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ON SUBSTRUCTURING TECHNIQUE: APPLICATION OF LAGRANGE MULTIPLIER METHOD ON CYCLIC STRUCTURES

B. Irwanto, H.-J. Hardtke, D. Pawandenat and R. Schmidt

Department of Mechanical Engineering, Technische Universität Dresden, D-D1062 Dresden, Germany, E-mail: irwanto@mfm.mw.tu-dresden.de

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1. INTRODUCTION

In references [1–2], Dowell has studied the method of Lagrange multipliers for free vibration with arbitrary constraints as well as the application of this method in terms of component modes to analyze complex structures. By using this approach, it is possible to obtain the solution for a complex structure from the mode shape of each component or substructure. Simpson and Tabarrok [4] have used a different approach which is based on Kron's procedure. The procedure consists of two steps. Firstly, the structure will be disjoined into free-ended substructures. Secondly, all substructures will be interconnected by applying displacement constraints at the junctions. Simpson has expanded his work [5] by using Hamilton's principle to obtain the equation of motion, which is similar to that proposed by Dowell [1, 2]. Of course, there are many publications related to substructuring techniques, but only a few selected references [6–10] are given here.

Most of the works related to symmetric structures take advantage of this by using the symmetric or antisymmetric modes. In the present work, a different approach is performed. A complete model is divided into substructures and assembled by using Lagrange multipliers without employing the symmetry. Consequently, some difficulties seem to be encountered by using this approach, if there are some confluent natural frequencies of both the substructures and the structure. The consideration behind this approach is that in a real system, or more specifically in a cyclic structure, there are always differences between substructures. Thus, the use of symmetric structure properties may not suffice and the complete structure must be modelled.

2. SUBSTRUCTURING OF CYCLIC STRUCTURE

A cyclic structure which can be divided into N substructures is shown in Figure 1. Each substructure has two constraint conditions, represented as a left constraint ($\mathbf{f}_L$) and a right constraint ($\mathbf{f}_R$). For the ith substructure, the constraint conditions will be $\mathbf{f}_{Li}$ und $\mathbf{f}_{Ri}$.

Using the modal matrix of each substructure $\boldsymbol{\phi}_i$ and following the co-ordinate transformation of displacement vector $\{\mathbf{s}_i\} = [\boldsymbol{\phi}_i]\{\mathbf{z}_i\}$, the total kinetic ($T$) and potential energy ($U$) of all substructures can be described as follows:

$$T = \frac{1}{2}\sum_{i=1}^{N} \dot{\mathbf{s}}_i^T[\mathbf{M}]_i\dot{\mathbf{s}}_i = \frac{1}{2}\sum_{i=1}^{N} \dot{\mathbf{z}}_i^T[\mathbf{I}]\dot{\mathbf{z}}_i, \tag{1}$$

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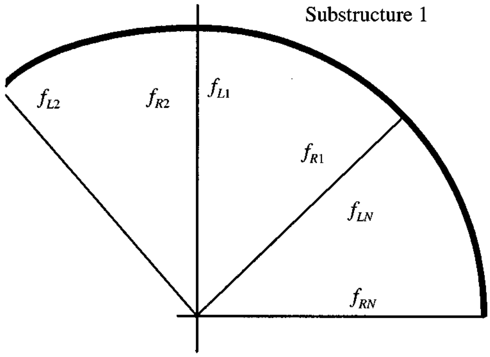

Figure 1. A structure and its constraint equation of each substructure.

$$U = \frac{1}{2}\sum_{i=1}^{N} \mathbf{s}_i^T[\mathbf{K}]_i\mathbf{s}_i = \frac{1}{2}\mathbf{z}_i^T[\omega_i^2]\mathbf{z}_i, \tag{2}$$

where $\mathbf{z}_i$ are the modal co-ordinates. $[\mathbf{M}_i]$ and $[\mathbf{K}_i]$ represent the mass and stiffness matrices of the ith substructure respectively. Thus, the condition $[\boldsymbol{\phi}_i]^{\mathrm{T}}[\mathbf{M}_i][\boldsymbol{\phi}_i] = [\mathbf{I}]$ and $[\boldsymbol{\phi}_i]^{\mathrm{T}}[\mathbf{K}_i][\boldsymbol{\phi}_i] = [\omega_i^2]$ should be fulfilled in which matrices $[\mathbf{I}]$ and $[\boldsymbol{\omega}_i^2]$ denote the identity matrix and the spectral matrix respectively.

