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SOUND PROPAGATION IN A MOVING FLUID CONFINED BY CYLINDRICAL WALLS—A COMPARISON BETWEEN AN EXACT ANALYSIS AND THE LOCAL-PLANE-WAVE APPROXIMATION

M. WILLATZEN

Mads Clausen Institute for Product Innovation, University of Southern Denmark, Grundtvigs Allé 150, *DK*-6400 *Sønderborg, Denmark. E-mail: willatzen@mci.sdu.dk*

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A discussion of sound propagation in a moving fluid confined by cylindrical walls is presented. Based on the continuity equation and the Euler equation, a single "exact" ordinary differential equation in the acoustic pressure is derived for the case where the medium flow $\mathbf{v}_0(r)$ depends on the radial co-ordinate only and points in the axial direction. This "exact" pressure wave equation is solved semi-analytically by means of the Frobenius method and compared with the conventional approximative wave equation known as the local-plane-wave (LPW) approximation for a range of flow values. In this way, information about mode phase-speed changes with flow and flow-meter performance is obtained. It is found that the LPW approximation works well only for mode propagation parallel or nearly parallel to the direction of flow. Based on the "exact" acoustic pressure wave equation, it is also concluded that flow-meter errors become independent of ultrasound frequency and cylinder radius, a point that the LPW approximation fails to predict. Furthermore, an "exact" procedure shows that flow-meter errors depend on the Reynolds number and the mode number only. In actual fact, it is found that flow measurement based on the fundamental mode is approximately free of errors while all other modes are characterized by the same (and, generally, non-vanishing) deviation of measurement.

1. INTRODUCTION

It has been a common practice in several papers to determine the influence of a background flow on mode phase speeds when discussing sound propagation characteristics in cylinders by using the so-called local-plane-wave approximation (LPW) [1–5]. In reference [1], a semi-analytical treatment of mode phase speeds in a cylindrical waveguide carrying a moving fluid is given for the case where the flow profile is laminar and parabolic. References [2, 3] continue along the same lines as in reference [1], although using a more restrictive condition $(\omega R/c)\sqrt{\bar{v}/c} \ll 1$, where ω, R, $\mathbf{v}$, and c denote the ultrasound frequency, cylinder radius, mean flow, and sound speed respectively), and consider flow measurement accuracy based on transit-time differentials in the more general case where the flow profile changes gradually from a parabolic profile in the laminar regime towards a flat profile in the turbulent regime as the Reynolds number increases. References [2, 3] point out that strong measurement errors generally result if higher order modes are excited by the transmitter, especially in the laminar regime, when using the standard expression in flow-meter

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applications for determining mean flow based on transit-time differentials,

$$\Delta t = 2b_a\bar{v}/c^2, \tag{1}$$

and b_a is the ultrasound transmission distance. In reference [4], a correction to the phase-speed changes with flow as obtained in references [2, 3] is pointed out, while the analysis of flow-measurement accuracy given in reference [5] is extended to cover arbitrary flow profiles.

In the present work, the LPW approximation employed in all the papers mentioned above [1–5] is compared with an "exact" analysis based on the equation of continuity and the Euler equation for the discussion of mode phase-speed changes with flow for an arbitrary flow profile. Following determination of phase-speed changes with flow, flow-meter performance is addressed. Numerical results are given for the particular case where the flow profile gradually changes from a parabolic profile to a flat profile as the Reynolds number increases. It is shown that the LPW approximation works well only if the mode considered propagates parallel or nearly parallel to the direction of flow.

2. THEORY

In the following, a differential equation describing sound propagation in a moving non-viscous fluid confined by cylindrical walls is derived in the low-flow regime, where flow velocity is much smaller than the speed of sound everywhere in the fluid. By the separation-of-variables method, upon assuming a harmonic dependence in time and the axial co-ordinate: $\exp[i(\beta z - \omega t)]$, a single ordinary differential equation in the radial co-ordinate r is obtained governing the acoustic pressure $p'(r)$. This differential equation in $p'(r)$ is solved by means of the Frobenius series expansion method [6].

In previous analyses on the same subject, the so-called LPW approximation has been employed in examining the influence of a background flow on ultrasound propagation in cylindrical waveguides [1–5]. The approximation made in LPW is that a background flow $v(r)$ along the axial direction modifies the local sound speed from c in the quiet medium to $c + v(r)$ in the moving medium. It is assumed that the medium is homogeneous so that c is a constant in space. The LPW is expected to be a good approximation for fundamental mode (plane-wave) propagation parallel to the direction of flow but less so as the mode number increases. This follows from the fact that the sound propagation direction becomes more and more tilted in comparison with the flow direction as the mode number increases.

