{(2)}\kappa_{jj}^{(2)}]^{-1}[T_{ij}^{(1)}\kappa_{jj}^{(1)}]$. Since the "point" motion transmissibility is unity, $T_{ii} \equiv 1$, and $(T_{ij})_G = T_{ij}^{\infty} < 1$ for $i \neq j$ owing to the fact that for physical systems the waves are attenuated with acoustic distance between the points i and j (see Appendix A), it can be argued that in a global sense, the off-diagonal elements of the matrix $[T_{ij}^{(2)}\kappa_{jj}^{(2)}]^{-1}[T_{ij}^{(1)}\kappa_{jj}^{(1)}]$ are smaller in magnitude than the diagonal. This effect is also enhanced the higher the frequency. Therefore, the coupling matrix may be estimated from $\mathbf{cp}_{21} \approx cp_{21}^{\infty}\,\mathrm{diag}[\kappa_{jj}^{(1)}/\kappa_{jj}^{(2)}]$.

To iillustrate explicitly the influence of the number of connection points on the global variation, $(\Delta S)_G$, the transfer mobility of the system to some arbitrary point q on substructure 2 can be considered. This is given by

$$Y_{qp} = (\mathbf{y}_{qJ}^{(2)})^{\mathrm{T}}(\mathbf{I} + \mathbf{cp}_{21})^{-1}(\mathbf{Y}^{(2)})^{-1}\mathbf{y}_{Jp}^{(1)}, \tag{38}$$

where $\mathbf{Y}^{(2)} = [Y_{ij}^{(2)}]$, $\mathbf{y}_{qJ}^{(2)} = \{Y_{q1}^{(2)}\ Y_{q2}^{(2)}\ \cdots\ Y_{qN}^{(2)}\}^{\mathrm{T}}$ and $\mathbf{y}_{Jp}^{(1)} = \{Y_{1p}^{(1)}\ Y_{2p}^{(1)}\ \cdots\ Y_{Np}^{(1)}\}^{\mathrm{T}}$. By making use of the expansion $(\mathbf{I} + \mathbf{cp}_{21})^{-1} = \mathbf{I} - \mathbf{cp}_{21} + \cdots$, the transfer mobility can be expanded as

$$Y_{qp} = (Y_{qp})_{\mathbf{cp}_{21}=[0]} - (\mathbf{y}_{qJ}^{(2)})^{\mathrm{T}}\mathbf{cp}_{21}(\mathbf{Y}^{(2)})^{-1}\mathbf{y}_{Jp}^{(1)} + \cdots, \tag{39}$$

where $(Y_{qp})_{\mathbf{cp}_{21}=[0]} = (\mathbf{y}_{qJ}^{(2)})^{\mathrm{T}}(\mathbf{Y}^{(2)})^{-1}\mathbf{y}_{Jp}^{(1)}$ is the first order approximation of Y_{qp}. The deviation, ΔS, is thus approximated by

$$\Delta S = -(\mathbf{y}_{qJ}^{(2)})^{\mathrm{T}}\mathbf{cp}_{21}(\mathbf{Y}^{(2)})^{-1}\mathbf{y}_{Jp}^{(1)} + \cdots. \tag{40}$$

For $cp_{21}^{\infty} \ll 1$, the higher order terms on the right-hand side of equation (40) can be neglected, leading to

$$\Delta S \approx -cp_{21}^{\infty}(\mathbf{y}_{qJ}^{(2)})^{\mathrm{T}}\,\mathrm{diag}(\kappa_{jj}^{(1)}/\kappa_{jj}^{(2)})(\mathbf{Y}^{(2)})^{-1}\mathbf{y}_{Jp}^{(1)}. \tag{41}$$

