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BOUNDARY CONDITION IDENTIFICATION BY SOLVING CHARACTERISTIC EQUATIONS

H. Ahmadian

Department of Mechanical Engineering, Iran University of Science and Technology, Narmak, Teheran 16844, *Iran. E-mail: ahmadian@iust.ac.ir*

J. E. Mottershead

*Department of Engineering, The University of Liverpool, Liverpool L*69 3*GH, England.*

AND

M. I. Friswell

*Department of Mechanical Engineering, University of Wales Swansea, Swansea SA*2 8*PP, Wales*

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The behaviour of a mechanical structure in the lower frequencies is dominated by constraints at the boundaries. Most structures have elastic supports and specifying the boundary conditions requires knowledge of the support parameters. In this paper, a new method is developed to determine the boundary parameters based on the solution of reduced order characteristic equations. The order of these non-linear equations is equal to the number of boundary degrees of freedom which is a small fraction of the order of the full structure and means that the amount of computation is not excessive. In this approach, no ill-conditioning occurs, which is a common problem in other identification procedures. The method is demonstrated by identifying the boundary parameters of a plate on elastic supports by using experimental test data.

1. INTRODUCTION

Boundary conditions have an important role in the dynamic behaviour of structures and should be considered carefully in theoretical studies and structural analysis. The finite element method is widely used to obtain a theoretical estimate of structural responses. The general procedure is to apply idealized constraints or to make use of previous experience in modelling apparently similar boundaries. The result of this approach is that many finite element models fail to represent boundary stiffnesses to an acceptable accuracy and consequently errors are produced in finite element predictions. One solution is to identify the boundary conditions from experiments carried out on the physical structure.

Boundary conditions may be determined by using conventional identification or updating methods [1]. Methods based on the error in the equations of motion, matrix perturbation or sensitivity are typical and can be used in either the modal or frequency domains. In the former approach, the equation of the motion of the system is formed at the measured frequencies using the measured displacements. These equations are rearranged to form a set of overdetermined linear equations in the unknown parameters. The spatial model is then obtained by minimizing the residual of the rearranged equations. The inconsistency between the order of the analytical model and that of the test model can be removed by expansion or reduction techniques. In many cases, measurement errors and ill-conditioning lead to unsatisfactory estimates of the spatial model [2, 3] by the equation error method. Measured displacements at "fixed" boundaries contain the highest levels of error because of the low level of displacement at these locations, i.e., the single-to-noise ratio is relatively low. Measurement errors at the boundaries produce large variations in estimates of boundary parameters.

In the matrix perturbation method described by Chen and Garba [4], it is usual for the number of parameters to exceed the number of equations of motion. Consequently, the equations are underdetermined and the analyst selects a solution based on a statistical estimate of the errors in the measurement and in the initial parameters. Therefore, the success of the method is highly dependent on experience and understanding of the test structure.

Perhaps the most popular method in the determination of boundary parameters is the sensitivity method [5]. The difference between model predictions and test observations is defined using linearized first order sensitivities. An iterative identification procedure is used to compensate the linearization effects. Although the sensitivity method in general produces better results than the other methods described above, it has also some shortcomings. It is known that sensitivity of modes to the boundary parameters is reduced as the higher-frequency modes are reached [6]. The insensitivity of higher modes along with linearization effects can cause convergence difficulties.

This paper considers a new method for the identification of structural boundary conditions which relies on the measured natural frequencies of the structure. A set of characteristic equations is formed. The number of these equations is equal to the number of measured natural frequencies but the order of each equation is equal to the order of the support or boundary model, which is much smaller than the order of the model of the complete structure. The method does not use the measured mode shapes, which are usually inconsistent with the order of the analytical model and may be in error especially at nodes close to a restrained boundary.

The essence of the method is as follows. The effect of an elastic support can be simulated using a set of nodal forces on the boundary of a free structure. Using the finite element model, a relationship may be established between nodal forces and nodal displacements at the boundary for each vibration mode. The support stiffness is also defined by a force–displacement relationship. The two systems of equations can then be combined to identify the parameters of the boundary support.