The starting point is the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{2}$$

and the Euler equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla p}{\rho}. \tag{3}$$

For the velocity, pressure, and density one can now write

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}, \qquad p = p_0 + p', \qquad \rho = \rho_0 + \rho', \qquad (4)\text{–}(6)$$

where $\mathbf{v}_0$, p_0, and ρ_0 denote the (background) flow velocity, pressure, and density in an undisturbed medium, respectively, and it is assumed that the background flow corresponds to a steady incompressible flow situation ($\rho_0 = constant$, $\nabla \cdot \mathbf{v}_0 = 0$). The primed quantities $\mathbf{v}'$, p', and ρ' represent small changes in flow velocity, pressure, and density due to the presence of, e.g., low-intensity ultrasound waves in the medium. To first order in the small quantities above, equations (2) and (3) read as

$$\frac{\partial p'}{\partial t} + \rho_0 c^2 \nabla \cdot \mathbf{v}' + (\mathbf{v}_0 \cdot \nabla) p' = 0, \tag{7}$$

$$\frac{\partial \mathbf{v}'}{\partial t} + (\mathbf{v}' \cdot \nabla)\mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla)\mathbf{v}' = -\frac{\nabla p'}{\rho_0} + \frac{p'}{(\rho_0 c)^2} \nabla p_0, \tag{8}$$

and use has been made of the isentropic relation

$$p' = \left(\frac{\partial p}{\partial \rho_0}\right)_S \rho' = c^2 \rho', \tag{9}$$

upon assuming adiabatic and reversible conditions.

Finally, applying the separation-of-variables method by assuming the following functional form of $p'(r, z, \theta; t)$:

$$p'(r, z, \theta; t) = f(r) \exp[\mathrm{i}(\beta z - \omega t)], \tag{10}$$

corresponding to monofrequency operation and axisymmetrical (ultrasound) excitation conditions (note that the latter condition implies independence of $p'(r, z, \theta; t)$ on the azimuthal angle θ), equations (7) and (8) can be simplified considerably so as to obtain

$$-\mathrm{i}\omega p' + \rho_0 c^2 \left(\frac{\partial v_r'}{\partial r} + \frac{v_r'}{r} + \mathrm{i}\beta v_z'\right) + \mathrm{i}\beta v_0 p' = 0, \tag{11}$$

$$-\mathrm{i}\omega v_z' + v_r' \frac{\partial v_0}{\partial r} + \mathrm{i}\beta v_0 v_z' = \frac{-\mathrm{i}\beta}{\rho_0} p', \tag{12}$$

$$-\mathrm{i}\omega v_r' + \mathrm{i}\beta v_0 v_r' = -\frac{1}{\rho_0} \frac{\partial p'}{\partial r}. \tag{13}$$

In deriving equations (11)–(13), it has furthermore been assumed that the background flow $\mathbf{v}_0$ points in the axial direction and depends on the radial co-ordinate only (note that equation (3) then implies $\nabla p_0 = -\rho_0 (\mathbf{v}_0 \cdot \nabla)\mathbf{v}_0 = 0$). The parameter β is left undetermined at this point; however, it will be shown in the following that an infinite but discrete set of β values are possible when invoking the boundary condition that the normal velocity component must vanish at the cylinder wall, also known as the rigid-wall approximation, so that equation (13) can be restated as

$$v_r' = \frac{\mathrm{i}}{\rho_0} \frac{1}{\beta v_0 - \omega} \frac{\partial p'}{\partial r}. \tag{14}$$

Inserting equation (14) into equation (12) yields

$$v_z' = -\frac{1}{\rho_0(\beta v_0 - \omega)^2}\frac{\partial p'}{\partial r}\frac{\partial v_0}{\partial r} - \frac{\beta}{\rho_0(\beta v_0 - \omega)}p'. \tag{15}$$

Next, inserting equations (14) and (15) into equation (11) concludes the derivation of an ordinary differential equation in the acoustic pressure p' [7–9]:

$$\frac{\partial^2 p'}{\partial r^2} + \left(\frac{1}{r} - \frac{2\beta}{(\beta v_0 - \omega)}\frac{\partial v_0}{\partial r}\right)\frac{\partial p'}{\partial r} + \left(\frac{(\beta v_0 - \omega)^2}{c^2} - \beta^2\right)p' = 0. \tag{16}$$