Furthermore, with the argument employed above, it is found that $\mathbf{Y}^{(2)} = Y_{\infty}^{(2)}[T_{ij}^{(2)}\kappa_{jj}^{(2)}] \approx Y_{\infty}^{(2)}\,\mathrm{diag}[\kappa_{jj}^{(2)}]$. Hence,

$$\Delta S \approx -cp_{21}^{\infty}Y_{\infty}^{(1)}(\{T_{qi}^{(2)}\kappa_{ii}^{(2)}\}^{\mathrm{T}}\,\mathrm{diag}[\kappa_{jj}^{(1)}/\kappa_{jj}^{(2)}]\,\mathrm{diag}[1/\kappa_{jj}^{(2)}]\{T_{ip}^{(1)}\kappa_{ii}^{(1)}\}). \tag{42}$$

Since both the motion transmissibilities, $(T_{qi})_G$ and $(T_{ip})_G$, are less than unity, the global value of the term within brackets should not be greater than unity. Thus, the global variation can be assessed from

$$(\Delta S)_G \leqslant cp_{21}^{\infty}Y_{\infty}^{(1)}. \tag{43}$$

As in the single-point case, the variation is negligible when the "characteristic" coupling strength, cp_{21}^{∞}, is sufficiently small. This indicates that for single components of excitation and motion, a multi-point interface behaves the same as the single point.

6.3. SINGLE POINT AND TWO COMPONENTS OF MOTION AND EXCITATION

The particulars of multi/component coupling can, for transparency, be examined by considering a single-point interface with two components of motion and excitation. If u, v are two components of motion involved in the transmission and F, M the associated (generalized) forces, then, with the mobility matrix

$$\mathbf{Y}_{ij} = \begin{bmatrix} Y_{uF,ij} & Y_{uM,ij} \\ Y_{vF,ij} & Y_{vM,ij} \end{bmatrix}, \tag{44}$$

the coupling strength parameter is given by $\mathbf{cp}_{21} = (\mathbf{Y}_{JJ}^{(2)})^{-1}\mathbf{Y}_{JJ}^{(1)}$. For a combined excitation, $\{F_p, M_p\}$, applied at p, the motion at point q can be written as

$$\begin{Bmatrix} u_q \\ v_q \end{Bmatrix} = \mathbf{Y}_{qJ}^{(2)}(\mathbf{I} + \mathbf{cp}_{21})^{-1}(\mathbf{Y}_{JJ}^{(2)})^{-1}\mathbf{Y}_{Jp}^{(1)}\begin{Bmatrix} F_p \\ M_p \end{Bmatrix}. \tag{45}$$

With the Taylor series expansion of $(\mathbf{I} + \mathbf{cp}_{21})^{-1}$, the deviation of the first order approximation from the complete response is obtained as

$$\Delta\left(\begin{bmatrix} u_q \\ v_q \end{bmatrix}\right) = -\,\mathbf{Y}_{qJ}^{(2)}\mathbf{cp}_{21}(\mathbf{Y}_{JJ}^{(2)})^{-1}\mathbf{Y}_{Jp}^{(1)}\begin{bmatrix} F_p \\ M_p \end{bmatrix} + \cdots. \tag{46}$$

For an assessment of this deviation, the point and transfer mobilities for each substructure are assumed to have the same spatial variation, an assumption which is likely to be fulfilled only in an overall sense. This means that $\mathbf{Y}^{(1)} = \mathbf{Y}^{(1\,\infty)}\kappa^{(1)}$ and $\mathbf{Y}^{(2)} = \mathbf{Y}^{(2\,\infty)}\kappa^{(2)}$, so that $\mathbf{cp}_{21} = \mathbf{cp}_{21}^{\infty}(\kappa^{(1)}/\kappa^{(2)})$, wherein $\mathbf{cp}_{21}^{\infty} = (\mathbf{Y}^{(2\,\infty)})^{-1}\mathbf{Y}^{(1\,\infty)}$ and

$$\mathbf{Y}^{(1\,\infty)} = \begin{bmatrix} Y_{uF}^{(1\,\infty)} & Y_{uM}^{(1\,\infty)} \\ Y_{vF}^{(1\,\infty)} & Y_{vM}^{(1\,\infty)} \end{bmatrix}, \quad \mathbf{Y}^{(2\,\infty)} = \begin{bmatrix} Y_{uF}^{(2\,\infty)} & Y_{uM}^{(2\,\infty)} \\ Y_{vF}^{(2\,\infty)} & Y_{vM}^{(2\,\infty)} \end{bmatrix}. \tag{47}$$

