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LETTERS TO THE EDITOR

DYNAMIC INTERACTION BETWEEN TRAIN WHEELS AND THE SUBSURFACE

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1. INTRODUCTION

Passing trains generate vibrations in the soil, caused by the train response to track irregularities. This is particularly the case in soft-soil regions, where the surface wavespeed is comparable to the train speed. Interaction between ground vibrations and the train can give rise to different effects. On the one hand, the waves generated by an axle can influence the vibration of adjacent axles. There is some concern that interaction between axles, through vibrations propagating in the near subsurface, might lead to stronger vibrations or even instabilities. This interaction mechanism is investigated in the present paper. On the other hand, the waves can propagate to nearby structures and cause annoyance to people living or working in these buildings (or even damage the buildings).

The effect of a train of vertical loads on a half-space has been discussed by Alabi [1]. Krylov [2] investigates the effect of track dynamics on the vibrations due to high-speed trains. The vibrations generated by the bending track are computed with the aid of a Green function formulation and its farfield asymptotic behavior in an elastic half-space. For trains travelling with a speed less than the Rayleigh-wave speed of the half-space, the presence of sleepers appears to be important for the generation of vibrations. In Krylov's study, the effect of dynamically induced loads was not considered. The effect of a moving load on a Timoshenko beam-half-plane system has been investigated by Suiker *et al.* [3]. Sheng *et al.* [4, 5] investigate the vibrations of a harmonic load moving over a layered earth. The track, including rails, rail pads, sleepers and ballast, is modelled by means of a layered beam structure. They find that harmonic loads give rise to significant vibrations.

In reference [6], Krylov investigates the effect of road humps on the excitation of axle–hop resonances. In this study of dynamically induced loads from a road vehicle, the interaction between different axles is not taken into account. Herman [7] considers the problem of vibrations generated by the presence of heterogeneities near the track. The interaction between a mass–spring system moving along a beam on an elastic half-space is investigated by Metrikine and Popp [8]. They demonstrate that instabilities can arise if the

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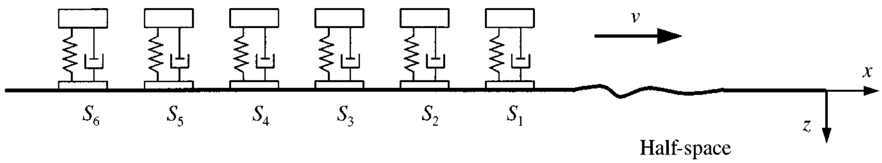

Figure 1. A "train", modelled by a sequence of rigid strips $S_1, \cdots, S_6$, moving with velocity v in the x direction along an elastic half-space containing irregularities at the surface. The strips support damped mass–spring systems in order to simulate the dynamic response of the train; coupling between strips through waves propagating in the half-space.

velocity of the mass–spring system exceeds the minimum phase velocity of waves in the beam.

In the present paper, the origin of the oscillation of the load is examined and the effect of the interaction of the train with track irregularities is analyzed. This problem resembles the one considered by Krylov [6], the main difference being that the interaction between axles through waves propagating in the subsurface is included. Concerning other aspects, however, the model is simplified since each wheel is represented by a rigid strip and not, for instance, as a moving point load on a beam, like in references [3–5, 8]. Another important assumption is the fact that the horizontal component of the displacement vector with respect to the vertical one can be neglected.

The dynamical behavior of the train itself is represented by a damped mass–spring system. In this way, the problem consists of two parts:

(1) The kinematical behavior of each mass–spring system.
(2) Coupling through wave propagation in the lower half-space.

Track irregularities can give to resonances and can be the source of vibrations, propagating away from the track.

2. FORMULATION OF THE PROBLEM

The train model consists of a sequence of N two-dimensional strips, parallel to the y-axis and moving on an elastic half-space with a horizontal velocity v ($v > 0$) in the positive x direction.