Next, the assumption that flow velocities v_0 are much smaller than the phase speed ω/β is made, i.e., $\beta v_0/\omega \ll 1$. In the case of ultrasonic flow-meter applications in, e.g., water-transport systems or district-heating systems, the flow speed is in the range 0·01–10 m/s while the sound speed is about 1500 m/s, and so the assumption $\beta v_0/\omega \ll 1$ is well justified. In this case ($\beta v_0/\omega \ll 1$), equation (16) can be approximated by

$$\frac{\partial^2 p'}{\partial r'^2} + \left(\frac{1}{r'} + \frac{2\beta}{\omega}\frac{\partial v_0}{\partial r'}\right)\frac{\partial p'}{\partial r'} + R^2\left(\frac{\omega^2 - 2\beta\omega v_0(r')}{c^2} - \beta^2\right)p' = 0, \tag{17}$$

where

$$r' = r/R \tag{18}$$

has been introduced and R denotes the cylinder radius. In the next section, this differential equation (equation (17)) is solved by using the Frobenius power series method.

As a remainder, note that in LPW the acoustic pressure wave equation is [4]

$$\nabla^2 p' + \frac{\omega^2}{(c + v_0(r))^2}p' = 0 \tag{19}$$

and by the separation-of-variables method, the associated LPW radial wave equation becomes

$$\frac{\partial^2 p'}{\partial r'^2} + \frac{1}{r'}\frac{\partial p'}{\partial r'} + R^2\left(\frac{\omega^2}{c^2} - \beta^2 - \frac{2v_0(r)}{c}\frac{\omega^2}{c^2}\right)p' = 0. \tag{20}$$

Observe that the present "exact" acoustic pressure wave equation (equation (17)) agrees with the LPW equation (equation (20)) for the case where $v_0(r) = 0$ as it should.

3. A POWER SERIES SOLUTION TO THE ACOUSTIC PRESSURE WAVE EQUATION USING THE FROBENIUS METHOD

Before applying the Frobenius method to solve equation (17), it can be assumed that the background flow $v_0(r')$ can be expanded in an infinite power series:

$$v_0(r') = \sum_{\lambda=0}^{\infty} v_{0\lambda} r'^{\lambda}. \tag{21}$$

Note that this assumption is a weak restriction since most radially dependent velocity profiles can be written as a power series in the radial co-ordinate including flat profiles, parabolic profiles, and logarithmic profiles as suggested by Nikuradse [5, 10].

The Frobenius method is based on the assumption that p' can be written as a series expansion in r',

$$p' = \sum_{\lambda=0}^{\infty} a_\lambda r'^{\lambda+k}, \tag{22}$$

where k is unspecified (in general, at this point, k can be any real constant). Insertion of equations (21) and (22) into equation (17) and demanding that $a_0 \neq 0$ gives $k = 0$ if terms proportional to r'^{k-2} are equated (corresponding to $\lambda = 0$). Again, employing the identity principle for infinite power series to terms proportional to r'^{λ} leads to the following recursion formula:

$$a_0 = 1, \qquad a_1 = 0,$$

$$a_{\lambda+2} = -\frac{2\beta}{\omega}\frac{\lambda+1}{(\lambda+2)^2} v_{01} a_{\lambda+1} + \frac{2R^2\beta\omega}{(\lambda+2)^2 c^2}\sum_{\lambda'=0}^{\lambda} a_{\lambda'} v_{0\lambda-\lambda'}$$
$$+ \frac{(\beta^2 R^2 - (R^2\omega^2/c^2))}{(\lambda+2)^2} a_\lambda - \frac{2\beta}{\omega(\lambda+2)^2}\sum_{\lambda'=1}^{\lambda} \lambda'(\lambda+2-\lambda') v_{0\lambda+2-\lambda'} a_{\lambda'} \quad \text{if } \lambda \geqslant 0. \tag{23}$$

Thus, a general solution to equation (17) is given by equation (22) where the coefficients a_λ obey the recursion formula expressed by equation (23). One is now in a position to determine the allowed values for β. The rigid-wall assumption implies that the radial (normal) velocity component v'_r must vanish at $r = R$: i.e., the radial component of the acoustic pressure gradient must obey (according to equation (14))

$$\partial p'/\partial r' = 0 \tag{24}$$

or

$$\sum_{\lambda=1}^{\infty} \lambda a_\lambda = 0. \tag{25}$$

By solving equation (25) numerically using equation (23), a set of discrete β_n values are found and so a discrete set of solutions p'_n exist. Each of the p'_n solutions represents a (sound) propagating mode.