Now, the constraint equation can be expressed as

$$\mathbf{f}_{Li} - \mathbf{f}_{R(i+1)} = \mathbf{0}, \quad i = 1, \ldots, N \tag{3}$$

and due to cyclic configuration, it follows that

$$\mathbf{f}_{R1} = \mathbf{f}_{R(N+1)}. \tag{4}$$

Before obtaining the equation of motion, one should define the right and left constraint equation of the ith substructure given by

$$\begin{Bmatrix} \mathbf{f}_{Li} \\ \mathbf{f}_{Ri} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\beta}_{Li} \\ \boldsymbol{\beta}_{Ri} \end{Bmatrix} \mathbf{z}_i, \tag{5}$$

where $\boldsymbol{\beta}_{Li}$ and $\boldsymbol{\beta}_{Ri}$ are the corresponding left and right constraint constants.
Substituting equations (4) and (5) into equation (3) and after arrangement, one obtains the following equation for the complete structure:

$$\begin{bmatrix} \boldsymbol{\beta}_{L1} & -\boldsymbol{\beta}_{R2} & \mathbf{0} & \mathbf{0} & \cdot\cdot & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\beta}_{L2} & -\boldsymbol{\beta}_{R3} & \mathbf{0} & \cdot\cdot & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\beta}_{L3} & \mathbf{0} & \mathbf{0} & : \\ : & : & \mathbf{0} & \cdot\cdot & -\boldsymbol{\beta}_{R(N-1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & : & \mathbf{0} & \boldsymbol{\beta}_{L(N-1)} & -\boldsymbol{\beta}_{RN} \\ -\boldsymbol{\beta}_{R1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\beta}_{LN} \end{bmatrix} \begin{Bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ : \\ : \\ \mathbf{z}_{N-1} \\ \mathbf{z}_N \end{Bmatrix} = \mathbf{0} \tag{6}$$

or it can be written as

$$\mathbf{f} = [\boldsymbol{\beta}_m]\{\mathbf{z}\} = \mathbf{0} \tag{7}$$

in which $\boldsymbol{\beta}_m$ represents the modified constraint constant for the complete structure.

From equation (6), it is obvious that each column in the left-hand matrix represents the constraint of each substructure.

The Lagrangian for the complete structure may be written as

$$L = T - U + \sum_r \boldsymbol{\lambda}^{\mathrm{T}}\mathbf{f}, \tag{8}$$

where $\boldsymbol{\lambda}$ are Lagrange multipliers and r expresses the number of constraints which is the same as the number of junction DOF.

On applying Lagrange's equation, the equation of motion and the constraint equations of the assembly are

$$[\mathbf{M}_g]\{\ddot{\mathbf{z}}\} + [\mathbf{K}_g]\{\mathbf{z}\} - [\boldsymbol{\beta}_m]^{\mathrm{T}}\boldsymbol{\lambda} = \mathbf{0}, \tag{9}$$

$$\mathrm{f} = \sum_{i=1}^{N} [\boldsymbol{\beta}]_i \mathbf{z}_i = [\boldsymbol{\beta}_m]\{\mathbf{z}\} \tag{10}$$

in which the global mass and stiffness matrices are given by

$$\mathbf{M}_g = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} \end{bmatrix} \quad \text{and} \quad \mathbf{K}_g = \begin{bmatrix} [\omega_1^2 & \mathbf{0} & \cdot\cdot & \mathbf{0} \\ \mathbf{0} & [\omega_2^2] & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cdot\cdot & \cdot\cdot & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & [\omega_N^2] \end{bmatrix}.$$