In the lower half-space ($z > 0$), an elastic solid is present. The particle displacement (u_x, u_y, u_z) satisfies the elastic wave equation. In this paper, the two-dimensional (P-SV) case is considered (which implies $u_y = 0$). The configuration is shown in Figure 1. Furthermore, the vertical displacement is assumed to be considerably larger than the horizontal displacement, i.e., $u_z \gg u_x$, and therefore the horizontal component is neglected with respect to the vertical one as suggested by Barends [9]. This approximation is illustrated in Figure 2 for the case of a vertical point source ($\rho = 2000$ kg/m^3, P-wave velocity = 280 m/s, S-wave velocity = 60 m/s and dominant frequency of the source = 10 Hz). In Figure 2, the vertical velocity is shown due to a vertical point source, both for the 2-D elastic case (a) and for the case where the horizontal component of the displacement in the elastic wave equation is neglected (b). In both cases, the response has been computed analytically with the modified Cagniard method [10]. From this comparison, one can conclude that the approximation is accurate at distances smaller than a wavelength from the source. Since interaction between axles takes place at relatively small distances, the scalar model is expected to be accurate enough to describe the interaction between axles if the distance

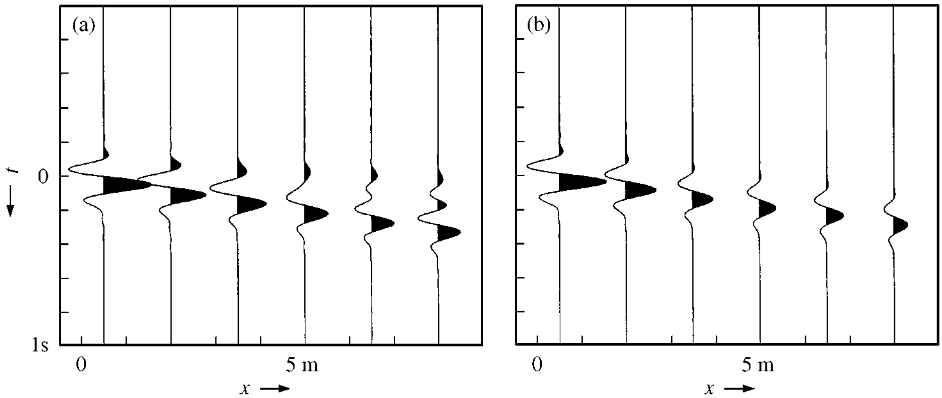

Figure 2. Comparison of the vertical velocity due to a vertical point source for the full 2-D elastic problem (a) with the same quantity obtained by neglecting the horizontal component. (b) Both results are displayed as functions of the horizontal distance x and time t. For the parameters used here, the dominant wavelength of shear waves is about 6 m. For distances smaller than 6 m, the two figures are similar.

between neighboring axles is not larger than the dominant wavelength of the vibrations. For the computation of the wavefield at larger distances from the track, however, the full elastic wave equation should be used [2]. In the computation of the wavefield at larger distances, the oscillating axles obtained from the present method could be used as a source term.

The elastic wave equation now simplifies to the following equation:

$$[\mu\partial_x^2 + (\lambda + 2\mu)\partial_z^2 - \rho\partial_t^2]w(x, z, t) = 0 \quad (z > 0), \tag{1}$$

where λ and μ are the Lamé coefficients of the solid, ρ is the density and w denotes the vertical displacement. In equation (1), ∂_x and ∂_z denote spatial derivatives and ∂_t is the temporal derivative. The relevant component of the stress is related to the vertical particle displacement by the relation

$$\tau_{zz}(x, z, t) = (\lambda + 2\mu)\partial_z w(x, z, t) \quad (z > 0), \tag{2}$$

where the horizontal components have been neglected also. The strips are assumed to be perfectly rigid and in friction-less contact with the half-space. This implies that they exert only a vertical force on the (otherwise traction-free) half-space. Therefore, the following boundary conditions at the half-space boundary ($z = 0$) are given:

$$w(x, 0, t) = w^{(i)}(t) \quad (x \in S_i) \tag{3}$$

and

$$\begin{aligned} \tau_{xz}(x, 0, t) &= \tau_{zx}(x, 0, t) = 0, \\ \tau_{zz}(x, 0, t) &= \mathrm{t}(x, t) \quad (x \in S_i), \\ &= 0 \quad (x \notin S_i), \end{aligned} \tag{4}$$