One can define the phase-speed difference $\Delta\phi_n$ between an upstream and a downstream sound propagation situation, in which the nth mode is the only mode excited, as

$$\Delta\phi_n = \omega b_a \left(\frac{1}{c_{pn}^-} - \frac{1}{c_{pn}^+}\right), \tag{26}$$

where b_a denotes the distance between the two transducers (the two transducer axes are assumed aligned with the cylinder axis), and $c_{pn}^- = \omega/\beta_n^-$ and $c_{pn}^+ = \omega/\beta_n^+$ are the phase speeds corresponding to an upstream sound propagation situation and a downstream sound propagation situation respectively. Similarly, β_n^- and β_n^+ denote the values of β_n in an upstream and a downstream sound propagation situation respectively. The transit time difference Δt_n between the two successive sound propagation situations then becomes

$$\Delta t_n = \frac{\Delta \phi_n}{\omega} \tag{27}$$

and the so-called deviation of measurement E_n, often used as a quality measure for a flow meter, can be written as

$$E_n = \frac{\Delta t_n - 2b_a\bar{w}/c^2}{2b_a\bar{w}/c^2}, \tag{28}$$

where $2b_a\bar{w}/c^2$ is the transit time difference of the fundamental mode.

4. NUMERICAL RESULTS AND DISCUSSIONS

In this section, results are given for mode phase-speed changes and the deviation of measurement as a function of mean flow based on solving the "exact" acoustic pressure wave equation (equation (17)) and the LPW wave equation (equation (20)) respectively.

Firstly, consider a background flow of the form

$$v_0(r) = \tau(\mathrm{Re})v_{turb}(r) + (1 - \tau(\mathrm{Re}))v_{lam}(r), \tag{29}$$

where

$$v_{turb}(r) = \bar{v}, \qquad v_{lam}(r) = 2\bar{v}\left(1 - \frac{r^2}{R^2}\right), \tag{30, 31}$$

$$\tau(\mathrm{Re}) = 1 - \frac{1}{1 + (\mathrm{Re}/\mathrm{Re}_0)^n}, \qquad \mathrm{Re} = \frac{2\bar{v}R}{v} \tag{32, 33}$$

and v denotes the medium viscosity. In equation (33), $\mathrm{Re}_0 = 2000$ is the Reynolds number locating the transition region between the laminar and turbulent flow regimes and the exponent $n = 4$ is chosen. Note that a flat profile is used in the turbulent regime as this profile approximates the logarithmic profile suggested by Nikuradse's measurements well [5, 10].

In Figure 1, calculated phase-speed changes with mean flow are shown for the fundamental mode (mode 1) and the first three higher order modes (modes 2–4) corresponding to the following parameter values (1) $f = \omega/2\pi = 4\,\mathrm{MHz}$, $R = 0{\cdot}01\,\mathrm{m}$, $c = 1543\,\mathrm{m/s}$, $v = 5{\cdot}471 \times 10^{-7}\,\mathrm{m^2/s}$. The values for the sound speed and viscosity correspond to water at 50°C. Two curves are shown for each mode. The solid and dash–dotted curves represent data based on solving equation (17) ("exact" wave equation) and equation (20) (LPW approximation) numerically respectively. Consider first, mode 1. It is evident that phase-speed changes with flow are equal to the mean flow in the flow range 0–0·2 m/s using equation (17) as well as equation (20). For modes 2–4, the situation is different. Although results based on equation (17) as well as equation (20) agree quite well, it is seen that phase-speed changes with flow are somewhat higher than the mean flow especially for the