Accordingly, the deviation is approximated by

$$\Delta\left(\begin{bmatrix} u_q \\ v_q \end{bmatrix}\right) \approx -\,\mathbf{Y}_{qJ}^{(2\,\infty)}\mathbf{cp}_{21}^{\infty}(\mathbf{Y}_{JJ}^{(2\,\infty)})^{-1}\mathbf{Y}_{Jp}^{(1\,\infty)}\begin{bmatrix} F_p \\ M_p \end{bmatrix}(\kappa^{(1)}/\kappa^{(2)})\kappa^{(1)} + \cdots. \tag{48}$$

Thus, if $\mathbf{cp}_{21}^{\infty}$ makes the global value of the deviation negligible, the coupling can be assumed to be weak and classified as subordinate. This occurs when each element of the matrix $\mathbf{cp}_{21}^{\infty}$ is sufficiently small.

The elements of $\mathbf{cp}_{21}^{\infty}$ are given by

$$\mathbf{cp}_{21}^{\infty} = \begin{bmatrix} \dfrac{1 - r_F^{(1\,\infty)} r_M^{(2\,\infty)}}{1 - r_F^{(2)} r_F^{(2)}}\, cp_{uu}^{\infty} & \dfrac{r_M^{(1\,\infty)} - r_M^{(2\,\infty)}}{1 - r_F^{(2)} r_F^{(2)}}\, cp_{vu}^{\infty} \\[2ex] \dfrac{r_F^{(1\,\infty)} - r_F^{(2\,\infty)}}{1 - r_F^{(2\,\infty)} r_F^{(2\,\infty)}}\, cp_{uv}^{\infty} & \dfrac{1 - r_M^{(1\,\infty)} r_F^{(2\,\infty)}}{1 - r_F^{(2\,\infty)} r_F^{(2\,\infty)}}\, cp_{vv}^{\infty} \end{bmatrix}, \tag{49}$$

where the coupling strength parameters, $cp_{uu}^{\infty} = Y_{uF}^{(1\,\infty)}/Y_{uF}^{(2\,\infty)}$ and $cp_{vv}^{\infty} = Y_{vM}^{(1\,\infty)}/Y_{vM}^{(2\,\infty)}$ represent the *direct coupling* for the same motion component at the two sides of the interface and $cp_{uv}^{\infty} = Y_{uF}^{(1\,\infty)}/Y_{vM}^{(2\,\infty)}$ and $cp_{vu}^{\infty} = Y_{vM}^{(1\,\infty)}/Y_{uF}^{(2\,\infty)}$ represent the *cross-coupling* between two different motion components on either side of the interface. The mobility ratios

$r_F^\infty = Y_{vF}^\infty / Y_{uF}^\infty$ and $r_M^\infty = Y_{uM}^\infty / Y_{vM}^\infty$ are introduced to assess the significance of the secondary motion component induced by each excitation respectively.

The values of cp_{uu}^∞, cp_{uv}^∞, cp_{vu}^∞ and cp_{vv}^∞ are generally different since each component of motion and excitation is associated with different characteristic mobilities. This means that a collective criterion for assessing the coupling strength over a frequency band is not readily developed. Conservatively, therefore, it is proposed that all "characteristic" coupling strength parameters simultaneously should be sufficiently small for the interface to be classified as subordinate. In cases where the cross-coupling between the components is negligible, it is sufficient that cp_{uu}^∞ and cp_{vv}^∞ are sufficiently small.