The theory of this new technique is developed in section 2. The method is then illustrated in section 3 using a numerical simulation of the boundary conditions of a beam with an unknown elastic support at one end. Sensitivity of the identification method to random measurement errors is examined by perturbation of the input data. To demonstrate the actual performance of the method an experimental set-up was developed. In the set-up, a steel plate was supported using a rubber seal with unknown stiffness properties. The test arrangement, experimental results and identification procedure are reported in section 4.

2. THE IDENTIFICATION METHOD

Consider the equilibrium equation of a freely vibrating mechanical structure with elastic supports. The boundary conditions may be introduced to the model by adding the support stiffness to the stiffness matrix of the model. The alternative is to constrain the boundary motion by applying reaction forces at the boundary nodes. Using the second option, the equation of motion can be written as

$$\begin{bmatrix} \mathbf{D}_{bb} & \mathbf{D}_{br} \\ \mathbf{D}_{br}^{\mathrm{T}} & \mathbf{D}_{rr} \end{bmatrix} \begin{Bmatrix} \mathbf{x}_b \\ \mathbf{x}_r \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_b \\ 0 \end{Bmatrix}, \tag{1}$$

where $\mathbf{D} = \mathbf{K} - \omega^2 \mathbf{M}$ is the known dynamical stiffness of an unconstrained structure, vector $\mathbf{x}_b$ contains the displacement of nodes on the boundary, $\mathbf{x}_r$ the displacement vector of nodes within the structure and $\mathbf{f}_b$ the unknown reaction force at the boundary. The force–displacement relationship on the boundary can be established from equation (1) as

$$(\mathbf{D}_{bb} - \mathbf{D}_{br}\mathbf{D}_{rr}^{-1}\mathbf{D}_{br}^{\mathrm{T}})\,\mathbf{x}_b = \mathbf{f}_b, \tag{2}$$

which is similar to the dynamic reduction formula. Also, the same boundary forces and displacements are related by the boundary stiffness $\mathbf{K}_b$ as

$$\mathbf{K}_b \mathbf{x}_b = -\mathbf{f}_b. \tag{3}$$

Note that in the above expression the mass properties of the support are neglected. To eliminate the unknown reaction forces, equations (2) and (3) may be added to obtain the following relationship:

$$(\mathbf{K}_b + \mathbf{D}_{bb} - \mathbf{D}_{br}\mathbf{D}_{rr}^{-1}\mathbf{D}_{br}^{\mathrm{T}})\,\mathbf{x}_b = \mathbf{0}. \tag{4}$$

Equation (4) can be re-arranged in terms of unknown parameters in stiffness matrix $\mathbf{K}_b$. Then these parameters can be identified using the resulting system of linear equations. However, the drawback of this approach is that it requires measured values $\mathbf{x}_b$. Unavoidable measurement errors in boundary displacements lead to an unstable identification procedure. The following offers an alternative approach to avoid using boundary displacements in the identification procedure.

In an elastic boundary $\mathbf{x}_b$ is a non-zero vector, therefore the coefficient matrix in equation (4) must be singular, i.e.,

$$|\,\mathbf{K}_b + \mathbf{D}_{bb} - \mathbf{D}_{br}\mathbf{D}_{rr}^{-1}\mathbf{D}_{br}^{\mathrm{T}}\,| = 0. \tag{5}$$

This characteristic equation can be formed for each mode and a set of non-linear equations is established to identify the entries of $\mathbf{K}_b$. The order of each polynomial in terms of unknown support parameters is equal to the order of the matrix $\mathbf{K}_b$. For example, the order of each characteristic equation, in the case of a beam with two degrees of freedom on each node elastically supported on one end and free on the other, is two. These equations may be solved by the usual non-linear algorithms.

The main advantage of the new method is that no ill-conditioning occurs during the identification procedure. A set of solutions for the boundary parameters is obtained by solving equation (5) for each measured mode. A unique solution is obtained by selecting the one that satisfies equation (5) for all measured modes. The following numerical study demonstrates practical aspects of the proposed method.