Thus, assuming harmonic solution the following equations can be obtained:

$$\boldsymbol{\beta}_m \mathbf{D}_g \boldsymbol{\beta}_m^{\mathrm{T}} \boldsymbol{\lambda} = \mathbf{0} \ \text{ or } \ \mathbf{R}\boldsymbol{\lambda} = \mathbf{0}, \tag{11a, b}$$

$$\det(\boldsymbol{\beta}_m \mathbf{D}_g \boldsymbol{\beta}_m^{\mathrm{T}}) = \mathbf{0}, \tag{12}$$

where $\mathbf{D}_g$ is defined by

$$\mathbf{D}_g = \begin{bmatrix} \left[\dfrac{1}{\omega_1^2 - \omega^2}\right] & 0 & 0 & 0 \\ 0 & \left[\dfrac{1}{\omega_2^2 - \omega^2}\right] & 0 & 0 \\ 0 & & \cdot\cdot & 0 \\ 0 & & & \left[\dfrac{1}{\omega_N^2 - \omega^2}\right] \end{bmatrix}.$$

By solving equations (11) and (12), the natural frequencies and the mode shapes of the complete structure can be attained. Moreover, it could be seen that the effort to determine the natural frequencies may be less because the matrix **R** has a reduced size ($r \times r$). It is also interesting to trace back how to solve these equations because they have been developed mainly only by some authors [11–15]. In addition, it has been addressed that a difficult condition may occur if the substructures have the same natural frequencies and the total structure has a multiplicity of natural frequencies. In this paper, these obstacles could be handled by using the same technique with some cautions as in references [13, 14, 16].

3. NUMERICAL EXAMPLES

Two examples are used to illustrate the accuracy of the formulation. Firstly, an annular plate of constant thickness will be considered and secondly, a mistuned bladed disc will be used. For both examples, the finite element method with shell elements is used to obtain the mass and stiffness matrices.

3.1. ANNULAR PLATE

In this example, a plate made of aluminium was considered. The plate has a thickness of 2 mm, inner radius of 25 mm and outer radius of 100 mm. It is clamped at its inner

TABLE 1

Comparison of natural frequencies (Hz) of annular disc

Mode	ANSYS	LMM
1	283·64	283·64
2	296·04	296·04
3	354·14	354·14
4	640·44	640·44

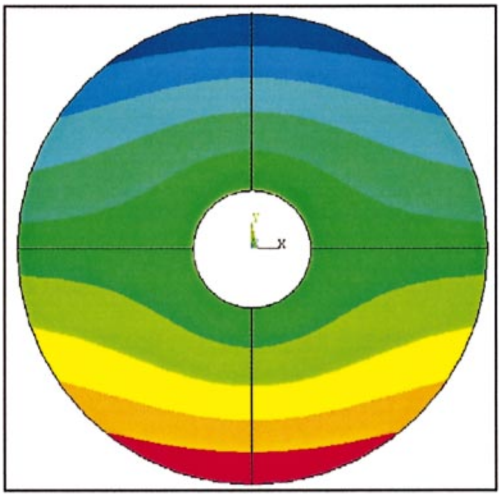

Figure 2. The mode shape of the first natural frequency (283·64 Hz).

boundary and free at its outer boundary. The annular plate was divided into four substructures and each substructure was discretized by applying the finite element method. The discretized substructure consists of 50 shell elements and 66 nodes. Due to the symmetry properties, each substructure has the same eigenvalues. The complete model was built from the substructures by using the Lagrange multiplier method (LMM). The commercial software ANSYS was used to calculate the natural frequencies and compare the results. As can be seen in Table 1, the comparison shows a good agreement. Except for the second natural frequency, the others are double frequencies. The mode shapes of three calculated modes of the plate are given in Figures 2–4 respectively.