By expanding equation (45) into the Taylor series with respect to $\mathbf{cp}_{21}^{-1}$ and repeating the analysis, a similar criterion for the other case of subordinate coupling is established.

If all the coupling strength parameters are close to unity, the global value of the deviation will not be negligible and thus the coupling should be classified as primary. If the coupling is such that only some of the parameters are close to unity, no manageable criterion can be derived for determining whether that coupling should be classified as primary or subordinate and for a reliable description of the system, it is referred to the primary group.

7. COMPARISONS WITH MEASUREMENTS

Comparisons were made with measured results for a system consisting of a cylindrical shell (length 700 mm, radius 128 mm, wall thickness 10 mm), two beams (length 750 mm, width 32 mm, thickness 20 mm) and a plate (length 1410 mm, width 900 mm, thickness 10 mm), all made of Perspex; see Figure 5. The shell, the beams and the plate were point-connected by means of bolts. The whole structure is placed on pieces of foam along two opposite edges of the plate and excited by a vertical force at point 1 on the shell, covering a frequency range of up to 6·4 kHz. The transfer mobility of the system to point 2 on the plate is measured. The density, Young's modulus and loss factor of Perspex were experimentally obtained as $\rho = 1041$ kg/m^3, $E = 4{\cdot}4$ GN/m^2 (by cantilever method), and $\eta = 0{\cdot}07$ (by half-power bandwidth method with a free–free Perspex beam).

With a vertical, external excitation, the transmission at low frequencies involves essentially the vertical components of motion and force at each joint. Although, at high frequencies, rotations and moments increase in importance, this case was not considered for brevity. Hence, by using well-established formulae for the characteristic mobilities of beams, plates and shells (12, 17), $cp_{shell\text{-}beam}^\infty$ and $cp_{beam\text{-}plate}^\infty$ are calculated. From this calculation, two frequency ranges are distinguished where all couplings in the system are weak in an overall sense, one between 200 and 1400 Hz, for which both the "characteristic" coupling strength parameters are well below unity, and the other between 3·2 and 6·4 kHz for which $cp_{shell\text{-}beam}^\infty$ is well above unity whereas $cp_{beam\text{-}plate}^\infty$ is well below.

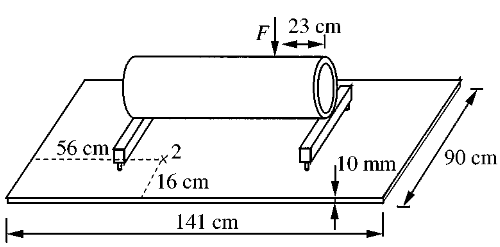

Figure 5. Configuration of the shell–beam–plate system.

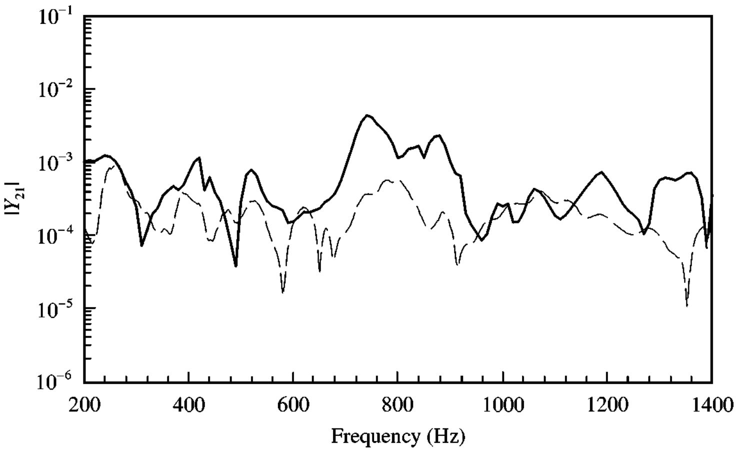

Figure 6. Transfer mobility between points 1 and 2 of the shell–beam–plate system: ——, predicted; ----, measured mobility.