3. NUMERICAL EXAMPLE

A uniform beam free at one end and elastically supported at the other end as shown in Figure 1 was chosen to demonstrate the capability of the proposed method in identifying elastic support parameters. The elastic support consists of a translational spring, $k_1 = 10$ and a rotational spring, $k_2 = 5$. The beam has a length of $L = 5$, flexural rigidity $EI = 1$, and mass per unit length of $\rho A = 1$. A finite element model of the structure with

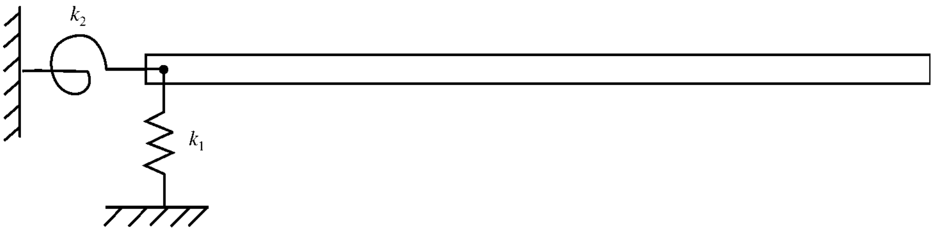

Figure 1. Beam with elastic support.

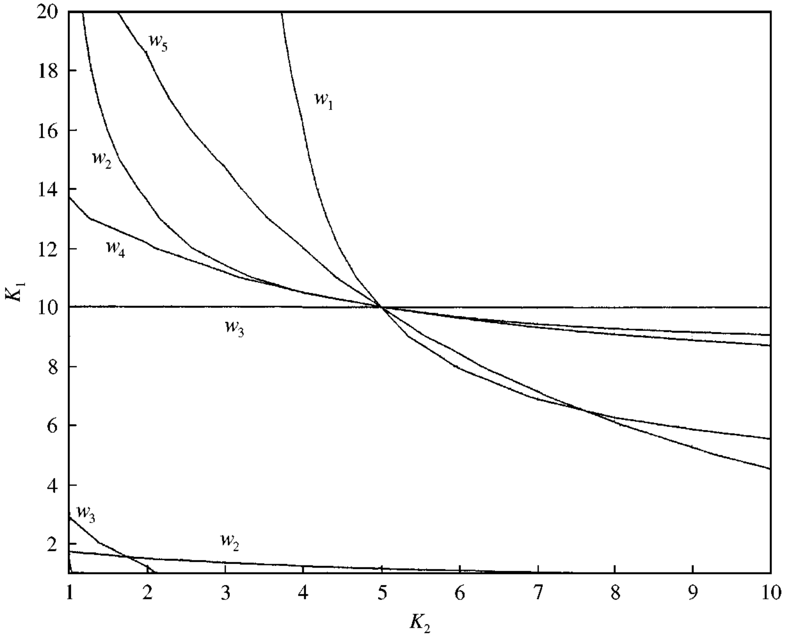

Figure 2. Solutions for k_1 and k_2 that satisfy the characteristic equation.

5 Euler–Bernoulli beam elements supported at one end with k_1 and k_2 was developed and its natural frequencies were determined. Using the stiffness and mass matrices of the beam when no boundary condition is applied and the natural frequencies of the restrained beam, the dynamic stiffness matrix **D** was developed for the first five modes. Then by solving

$$|\mathbf{K}_b + \mathbf{D}_{bb} - \mathbf{D}_{br}\mathbf{D}_{rr}^{-1}\mathbf{D}_{br}^{\mathrm{T}}| = 0,$$

which can be rewritten as

$$(k_1 + d_{11})(k_2 + d_{22}) - d_{12}^2 = 0,$$

where

$$\mathbf{K}_b = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \quad \text{and} \quad \mathbf{D}_{bb} - \mathbf{D}_{br}\mathbf{D}_{rr}^{-1}\mathbf{D}_{br}^{\mathrm{T}} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix},$$

a set of boundary stiffnesses k_1 and k_2 is determined for each mode. Figure 2 shows the acceptable solutions for each of the first five modes. The solution which satisfies all modes requirements namely, $k_1 = 10$, $k_2 = 5$ is the answer to the problem.