where t is the vertical traction under the strips and $w^{(i)}$ is the displacement directly under the rigid strip, associated with wave motion in the lower half-space. For the dynamic behavior of each strip and its associated mass–spring system, the configuration shown in Figure 1 is considered again. In order to relate the strip displacement $u^{(i)}$ to the ground motion of the underlying half-space, the displacement of each strip is assumed to have the form

$$u^{(i)}(t) = w^{(i)}(t) + h^{(i)}(t), \tag{5}$$

where $w^{(i)}$ is defined in equation (3) and $h^{(i)}(t)$ is the contribution to the strip displacement due to irregularities in the track itself. These are associated with roughness or bumps in the track that are not due to wave propagation effects in the lower half-space. This implies that the track irregularities are considered as the source term generating the vibrations. In the present paper, a method is developed for computing the wavefield w due to an irregular track geometry h.

3. INTEGRAL REPRESENTATION OF THE WAVEFIELD

The wave equation (1) is first transformed to a co-moving co-ordinate system (x', z). The co-moving horizontal co-ordinate x' is given by

$$x' = x - vt. \tag{6}$$

Subsequently, after performing a temporal Fourier transform, one obtains

$$\left[\left(1 - \frac{\rho v^2}{\mu}\right)\partial^2_{x'} + \frac{(\lambda + 2\mu)}{\mu}\,\partial^2_z + 2\mathrm{j}\omega\,\frac{\rho v}{\mu}\,\partial_{x'} + \omega^2\,\frac{\rho}{\mu}\right] w(x', z, \omega) = 0. \tag{7}$$

The first derivative can be eliminated by expressing the vertical displacement in the form

$$w(x', z, \omega) = \mathrm{e}^{\mathrm{j}\alpha x'}\, W\;(x', z, \omega) \tag{8}$$

with α given by

$$\alpha = -\,\omega v/(c^2 - v^2) \quad (v \neq c) \tag{9}$$

and c the wavespeed in the medium, given by

$$c = \sqrt{\frac{\mu}{\rho}}. \tag{10}$$

Upon substitution of equation (8) into equation (7), the following equation is obtained:

$$\left[\left(1 - \frac{v^2}{c^2}\right)\partial^2_{x'} + \frac{(\lambda + 2\mu)}{\mu}\,\partial^2_z + \frac{\omega^2}{(c^2 - v^2)}\right] W(x', z, \omega) = 0 \quad (v \neq c). \tag{11}$$

If the train speed v is less than the wave speed in the half-space, i.e., if $v < c$, the above equation is elliptic. If, on the other hand, the train speed v exceeds c, the equation is hyperbolic and the solutions behave very differently (see, for instance, also the discussion given by Krylov [2]). In the present paper, only the case $v < c$ is considered, being the one most important in practice. Then, equation (11) can be written as a Helmholtz equation

$$[\partial^2_{\tilde{x}} + \partial^2_{\tilde{z}} + k^2] \quad W(\tilde{x}, \tilde{z}, \omega) = 0 \tag{12}$$

with the scaled spatial co-ordinates given by

$$\tilde{x} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} x' \tag{13}$$

and

$$\tilde{z} = \left(\frac{\lambda + 2\mu}{\mu}\right)^{-1/2} z \tag{14}$$

and the scaled wavenumber k by

$$k = \omega / \sqrt{c^2 - v^2}. \tag{15}$$

With the aid of the integral representation for the Helmholtz equation and the boundary conditions (4) at the surface, the following integral relation is obtained between the surface displacement w and vertical traction T in the co-ordinate system of the moving strips:

$$w(\tilde{x}, \tilde{z}, \omega) = -\, \mathrm{e}^{\mathrm{j}\alpha(1 - v^2/c^2)^{1/2}\tilde{x}} \sum_{l=1}^{N} \int_{\tilde{x}' \in S_l} W^G(\tilde{x}, \tilde{x}', \tilde{z}, 0, \omega)[\mu(\lambda + 2\mu)]^{-1/2} T(\tilde{x}', 0, \omega)\, \mathrm{d}\tilde{x}', \tag{16}$$