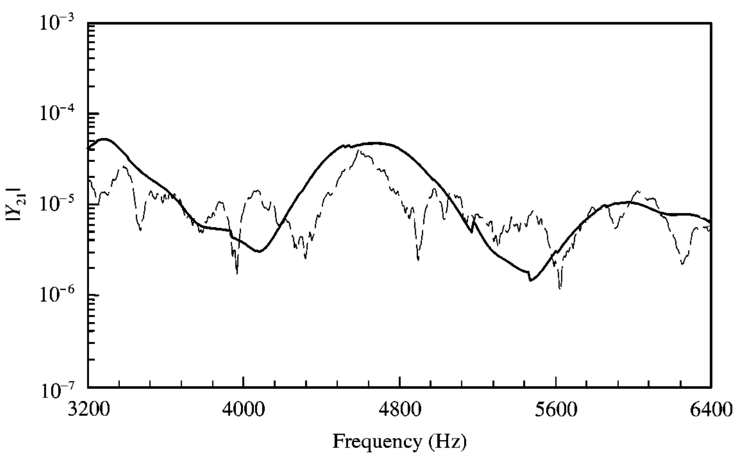

Figure 7. Transfer mobility between points 1 and 2 of the shell–beam–plate system: ——, predicted; ----, measured mobility.

Shown in Figures 6 and 7 are the comparisons between the predicted transfer mobility and the measured one. Due to the difficulties in obtaining accurate dynamic characteristics from simplified shell theories below the ring frequency at about 2500 Hz, the mobility for the first frequency range was obtained by considering the shell a primary substructure and describing it by using a set of experimental data, obtained for the freely suspended shell. In contrast, the beams and plate were set to be subordinate substructures such that they can be described by semi-infinite beam and quarter-infinite plate models respectively (see Appendix B). For the second frequency range within which the coupling can be considered weak, following the previous sections, the predicted transfer mobility was obtained by considering the shell and the plate subordinate and they were accordingly modelled as a semi-infinite shell and a quarter-infinite plate respectively. In this range, the beams were found to be primary substructures and described by Euler–Bernoulli beam theory (see Appendix B).

From Figures 6 and 7 it is seen that for both frequency ranges, the predicted curves capture the major trends with respect to global maxima and minima. The main discrepancy can be observed in the first frequency range at around 800 Hz, where the predicted level is an order of magnitude higher than measured. Upon revisiting the mobilities, however, the dynamic characteristics of the beams and the shell approach a matched condition in a narrowband centred at 820 Hz, i.e., $cp_{shell\text{-}beam} \rightarrow 1$ and thus, those couplings should therein strictly be regarded primary. This observation, in a reciprocal way, corroborates the applicability of the first order approximation to the dynamic characteristics of subordinate subsystems.

8. CONCLUDING REMARKS

A procedure to extract the dominant dynamic characteristics of a built-up structure has been described. It can be concluded that for built-up structures, assembled from dynamically weakly coupled, elementary substructures (e.g., rods, beams, plates and cylindrical shells), some of the substructures and couplings play major roles for the system behaviour whilst others have a secondary influence. The former are therefore termed and treated as primary and the latter termed subordinate and given simplified descriptions. Subordinate substructures can be identified in that only their motion transmissibilities are involved in the description of the complete system. This means that information of the vibration distribution is sufficient. Moreover, subordinate couplings are weak which means that the dynamic characteristics of the subsystems connected at such coupling are widely mismatched. For broadband excitation, the coupling strength is assessed from the geometric average of the upper and lower envelopes to the dynamic characteristics.

Multi-point coupling for one component of motion and excitation, behaves as single-point, single-component coupling. When a coupling involves multiple components of motion and excitation; however, no single value parameter has been found for the assessment of the coupling strength. Therefore, it is tentatively concluded that such couplings can be considered subordinate only when all direct and cross-couplings are weak simultaneously.

