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NAILING DOWN A VIBRATING MEMBRANE

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In a previous letter [1], the exact solution of a vibrating annular membrane is considered. The asymptotic form for the fundamental frequency k can be shown to be

$$k = k_0 + \frac{\pi Y_0(k_0)}{2J_1(k_0)} \frac{1}{|\ln c|} + O(|\ln c|^{-2}), \tag{1}$$

where the outer radius is one, c is the core radius, and k_0 is the fundamental frequency of the unit circle without the constraint. When $c \to 0$ or when the inner core shrinks to zero (a centered pin-point constraint), the frequency is the same as that of a circular membrane without the constraint. Laura *et al.* [2, 3] showed that the higher frequencies of a circular membrane are also unaffected by a centered pin-point constraint. The problem was further discussed by Gottlieb [4], who noted that Rayleigh [5] had conjectured that this phenomena would apply to any number of pin-points on a membrane of any shape.

The second term of equation (1) is the first correction due to the small, finite size of the point constraint. Since $(\mathrm{d}k/\mathrm{d}c) \to \infty$ as $c \to 0$, the rise of frequency is singular as c is increased from zero. From reference [1] one can show that the increase from the unconstrained case is about 5% when c is 10^{-6} and 10% when c is 10^{-3}. Since no real constraint can have infinitesimal size, this correction term is important.

The purpose of this letter is to find the size correction term in the case of an arbitrary vibration mode of an arbitrary membrane with an arbitrary number, locations, and (small) sizes of internal point constraints. The method used is similar to that of Rayleigh [5], and may be well known to some researchers.

Consider a membrane of arbitrary shape with N small internal circular constraints with radius $c_i \ll 1$ centered at P_i. Let w_0 and k_0 solve the membrane problem without the constraints, and w and k solve the problem the constraints. Thus

$$\nabla^2 w_0 + k_0^2 w_0 = 0, \qquad \nabla^2 w + k^2 w = 0. \tag{2, 3}$$

Multiply equation (2) by w and equation (3) by w_0. The difference is then integrated over the multiply-connected area of the membrane:

$$\iint (w \nabla^2 w_0 - w_0 \nabla^2 w)\,\mathrm{d}A + (k^2 - k_0^2) \iint w w_0\,\mathrm{d}A = 0. \tag{4}$$

Using Green's second theorem and noting that w_0 is zero on the outer boundary, w is zero both on the outer boundary and the internal constraints, equation (4) gives

$$(k^2 - k_0^2) \iint w w_0\,\mathrm{d}A = -\sum_1^N \oint w_0 \frac{\partial w}{\partial n}\,\mathrm{d}s_i, \tag{5}$$

where s_i is the boundary of the ith constraint and n is the outward normal of the domain. Let (ρ_i, ϕ_i) be polar co-ordinates centered at P_i. Then, on each constraint $\mathrm{d}s_i = c_i\,\mathrm{d}\phi_i$, $\partial/\partial n = -\partial/\partial p_i$, $w_0 \approx w_0|_{P_i}$. Since c_i is small, w approximately satisfies the Laplace equation. The non-periodic solution is

$$w \approx w_0|_{P_i}\left(1 - \frac{\ln \rho_i}{\ln c_i}\right), \tag{6}$$

where the constants are adjusted such that w is zero on $\rho_i = c_i$, and $w \approx w_0|_{P_i}$ when $c_i \to 0$. The derivative on the boundary is thus

$$\frac{\partial w}{\partial \rho_i} = \frac{-\,w_0|_{P_i}}{c_i \ln c_i}. \tag{7}$$

Equation (5) then gives

$$k^2 = k_0^2 + \frac{2\pi}{\iint w_0^2\,\mathrm{d}A} \sum_1^N \frac{(w_0|_{P_i})^2}{(-\ln c_i)} + \cdots, \tag{8}$$

where $w \approx w_0$ in the area integral. The corresponding eigenfunction is

$$w = w_0 - \sum_1^N w_0|_{P_i} \frac{\ln \rho_i}{\ln c_i} + \cdots. \tag{9}$$

For the fundamental mode of a circular boundary, $w_0 = J_0(k_0 r)$ and equation (8) simplifies to

$$k^2 = k_0^2 + \frac{2}{J_1^2(k_0)} \sum \frac{J_0^2(k_0 r_i)}{(-\ln c_i)} + \cdots. \tag{10}$$

For a single constraint at the center, $r_i = 0$, $c_i = c$ and equation (10) becomes

$$k = k_0 + \frac{1}{k_0 J_1^2(k_0)(-\ln c)} + \cdots. \tag{11}$$

which is identical to equation (1) after using the Wronskin identity

$$J_1(z)Y_0(z) - J_0(z)Y_1(z) = 2/\pi z. \tag{12}$$

We shall give some examples. Consider a square membrane of side length one and nailed at the center. For the fundamental mode $k_0 = \sqrt{2}\pi$ and $w_0 = \sin(\pi x)\sin(\pi y)$. From equation (8), the frequency with the constraint is

$$k^2 = 2\pi^2 + 8\pi \frac{1}{|\ln c|} + \cdots, \tag{13}$$

where $c \ll 1$ is the radius of the constraint. Consider next a circular membrane nailed at seven evenly points: $r_1 = 0$, $r_i = 0{\cdot}5$, $\theta_i = \alpha + (i-2)\pi/3$, $c_i = c$ for $i = 2$–7, Then for the fundamental mode equation (10) gives

$$k^2 = k_0^2 + \frac{2}{J_1^2(k_0)}[1 + 6J_0^2(k_0/2)]\frac{1}{|\ln c|} + \cdots = 5.783 + \frac{27{\cdot}40}{|\ln c|} + \cdots \tag{14}$$

Now for the second mode $k_0 = 3{\cdot}8317$ and $w_0 = \sin\theta\, J_1(k_0 r)$. Equation (8) gives

$$k^2 = k_0^2 + \frac{4}{J_0^2(k)} J_1^2(k_0/2) \sum_2^7 \sin^2[\alpha + (i-2)\pi/3] \frac{1}{|\ln c|} + \cdots$$

$$= 14{\cdot}682 + \frac{24{\cdot}95}{|\ln c|} + \cdots \tag{15}$$

For both modes, the phase angle α does not enter the frequency correction.

Finally, guided by comparisons with the exact solution in reference [1], the constraint radius c should be less than about 0·001 for equation (8) to have an error or less than 1%. Also, the constraint locations should be more than $O(c)$ distance from the membrane boundary and from each other.

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