where T is related to the traction t by a relation of the same form as equation (8). The Green function W^G of the traction-free half-space is given by

$$W^G(\tilde{x}, \tilde{x}', \tilde{z}, \tilde{z}', \omega) = -\frac{\mathrm{j}}{4}[H_0^{(2)}(kr^-) + H_0^{(2)}(kr^+)] \tag{17}$$

with $H_0^{(2)}$ being the Hankel function of second kind and order zero and the distance $r^{\pm}$ given by

$$r^{\pm} = \{(\tilde{x} - \tilde{x}')^2 + (\tilde{z} \pm \tilde{z}')^2\}^{1/2}. \tag{18}$$

The Green function W^G satisfies the Helmholtz equation (12) for a point source in the lower half-space and the traction-free boundary condition at the surface ($z = 0$). Equation (16) constitutes a relation between the displacement under the strips and the forces exerted by the strips on the half-space.

4. DYNAMIC BEHAVIOR OF THE STRIPS

The dynamic behavior of the strips is accounted for by the equation of motion. The analysis here closely follows the one given by Tan [11]. He considers the two-dimensional, full elastic problem for the case $v = 0$ (no movement). After performing the temporal Fourier transform, the equation of motion, for each strip i, is given by

$$F^{(i)}(\omega) + \int_{\tilde{x} \in S_i} t(\tilde{x}', \omega)\, \mathrm{d}\tilde{x}' = -\, m^{(i)}\omega^2 u^{(i)}(\omega), \tag{19}$$

where $m^{(i)}$ is the mass per unit length of strip i, $u^{(i)}$ is the displacement of the strip, t is the traction exerted by the half-space on the strip and $F^{(i)}$ is the force exerted by the spring–damper on the strip, see also Figure 1. The dynamic behavior is also dependent on the displacement of the hold-down mass with its mass per unit length $M^{(i)}$. After eliminating the displacement of the hold-down mass, the following equation results, describing the dynamic behavior of the strip [11]:

$$\int_{\tilde{x} \in S_i} t(\tilde{x}', \omega)\, \mathrm{d}\tilde{x}' = -\, M_e^{(i)}(\omega)\omega^2 u^{(i)}(\omega). \tag{20}$$

The effective mass, $M_e^{(i)}(\omega)$, is given by

$$M_e^{(i)}(\omega) = m^{(i)} - \frac{M^{(i)}\sigma^{(i)}}{(M^{(i)}\omega^2 - \sigma^{(i)})} \tag{21}$$

with $\sigma^{(i)}$ the spring constant which can be taken complex in order to take damping into account. The temporal Fourier transform of equation (5) is substituted into the equation of motion (20) and, after some manipulations, one obtains the following relation:

$$\int_{\tilde{x} \in S_i} \mathrm{e}^{\mathrm{j}\alpha(1 - v^2/c^2)^{1/2}\tilde{x}'} T(\tilde{x}', 0, \omega)\, \mathrm{d}\tilde{x}' = -M_e^{(i)}(\omega)\omega^2 \{w^{(i)}(\omega) + h^{(i)}(\omega)\}. \tag{22}$$

This represents the second relation between the displacement w and traction under each strip. In the next section, it is discussed how the resulting equations are solved numerically.

5. METHOD OF SOLUTION

In order to solve the boundary value problem given by equations (16) and (22), it is assumed that the traction $T(\tilde{x}', 0, \omega)$ under strip i can be replaced by its value in the center of the strip, i.e., $T(\tilde{x}', 0, \omega) \approx T(\tilde{x}^{(i)}, 0, \omega) = t^{(i)}(\omega)$. This approximation is valid if the width of the strip is small compared to the wavelength. The integral representation is evaluated at each strip center using equation (3). Then, the displacements $w^{(i)}(\omega)$ are eliminated and a linear system of equations results for the unknown discretized stress values $t^{(i)}(\omega)$:

$$\sum_{l=1}^{N} G^{(i,l)}(\omega) t^{(l)}(\omega) = b^{(i)}(\omega) \quad (i = 1, 2, \ldots, N). \tag{23}$$