The dominating dynamic characteristics of a built-up structure can be obtained from the first order approximation of the system description with respect to weakly coupled subsystems and, at the same time, employing simplified descriptions for the subordinate subsystems. Thereby, simplified descriptions can be established from models based on the corresponding, partial infinite system which, for one-dimensional ones, is of semi-infinite extent and, for two-dimensional ones, is of quarter-infinite extent.

Finally, it can be concluded that this approach is not applicable for energetic formulations because the first order approximations of such system descriptions are invalid.

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APPENDIX A: EVALUATION OF $(T_{ij})_G$

For elementary structures, such as rods, beams, plates and shells, the motion transmissibility T_{ij} can be estimated from the corresponding partial infinite structures [9]. If the ratio of the reflected wave at an arbitrary point i to the direct wave at the same location is given by R, then

$$T_{ij} \approx \frac{v_{ij}^{\infty} + R_j v_{ij}^{\infty}}{v_{jj}^{\infty} + R_i v_{jj}^{\infty}} = T_{ij}^{\infty} \, \frac{1 + R_j}{1 + R_i}. \tag{A.1}$$

Thus,

$$(T_{ij})_G \approx T_{ij}^{\infty} \sqrt{\left|\frac{1 + R_j}{1 + R_i}\right|_{max} \left|\frac{1 + R_j}{1 + R_i}\right|_{min}} = T_{ij}^{\infty} \sqrt{\left|\frac{1 + |R_j|}{1 - |R_i|}\right| \cdot \left|\frac{1 - |R_j|}{1 + |R_i|}\right|} = T_{ij}^{\infty} \sqrt{\frac{1 - |R_j|^2}{1 - |R_i|^2}}. \tag{A.2}$$

Since $|R_i| < 1$ and $|R_j| < 1$ due to absorption at the boundaries, dissipation and wave divergence, the approximation $\sqrt{(1 - |R_j|^2)/(1 - |R_i|^2)} \approx 1$ can be employed, so that

$$(T_{ij})_G \approx T_{ij}^{\infty}. \tag{A.3}$$

APPENDIX B: TRANSFER MOBILITY OF BEAM–PLATE–SHELL STRUCTURES

With reference to the system depicted in Figure 5, the subscripts U and L are introduced to denote the interface between the shell and the beams and that between the beams and the plate respectively. From continuity of velocity at the joints, two equations can be established for the interconnection forces along the U and L interfaces:

$$\begin{aligned} (\mathbf{Y}_{UU}^{shell} + \mathbf{Y}_{UU}^{beam})\mathbf{F}_U - \mathbf{Y}_{UL}^{beam}\mathbf{F}_L &= \mathbf{y}_{Up}^{shell} F_p, \\ \mathbf{Y}_{LU}^{beam}\mathbf{F}_U - (\mathbf{Y}_{LL}^{shell} + \mathbf{Y}_{LL}^{plate})\mathbf{F}_L &= 0, \end{aligned} \tag{B.1}$$

where $\mathbf{y}_{UU}^{shell} = \{Y_{U_i p}^{shell}\}$, $\mathbf{Y}_{UU}^{shell} = [Y_{U_i U_j}^{shell}]$, $\mathbf{Y}_{UU}^{beam} = [Y_{U_i U_j}^{beam}]$, $\mathbf{Y}_{UL}^{beam} = [Y_{U_i L_j}^{beam}]$, $\mathbf{Y}_{LU}^{beam} = [Y_{L_i U_j}^{beam}]$, $\mathbf{Y}_{LL}^{beam} = [Y_{L_i L_j}^{beam}]$, $\mathbf{Y}_{LL}^{plate} = [Y_{L_i L_j}^{plate}]$. By describing the couplings between the shell and beam using $\mathbf{cp}_{21} = (\mathbf{Y}_{UU}^{beam})^{-1}\mathbf{Y}_{UU}^{shell}$, and that between the beams and the plate by using $\mathbf{cp}_{32} = (\mathbf{Y}_{LL}^{plate})^{-1}\mathbf{Y}_{LL}^{beam}$, the first order approximation of the forces at the lower interface, $\mathbf{F}_L$, for the first frequency range is found to be