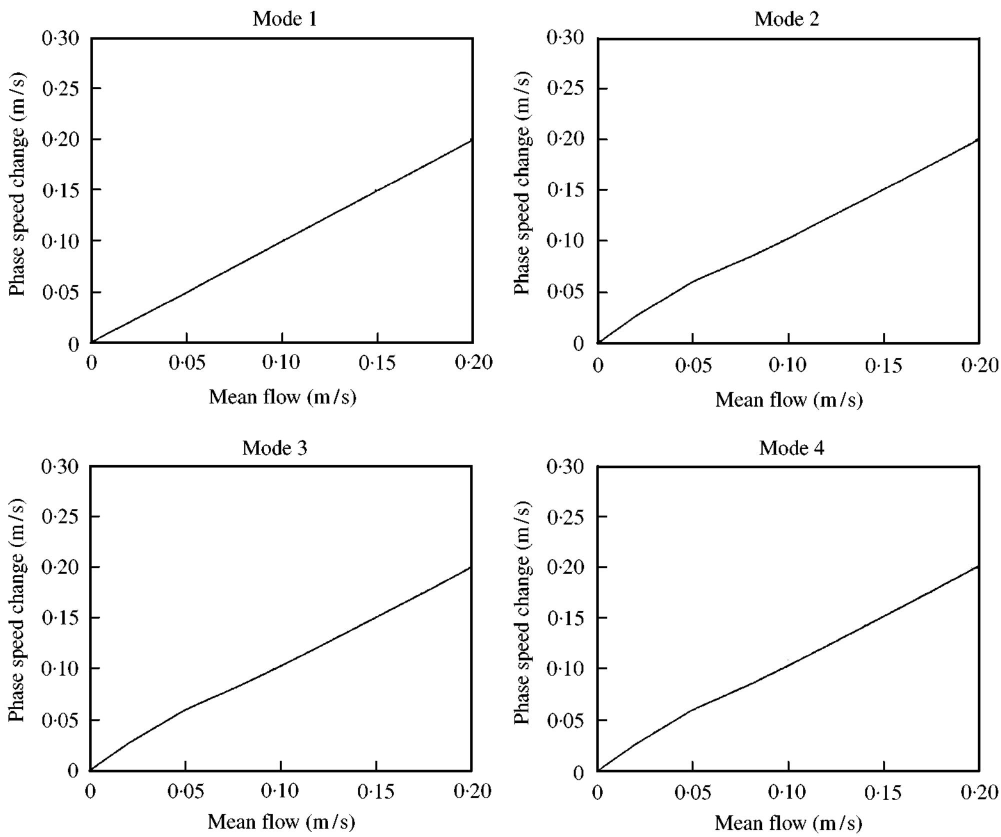

Figure 1. Mode phase-speed changes with mean flow for the first four modes allowed to propagate. The parameter values used in the calculation are: $f = 4 \times 10^6$ Hz, $R = 0{\cdot}01$ m, $c = 1543$ m/s, and $\nu = 5{\cdot}471 \times 10^{-7}$ m^2/s. The (——) and (–·–·–) curves are found by solving the "exact" acoustic pressure wave equation (equation (17)) and the LPW approximate equation (equation (20)) respectively.

lowest flow values, i.e., for a mean flow below 0·05 m/s. This behavior can be understood from Figure 2. The deviation of measurement E_1 (mode 1) is approximately 0 in the flow range: 0–20 m/s, but E_2, E_3, and E_4 (modes 2, 3, and 4) are all approximately $+1/3$ in the flow range 0–0·02 m/s, and at higher flow values they drop to approximately zero (note the logarithmic scale on the x-axis). The 33·33% phase-speed change overshoot with flow found in the low-flow regime for modes 2–4 (Figure 1) leads to deviation of measurement values of the same amount: 33·33% (Figure 2). Analytical LPW results based on Sodha's paper [1] agree with the E_n values found in the present work [2–5]. In actual fact, equation (28) in reference [4] shows that in the laminar region ($\tau(\mathrm{Re}) = 0$), E_n becomes 0 for the fundamental mode and approximately 1/3 for all the other modes characterized by mode phase-speed values c_{pn} close to the (thermodynamic) sound speed c. As mean flow approaches 0·1 m/s (onset of turbulent regime), deviation of measurement values approaches 0 for all modes whether based on equation (17) or equation (20).

In Figure 3, calculated phase-speed changes with mean flow are shown for modes 1–4 corresponding to the parameter values (2) $f = \omega/2\pi = 1$ MHz, $R = 0{\cdot}0055$ m, $c = 1543$ m/s, $\nu = 5{\cdot}471 \times 10^{-7}$ m^2/s. Again, the solid and dash-dotted curves represent data based on solving equation (17) ("exact" wave equation) and the approximate LPW equation (equation (20)) numerically respectively. The fundamental mode is still characterized by