The system matrix $G^{(i,l)}$ is given by

$$G^{(i,l)}(\omega) = -M_e^{(i)}(\omega)\omega^2 \mathrm{e}^{\mathrm{j}\alpha(1 - v^2/c^2)^{1/2}\tilde{x}^{(i)}} \int_{\tilde{x}' \in S_l} W^G(\tilde{x}^{(i)}, \tilde{x}', 0, 0, \omega)\, [\mu(\lambda + 2\mu)]^{-1/2}\, \mathrm{d}\tilde{x}'$$

$$+ \delta^{(i,l)} \int_{\tilde{x}' \in S_i} \mathrm{e}^{-\mathrm{j}\alpha(1 - v^2/c^2)^{1/2}\tilde{x}'}\, \mathrm{d}\tilde{x}', \tag{24}$$

where $\delta^{(i,l)}$ denotes the Kronecker delta, given by

$$\delta^{(i,l)} = \begin{cases} 1 & (i = l) \\ 0 & (i \neq l). \end{cases} \tag{25}$$

The right-hand side of the system, the source term $b^{(i)}$, is given by

$$b^{(i)}(\omega) = -M_e^{(i)}(\omega)\omega^2 h^{(i)}(\omega) \tag{26}$$

and is determined by the track irregularities h. By solving system (23), the stress values $t^{(i)}$ can be determined, for each frequency separately, if the track irregularity h is known. The

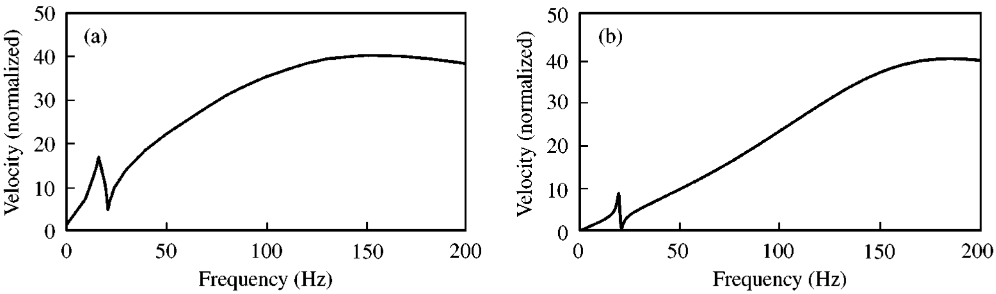

Figure 3. Comparison of the modulus of the full elastic solution for the vertical velocity of single strip with velocity $v = 0$, obtained by Tan (a), with the same quantity obtained by our method neglecting the horizontal component (b).

displacement w can then be computed with the aid of integral relation (16), after which the strip displacement follows from equation (5).

6. NUMERICAL RESULTS

In order to assess the validity of the assumption that the horizontal displacement can be neglected close to the strip, the method developed in this paper is first compared to the method of Tan [11] for a single strip with velocity $v = 0$. Tan considered the two-dimensional elastic case. In Figure 3(a), the vertical velocity of the strip is shown according to Tan. Since the method of Tan applies to a problem typical for seismic exploration for oil and gas, the parameters used are different from the ones relevant for the train vibration problem. Apart from the fact that Tan considers $v = 0$, this is also apparent from the large frequency range Tan considers and the choice of resonance frequency. Nevertheless, from a comparison of the results from our method in Figure 3(b) to Tan's results, one can observe a reasonable, qualitative correspondence indicating that neglecting the horizontal components does not lead to major errors. This is consistent with the results shown in Figure 2.