$$\begin{aligned} \mathbf{F}_L &= [(\mathbf{cp}_{21} + \mathbf{I})(\mathbf{Y}_{LU}^{beam})^{-1}\mathbf{Y}_{LL}^{plate}(\mathbf{cp}_{32} + \mathbf{I}) - (\mathbf{Y}_{UU}^{beam})^{-1}\mathbf{Y}_{UL}^{beam}]^{-1}(\mathbf{Y}_{UU}^{beam})^{-1}\mathbf{y}_{Up}^{shell}F_p \\ &\approx [(\mathbf{Y}_{LU}^{beam})^{-1}\mathbf{Y}_{LL}^{plate} - (\mathbf{Y}_{UU}^{beam})^{-1}\mathbf{Y}_{UL}^{beam}]^{-1}(\mathbf{Y}_{UU}^{beam})^{-1}\mathbf{y}_{Up}^{shell}F_p \end{aligned}$$

$$= [\mathbf{I} - \mathbf{cp}_{32}(\mathbf{Y}_{LL}^{beam})^{-1}\mathbf{Y}_{LU}^{beam}(\mathbf{Y}_{UU}^{beam})^{-1}\mathbf{Y}_{UL}^{beam}]^{-1}(\mathbf{Y}_{LL}^{plate})^{-1}\mathbf{Y}_{LU}^{beam}(\mathbf{Y}_{UU}^{beam})^{-1}\mathbf{y}_{Up}^{shell}F_p$$

$$\approx (\mathbf{Y}_{LL}^{plate})^{-1}\mathbf{Y}_{LU}^{beam}(\mathbf{Y}_{UU}^{beam})^{-1}\mathbf{y}_{Up}^{shell}F_p \tag{B.2}$$

and

$$\mathbf{F}_L = [(\mathbf{cp}_{21}^{-1} + \mathbf{I})(\mathbf{Y}_{LU}^{beam})^{-1}\mathbf{Y}_{LL}^{plate}(\mathbf{cp}_{32} + \mathbf{I}) - (\mathbf{Y}_{UU}^{shell})^{-1}\mathbf{Y}_{UL}^{beam}]^{-1}(\mathbf{Y}_{UU}^{shell})^{-1}\mathbf{y}_{Up}^{shell}F_p$$

$$\approx [(\mathbf{Y}_{LU}^{beam})^{-1}\mathbf{Y}_{LL}^{plate} - (\mathbf{Y}_{UU}^{shell})^{-1}\mathbf{Y}_{UL}^{beam}]^{-1}(\mathbf{Y}_{UU}^{shell})^{-1}\mathbf{y}_{Up}^{shell}F_p$$

$$= [\mathbf{I} - \mathbf{cp}_{32}(\mathbf{Y}_{LL}^{beam})^{-1}\mathbf{Y}_{LU}^{beam}(\mathbf{Y}_{UU}^{shell})^{-1}\mathbf{Y}_{UL}^{beam}]^{-1}(\mathbf{Y}_{LL}^{plate})^{-1}\mathbf{Y}_{LU}^{beam}(\mathbf{Y}_{UU}^{shell})^{-1}\mathbf{y}_{Up}^{shell}F_p$$

$$\approx (\mathbf{Y}_{LL}^{plate})^{-1}\mathbf{Y}_{LU}^{beam}(\mathbf{Y}_{UU}^{shell})^{-1}\mathbf{y}_{Up}^{shell}F_p \tag{B.3}$$

for the second.