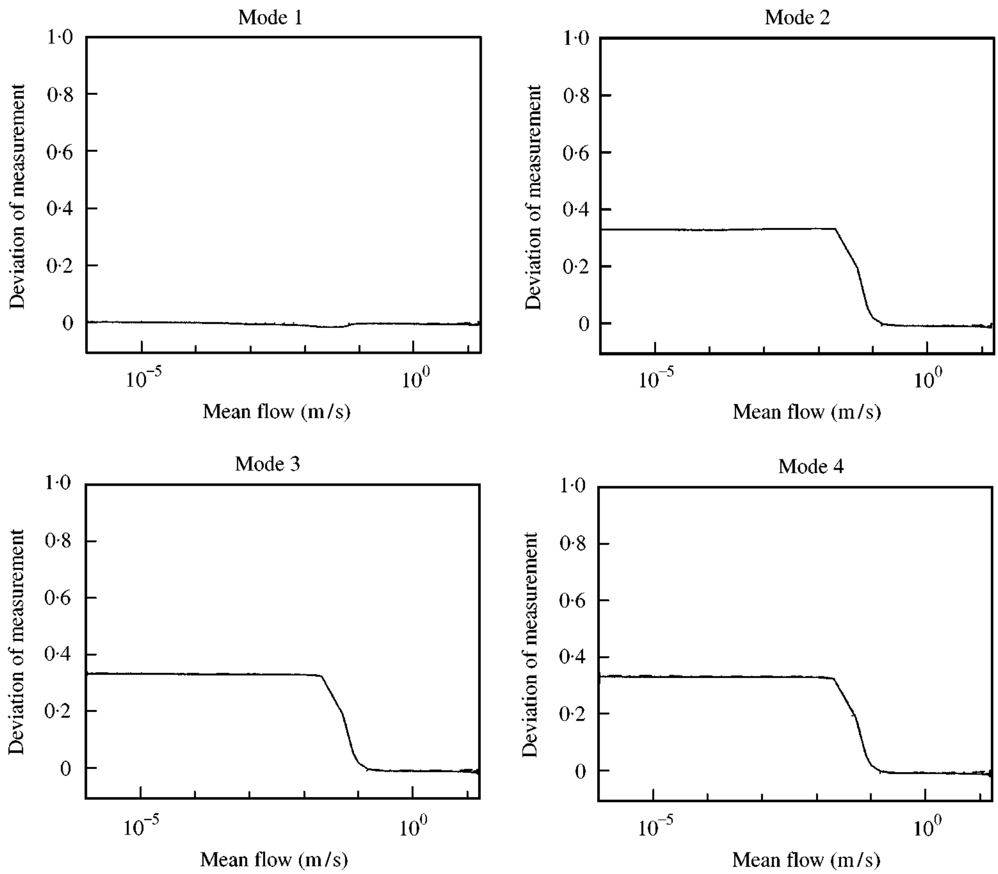

Figure 2. Deviation of measurement as a function of mean flow for the first four modes allowed to propagate. The parameter values used in the calculation are: $f = 4 \times 10^6$ Hz, $R = 0{\cdot}01$ m, $c = 1543$ m/s, and $\nu = 5{\cdot}471 \times 10^{-7}$ m^2/s. The (——) and (–·–·–) curves are found by solving the "exact" acoustic pressure wave equation (equation (17)) and the LPW approximate equation (equation (20)) respectively.

a phase-speed change with flow equal to the mean flow. This results was found in previous analyses using the LPW approximation as already mentioned ($E_1 = 0$ according to equation (28) in reference [4]). The present work shows that an "exact" analysis yields the same result as in an LPW analysis. However, considering modes 2–4, it is evident that increasing discrepancy is found between phase-speed change results with mean flow based on using the "exact" (equation (17)) and LPW (equation (20)) wave equations. This is expected since the LPW approximation is good approximation for small tilt angles only and the mode propagation direction becomes increasingly tilted with respect to the flow direction as the mode number increases from 2 to 4. The reason that this discrepancy is (much) more pronounced for the parameter choice in (2) as compared to (1) is that the lower frequency and radius in the former case leads to much higher c_{pn} values and diffraction angles θ ($\cos\theta = c/c_{pn}$) for a given mode $n > 1$ as compared to the latter case.

In Figure 4, deviation of measurement values as a function of mean flow are shown for the parameter choice (2). Figure 3 suggests that E_1 must be approximately 0 in the (laminar) flow regime 0–0·2 m/s. Data in Figure 4 reveal that E_1 is approximately 0 in the larger flow range 0–15 m/s. The fact that E_1 is approximately 0 in the turbulent regime is a consequence of the flat-profile assumption (refer again to equation (28) in reference [4]). The previously