In order to examine the influence of interaction through waves propagating in the subsurface, first the case is considered without interaction. The model consists of two strips having a width of 1 m each, with a spacing of 18 m between the strips. Interaction between the strips is neglected by putting the non-diagonal terms of the matrix $G^{(i,l)}$ equal to zero in equation (23). The wave speed c equals 65 m/s and the speed v of the train equals 30 m/s. The eigenfrequency of each mass–spring system is 4·26 Hz. The damping ratio ($= \mathrm{Im}\{\sigma^{(i)}\}/2\sqrt{M^{(i)}\mathrm{Re}\{\sigma^{(i)}\}}$) is 0·1. In Figure 4, the vertical displacement $u^{(i)}(t)$ of the two strips is displayed when the track irregularity h is a single bump, defined by a $\cos^2$ function with a width of 10 m. Due to the absence of interaction, the two responses are similar but shifted due to the fact that the second strip reaches the bump later than the first strip. In Figure 4, one can also observe that each strip initially follows the shape of the bump, after which it excites waves propagating away from the bump (this is accounted for by the term $w^{(i)}(t)$; see also equation (5). Also clearly visible are the resonance frequency of the mass–spring system and the damping, partially caused by the spring damping, but mainly caused by radiation damping due to the fact that vibration energy is being radiated into the half-space. In Figure 5, the same quantities are displayed when the interaction between the strips, described by the non-diagonal terms of $G^{(i,l)}$, is taken into account. Upon comparison of Figures 4(a) and 5(a), one can observe that the resonance of the first strip damps out more

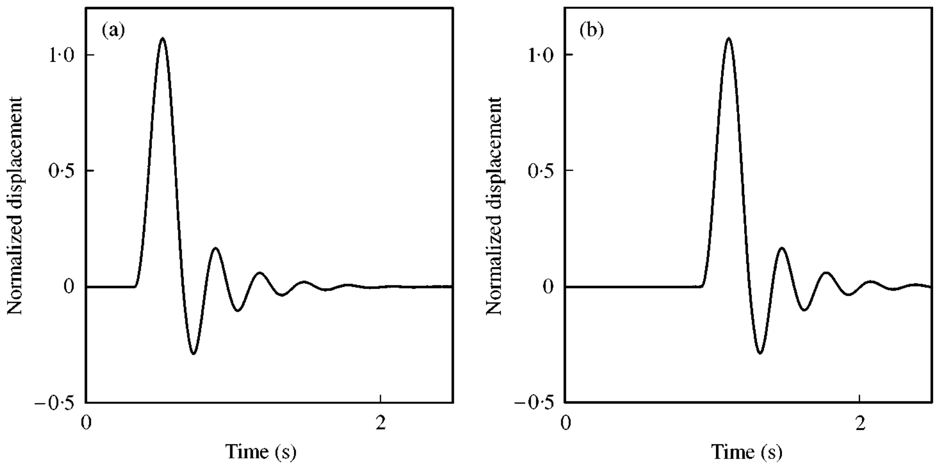

Figure 4. Vertical displacement of the first strip (a) and the second (and last) strip (b) when the track irregularity is formed by a bump. In both figures, the displacement is normalized with respect to the height of the bump. The interaction is not taken into account here.

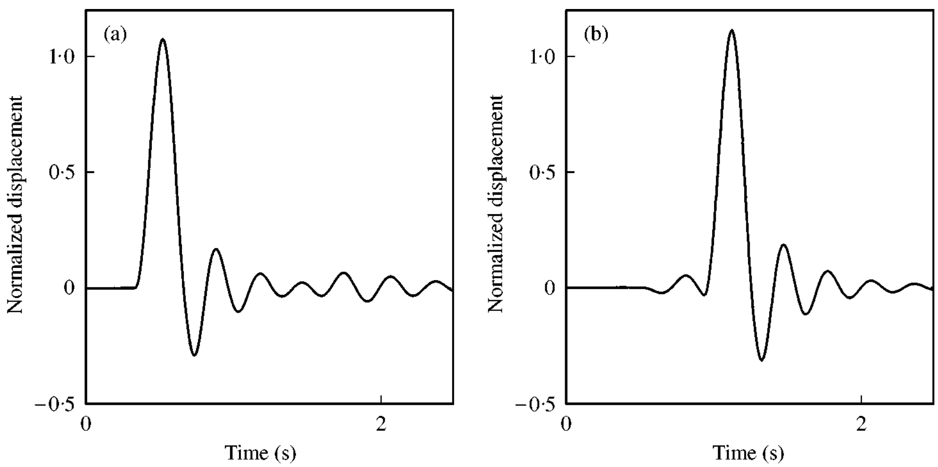

Figure 5. Vertical displacement of the first strip (a) and the second (and last) strip (b) when the track irregularity is formed by a bump. In both figures, all parameters are similar to the ones used in Figure 4, the only difference being that now, interaction is taken into account.