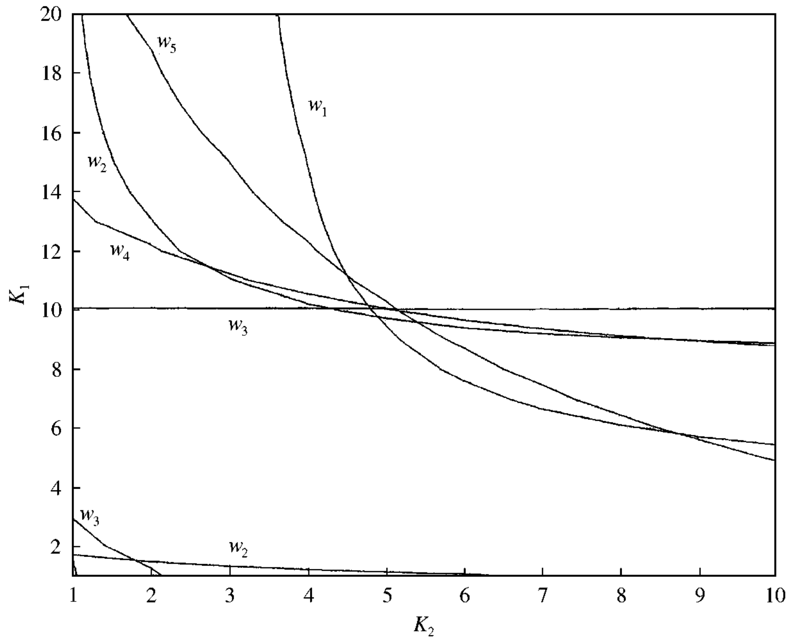

Figure 3. Acceptable solutions for k_1 and k_2—noisy data.

Next, the natural frequencies were perturbed each within $\pm 2\%$ in a random manner. The results of the identification procedure are shown in Figure 3. As may be expected there is no unique solution that satisfies all the requirements for every mode. However, the results indicate a set of solutions close to the exact values. A solution may be chosen from these results based on the quality of measurement and the accuracy of each measured mode. In general, more weight should be put on the lower modes as they are more sensitive to the boundary parameters. The following section deals with the problem of boundary-condition parameter identification from a physical experiment.

4. IDENTIFICATION OF A RUBBER SEAL MODEL

In the next step, the capability of the proposed method is verified by the identification of a rubber seal model. The rubber seal in the experiment is the same type that is used to hold the windscreen to a car body frame. The identification of the rubber seal parameters is important because of noise produced by windscreen vibration. These parameters have been identified by the authors using a sensitivity approach [7]. Here the same parameters are identified using the new method.

The experimental set-up used in this study consisted of a 0·5 m $\times$ 0·8 m steel plate with a thickness of 2·5 mm grounded using a rubber seal as shown in Figure 4. The first five lateral vibration modes of the plate restrained with the rubber windscreen-seal were measured and are listed in Table 1. A finite element model of the plate with mesh of 5×8 was also developed. A model with the lowest level of discretization error [8] was used for each plate element. Such elements can be considered to be one of a family having a generic formulation [9]. To complete the model it is necessary to specify the rubber seal. In the following, it is shown how such a model can be identified using the method proposed in section 2.

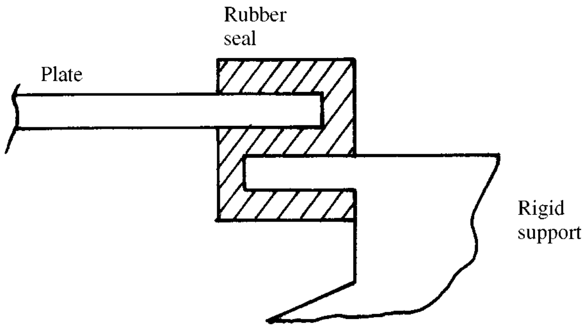

Figure 4. Experimental structure.