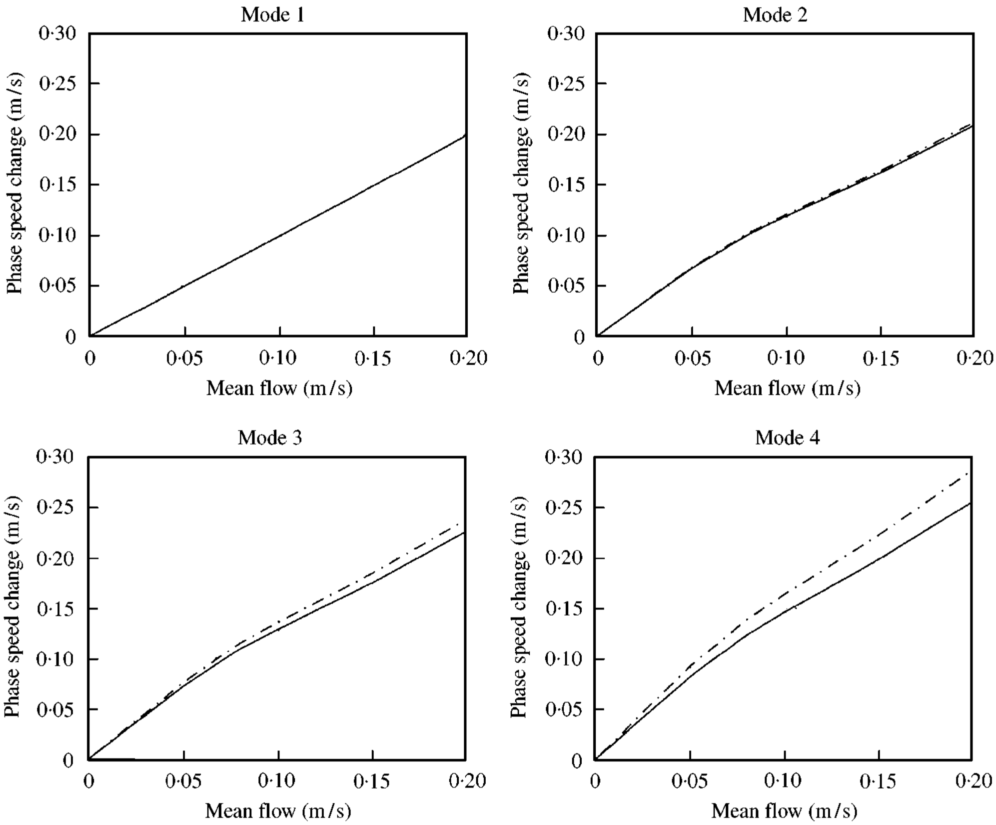

Figure 3. Mode phase-speed changes with mean flow for the first four modes allowed to propagate. The parameter values used in the calculation are: $f = 1 \times 10^6$ Hz, $R = 0{\cdot}0055$ m, $c = 1543$ m/s, and $\nu = 5{\cdot}471 \times 10^{-7}$ m^2/s. The (——) and (–·–·–) curves are found by solving the "exact" acoustic pressure wave equation (equation (17)) and the LPW approximate equation (equation (20)) respectively.

mentioned discrepancy (Figure 3) between results based on the "exact" wave equation (equation (17)) and the approximate LPW equation (equation (20)) is also evident in Figure 4 for the higher order modes $n > 1$. It is interesting to observe that E_n becomes 1/3 in the laminar-flow regime for all higher order modes ($n > 1$) in case (2) (Figure 4) as well as in case (1) (Figure 2) based on the "exact" wave equation. This result suggests that E_n, in general, depends on flow profile and mode number only, and that E_n is independent of frequency and radius (this statement has been checked for numerically using various parameter choices!). In the LPW approximation, on the other hand, the deviation of measurement depends on all four quantities: flow profile, mode number, frequency, and radius [4, 5]. The present analysis clearly shows, as expected, that LPW works well only for modes characterized by phase speed c_{pn} close to the thermodynamic sound speed c.

5. CONCLUSIONS

Sound propagation in a moving fluid confined by cylindrical walls is discussed. Based on the equation of continuity and the Euler equation, a single "exact" ordinary differential equation is derived in the acoustic pressure. This differential equation is solved by means of