slowly in Figure 5(a). This is due to the excitation by waves originating from the second strip when it hits the bump. From a comparison of Figures 4(b) and 5(b), one can draw a similar conclusion and, in addition, observe that the response of the second strip has a precursor in Figure 5(b) due to waves, excited by the first strip upon passing the bump and that reach the second strip, before it actually has reached the bump itself.

In Figure 6, the normalized displacement of the first and last strips of a sequence of 20 strips is shown. All parameters are the same as in Figures 4 and 5, the only difference being that the width of the bump is now 13·3 m. The results are similar to the ones shown in the previous two figures, but the resonance damps out more slowly and the precursor of the last strip is much longer than in the case of two strips.

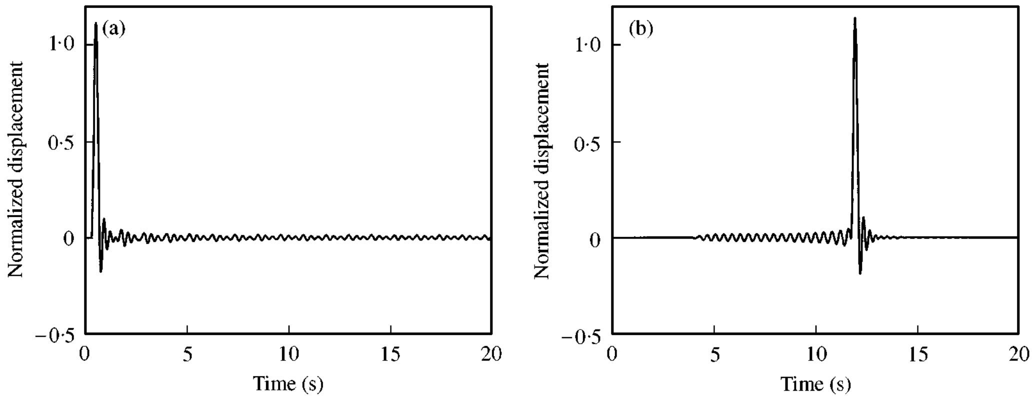

Figure 6. Vertical displacement of the first strip (a) and the 20 (and last) strip (b) when the track irregularity is formed by a bump. In both figures, the displacement is normalized with respect to the height of the bump.

7. CONCLUSIONS

Track irregularities can excite vibrations of the train axles. There is concern that interaction between the vibrations of different axles, through waves propagating in the near subsurface, can give rise to strong vibrations or even instabilities. In order to model this interaction mechanism, a two-dimensional model is studied where each wheel is represented by a rigid, two-dimensional strip. The dynamical behavior of the train itself is represented by a damped mass–spring system. Due to the fact that the subsurface is assumed to be a homogeneous half-space and a two-dimensional scalar formulation is used, only qualitative conclusions can be drawn from the model presented here. In order to be able to draw more quntitative conclusions, the model should be extended with the presence of a track structure and the presence of layering in the subsurface. The interaction is expected to be much stronger in the case of a soft weathered layer overlying a faster half-space. In that case, one would expect that much more vibration energy would remain trapped in the upper layer. In the current model, most energy is radiated into the half-space without contributing to the interaction between the axles. Nevertheless, it seems that interaction between train wheels and subsurface vibrations can result in an increased vibration of the mass–spring systems. In principle, the method presented here can be extended to the case of a layered half-space by considering the appropriate Green function. This is one of our current topics of investigation. Also, the case of supercritical train speed is being considered at the moment.

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