3.2. MISTUNED BLADED DISC

A simple model of a bladed disc will be used to prove the above derivation as well. Figure 5 shows the FE Model which consists of one disc and eight blades. No constraint is

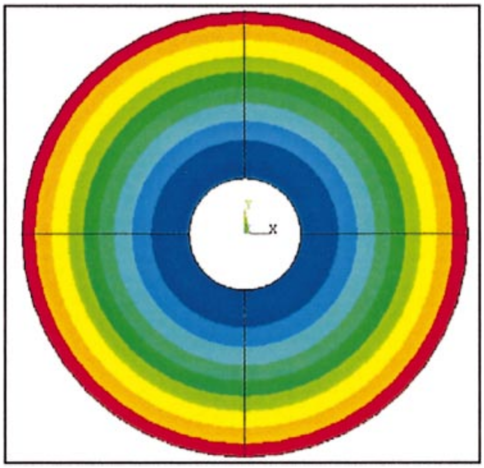

Figure 3. The mode shape of the second natural frequency (296·04 Hz).

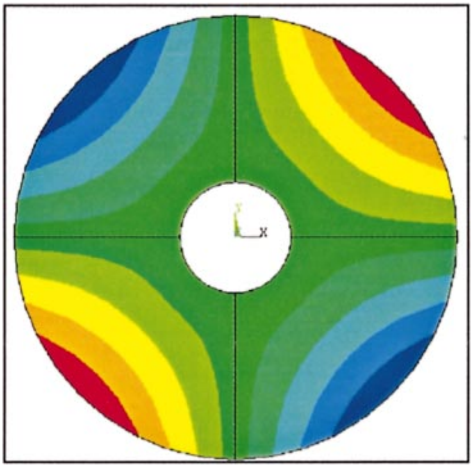

Figure 4. The mode shape of the third natural frequency (354·14 Hz).

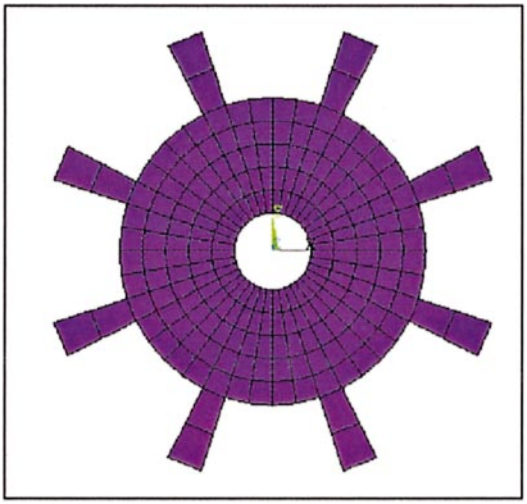

Figure 5. FE Model of mistuned bladed disc.

TABLE 2

Comparison of the natural frequencies (Hz) of bladed disc

Mode	ANSYS	LMM
1	143·31	143·31
2	144·90	144·90
3	196·30	196·30
4	205·83	205·83
5	209·78	209·78
6	220·22	220·22
7	280·49	280·49
8	287·03	287·03

applied to the model. To introduce mistuning, Young's modulus of each blade was changed. By doing this, different dynamic properties of each substructure were obtained. The geometry and material properties of the disc are the same as those of the plate mentioned before. For reasons of consistency, four substructures were also used. Each substructure has two blades. Again, from the comparison the method yields excellent results (Table 2).

Based on two examples, there are two important points to note. In the above calculations, a complete set of normal modes were used. Special attention should be paid if one utilizes a truncated set of normal modes instead of the complete set of modes. Although the computational time will be lower, the truncation of modes might introduce a certain degree of error and difficulties in finding the location of eigenvalues. The second point is that the expected results should be exact relative to the calculated results for the parent finite element model, since the complete set of modes of each substructure have been used.

4. CONCLUSION

A substructuring technique based on a Lagrange multiplier formalism has been presented. From the present work, it can be shown that the Lagrange multiplier method might be employed to form a complete structure (in this case it consists of shell elements) without using the symmetric property. In future work, the method will be applied to solid element models of bladed discs and optimization of this method will be carried out to reduce the computational time for large models.

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