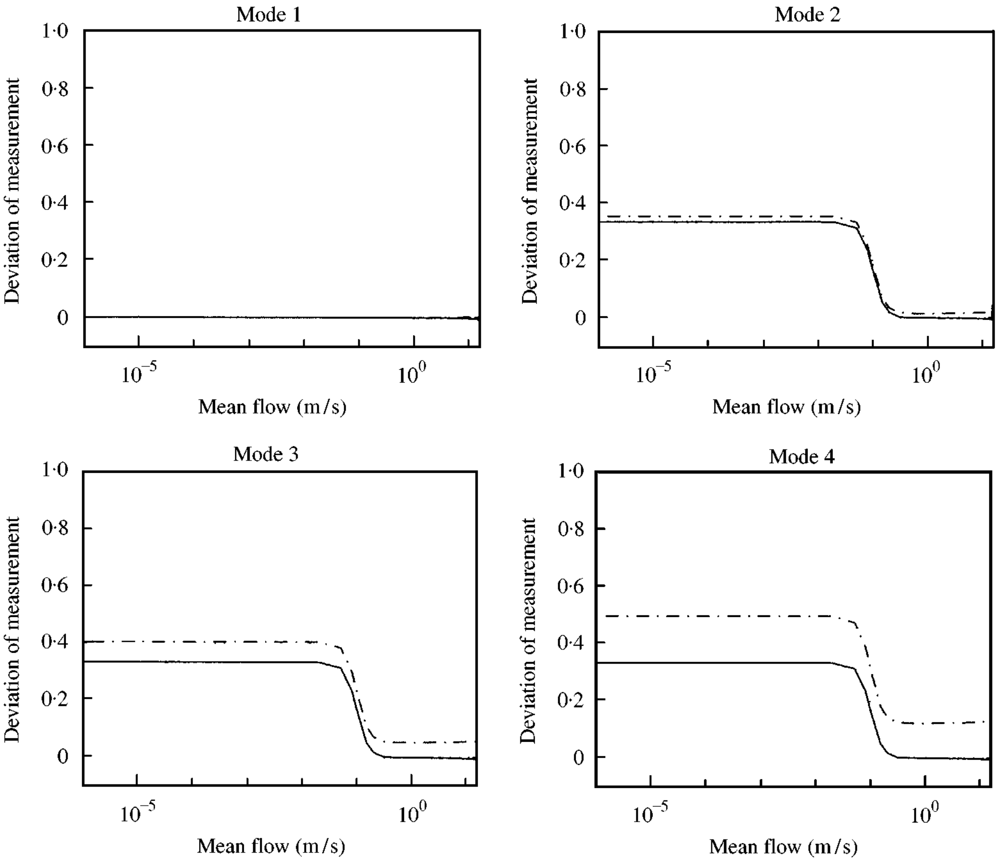

Figure 4. Deviation of measurement as a function of mean flow for the first four modes allowed to propagate. The parameter values used in the calculation are: $f = 1 \times 10^6$ Hz, $R = 0{\cdot}0055$ m, $c = 1543$ m/s, and $\nu = 5{\cdot}471 \times 10^{-7}$ m^2/s. The (——) and (–·–·–) curves are found by solving the "exact" acoustic pressure wave equation (equation (17)) and the LPW approximate equation (equation (20)) respectively.

the semi-analytical Frobenius method and a discrete set of axial wave vectors β_n is found thereby enabling the determination of mode phase-speed changes with flow and an assessment of flow-meter performance/accuracy. A comparison with the conventional but approximative LPW approximation shows that the LPW approximation works well only in cases where the mode considered propagates parallel or nearly parallel to the direction of flow. It is also concluded that the deviation of measurement E_n depends on the Reynolds number only for a given mode number and not on the ultrasound frequency or cylinder radius, a result that the LPW approximation fails to predict.

REFERENCES

1. M. S. Sodha, A. Kumar, I. C. Goyal and A. K. Ghatak 1979 *Acustica* **41**, 232–237. Sound wave propagation in cylindrical inhomogeneus waveguides.
2. H. Lechner 1983 *Journal of Acoustical Society of America* **74**, 955–959. Ultrasonic flow metering based on transit time differentials which are insensitive to flow profile.
3. H. Lechner 1982 *ACTA IMEKO*, 279–288. Ultrasonic measurement of volume flow independent of velocity distribution.

4. M. Willatzen, 2000 *Journal of Acoustical Society of America* **107**, 1762–1765. Comments on "Ultrasonic flow metering based on transit time differentials which are insensitive to flow profile" (*Journal of Acoustical Society of America* **74**, 955–959 (1983)).
5. M. Willatzen 2001 *Journal of Acoustical Society of America* **109**, 102–107. Perturbation theory applied to sound propagation in flowing media confined by a cylindrical waveguide.
6. G. Arfken 1985 *Mathematical Methods for Physicists*, 113–115. Orlando: Academic Press, Inc.; third edition.
7. P. Mungur and H. E. Plumblee 1969 *NASA SP*-207, 305–322. Propagation and attenuation of sound in a soft walled annular duct containing shear flow.
8. N. K. Agarwal and M. K. Bull 1989 *Journal of Sound and Vibration* **132**, 275–298. Acoustic wave propagation in a pipe with fully developed turbulent flow.
9. G. R. Gogate and M. L. Munjal 1993 *Journal of Sound and Vibration* **160**, 465–484. Analytical solution of sound propagation in lined or unlined circular ducts with laminar mean flow.
10. J. Nikuradse 1932 *Forschung auf dem Gebiete des Ingenieurwesens*, Forschungsheft **356**, **B**. Gesetzmaessigkeiten der turbulenten Stroemung in glatten Rohren.