Accordingly, the first order approximation of the transfer mobility for the first frequency range is obtained as

$$\mathbf{Y}_{qp} \approx (\mathbf{y}_{qL}^{plate})^{\mathrm{T}}(\mathbf{Y}_{LL}^{plate})^{-1}\mathbf{Y}_{LU}^{beam}(\mathbf{Y}_{UU}^{beam}))^{-1}\mathbf{y}_{Up}^{shell}$$

$$= \{(\mathbf{t}_{qL}^{plate})^{\mathrm{T}}(\mathbf{T}_{LL}^{plate})^{-1}\}\{\mathbf{T}_{LU}^{beam}(\mathbf{T}_{UU}^{beam})^{-1}\}\,\mathbf{y}_{Up}^{shell} \tag{B.4}$$

and

$$\mathbf{Y}_{qp} \approx (\mathbf{y}_{qL}^{plate})^{\mathrm{T}}(\mathbf{Y}_{LL}^{plate})^{-1}\mathbf{Y}_{LU}^{beam}(\mathbf{Y}_{UU}^{shell})^{-1}\mathbf{y}_{Up}^{shell}$$

$$= \{(\mathbf{t}_{qL}^{plate})^{\mathrm{T}}(\mathbf{T}_{LL}^{plate})^{-1}\}\{\mathbf{Y}_{LU}^{beam}\}\{((\mathbf{T}_{UU}^{shell})^{\mathrm{T}})^{-1}\mathbf{t}_{pU}^{shell}\} \tag{B.5}$$

for the second. Herein, $\mathbf{t}_{qL}^{plate} = \{T_{qL_i}^{plate}\}$, $\mathbf{T}_{LL}^{plate} = [T_{L_iL_j}^{plate}]$, $\mathbf{T}_{LU}^{beam} = [T_{L_iU_j}^{beam}]$, $\mathbf{T}_{UU}^{beam} = [T_{U_iU_j}^{beam}]$, $\mathbf{T}_{UU}^{shell} = [T_{U_iU_j}^{shell}]$ and $\mathbf{t}_{pU}^{shell} = \{T_{pU_i}^{shell}\}$. Since the system is excited vertically, the joints predominantly at low frequencies transmit the flexural waves in translation. Invoking the discussions of multi-point couplings via one component of motion (section 6.2), the dominating part of the transfer mobility can, for the first frequency range, be obtained from

$$Y_{qp}^{Dominating} = ((\mathbf{t}_{qL}^{plate,simplified})^{\mathrm{T}}(\mathbf{T}_{LL}^{plate,simplified})^{-1})\,(\mathbf{T}_{LU}^{beam,simplified}(\mathbf{T}_{UU}^{beam,simplified})^{-1})\,\mathbf{y}_{Up}^{shell} \tag{B.6}$$

and, in the second, from

$$Y_{qp}^{Dominating} = ((\mathbf{t}_{qL}^{plate,simplified})^{\mathrm{T}}(\mathbf{T}_{LL}^{plate,simplified})^{-1})\,\mathbf{Y}_{LU}^{beam}\,(((\mathbf{T}_{UU}^{shell,simplified})^{\mathrm{T}})^{-1}\mathbf{t}_{pU}^{shell,simplified}). \tag{B.7}$$

APPENDIX C: NOMENCLATURE

cp_i	coupling strength parameter for ith coupling
$\mathbf{CP} = \{cp_i \mid i = 1, 2, \dots\}$	set of coupling strength parameters
$\mathbf{CP}_{primary}$	set of primary couplings
$\mathbf{CP}_{subordinate}$	set of subordinate couplings
e_i	ith subsystem description
$\mathbf{E} = \{e_i \mid i = 1, 2, \dots\}$	set of subsystem descriptions
$\mathbf{E}_{primary}$	primary subsystem descriptions
$\mathbf{E}_{subordinate}$	subordinate subsystem descriptions
S	system characteristics
ΔS	deviation between approximate and exact characteristics
T_{qp}	motion transmissibility from point p to q
$\mathbf{y}$	column vector of mobilities
Y_{qp}	transfer mobility from point p to q
$\mathbf{Y}$	mobility matrix