TABLE 1

Measured and predicted modes

Mode no.	Measured (Hz)	Predicted (Hz)	Error (%)
1	33·39	32·37	− 3·3
2	60·61	58·80	− 2·9
3	100·71	100·23	− 0·5
4	106·23	102·69	− 3·3
5	128·19	124·43	− 2·9
6	162·33	163·90	0·1

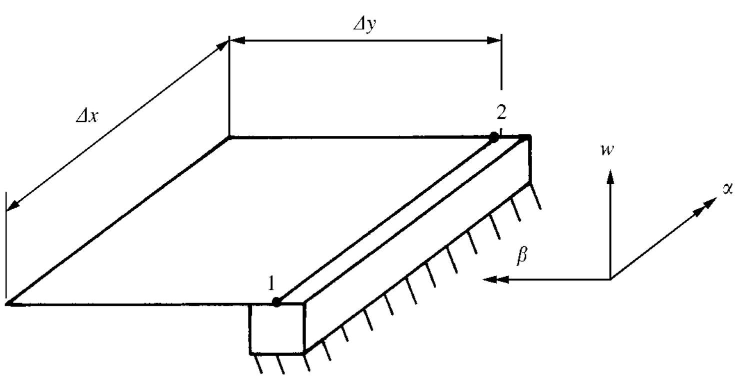

Figure 5. Elastic support element.

The first step in the identification procedure is the modelling of the rubber seal. To choose an appropriate model a rubber-seal element with the degrees of freedom w, $\alpha = \mathrm{d}w/\mathrm{d}y$, $\beta = \mathrm{d}w/\mathrm{d}x$ as shown in Figure 5, may be considered.

The rubber seal acts as a distributed elastic support along the edge of the plate and displacement functions for the beam are chosen which match those of the plate. Thus, for motion in w, β

$$w(x) = N_1(x)w_1 + N_2(x)\beta_1 + N_3(x)w_2 + N_4(x)\beta_2, \tag{6}$$

where

$$N_1(x) = (1 - 2\xi)(1 - \xi)^2, \qquad \xi = x/\Delta x,$$

$$N_2(x) = -\xi(1 - \xi)^2\,\Delta x, \quad N_3(x) = \xi^2(3 - 2\xi), \quad N_4(x) = \xi^2(1 - \xi)\Delta x$$

and, for motion in α,

$$\alpha(x) = N_5(x)\alpha_1 + N_6(x)\alpha_4, \tag{7}$$

where

$$N_5(x) = \Delta y(1 - \xi)^2(1 + 2\xi), \qquad N_6(x) = \Delta y(3 - 2\xi)\xi^2.$$

The above shape functions are consistent with well-known plate element formulations [8, 10–12]. The characteristic dimensions Δx and Δy are described in Figure 4. The following positive-definite matrices are obtained:

$$\frac{k_w\,\Delta x}{420}\begin{bmatrix} 156 & -22 & 54 & 13 \\ & 4 & -13 & -3 \\ & & 156 & 22 \\ Sym & & & 4 \end{bmatrix}\begin{Bmatrix} w_1 \\ \Delta x\beta_1 \\ w_2 \\ \Delta x\beta_2 \end{Bmatrix}, \tag{8}$$

$$\frac{k_\alpha\,\Delta x}{420}\begin{bmatrix} 156 & 54 \\ Sym & 156 \end{bmatrix}\begin{Bmatrix} \Delta y\alpha_1 \\ \Delta y\alpha_2 \end{Bmatrix}. \tag{9}$$

Equations (8) and (9) have a form similar to the mass matrix of a conventional Hermitian beam element because they represent an elastic foundation. This is because the shape functions, rather than their spatial derivatives, are used in the element formulation, and since the rubber-seal element is grounded on one side it has no rigid-body modes.

Twenty-six, $2 \times (5 + 8)$, rubber-seal elements were used to support the plate. The support stiffness, $\mathbf{K}_b$, formed by assembling the rubber-seal elements, has two unknown parameters, k_w and k_α, which represent distributed lateral and torsional stiffnesses. Negligible torsional stiffness is provided by the rubber seal because of its geometry and therefore k_α is set to zero. The remaining parameter is identified by the solution of characteristic equation

$$|\,\mathbf{K}_b + \mathbf{D}_{bb} - \mathbf{D}_{br}\mathbf{D}_{rr}^{-1}\mathbf{D}_{br}^{\mathrm{T}}| = 0$$

for the first six measured modes of the experimental set-up. The matrix $\mathbf{D}$ in the above expression is the dynamic stiffness matrix of the free plate at measured frequencies. Figure 6 shows the solution of the above expression for the first six measured modes. Singularity of the matrix $\mathbf{K}_b + \mathbf{D}_{bb} - \mathbf{D}_{br}\mathbf{D}_{rr}^{-1}\mathbf{D}_{br}^{\mathrm{T}}$ is denoted by the peaks on each of the six curves, which represent the characteristic equations for the first six modes. A cluster of peaks can be observed between values of 20 and 30 for the non-dimensionalized support parameters. Five of the six peaks lie between 25 and 30. The slightly outlying mode is the sixth mode (the highest frequency). It is not particularly significant that this mode has two peaks (two solutions of the determinantal equation) close to 20.

The support parameters are most important in the lower modes and are less significant in the higher modes. For this reason the first six measured modes were chosen in the identification procedure. When the eigenvalues are found from the model with the identified parameters, a good correlation can be seen between the predictions and test data. Table 1 shows the first six modes of the model with an identified non-dimensional support parameter of 26. The support stiffness is non-dimensionalised by dividing by the

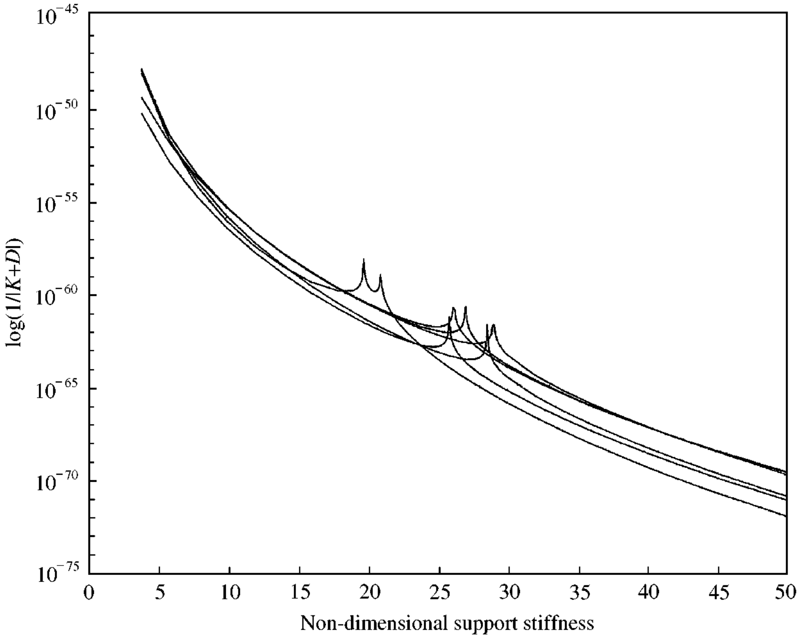

Figure 6. Solution of the characteristic equation—rubber-seal identification.

unconstrained plate stiffness between two adjacent boundary nodes. But, from Figure 6 it is clear that the solutions of the characteristic equations are scattered, though fairly closely scattered, around the value of 26 for the different modes. Therefore the effect of a measurement error in the range of $\pm 3\%$ on the natural frequencies was investigated. The most sensitive modes were 3 and 6 which showed a range of identified support stiffnesses of ± 10 (non-dimensional) over the frequency range. For the other modes very large parameter changes produce similarly small deviations in the natural frequencies. It can therefore be concluded that the scatter shown in the peaks of the curves in Figure 6 can be attributed to very small errors in the measured natural frequencies.

5. CONCLUSION

A method is developed to identify the boundary conditions of a structure using modal testing data. The method requires a finite element model before applying boundary conditions, and the natural frequencies of the physical structure constrained at its boundaries. A boundary stiffness matrix is identified to reconcile the two sets of inputs. It is shown using test cases that the method is robust in dealing with measurement errors and the resulting set of equations is well conditioned.

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