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ELASTIC WAVE SCATTERING ON A SYSTEM OF ROD-LIKE INCLUSIONS

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Scattering of elastic wave on two collinear cylindrical inclusions is considered. Analysis is restricted by the case of normal incidence. Inclusions are thin; the aspect ratio is small. The length of the incident wave is comparable with the length of inclusion, and the distance between them. Inclusions and the surrounding medium are homogeneous, isotropic, and linearly elastic. They differ only in the mass density. Direct numerical analysis (such as FEM, BEM, FDM, etc.) of scattering on thin deformable inclusions is connected with the principal difficulty caused by degeneration of the domain occupied by inclusions into a set of segments. A two-dimensional (2-D) approach, where the length is assumed to be infinite, is inefficient at low frequencies. An engineering approach based on beam theory equations (for inclusions) would lead to considerable errors. An original asymptotic approach is proposed. The integral equation of stationary motion of an inhomogeneous elastic medium is derived and then asymptotically simplified. The original 3-D dynamic problem is decomposed to the combination of two problems of reduced dimension. The first one is governed by the integral equation over the mid-line contour. The second one is a 2-D quasi-static problem for the cross-section of inclusion. In such a way the separation of variables is made. The averaged (over the cross-section) displacement of inclusions is calculated numerically. Results obtained are compared with the corresponding ones for the single inclusion. Displacement and stress fields inside inclusions are to be determined through solving a quasi-static 3-D (2-D at the points of middle region of the inclusion) problem.

1. INTRODUCTION

Dynamic interaction of a plane harmonic wave with a system of thin rod-like inclusions, shown in Figure 1, is under study. It is assumed that the length of the inclusions, and the distance between them is comparable with the length of the incident wave. The medium and inclusions are homogeneous, isotropic, and linearly elastic, and they differ only in mass density.

As the scattering problem is very complicated, direct numerical methods and simplified approaches are mainly used for its solution. As to the latter methods it should be noted that there the first significant results were obtained by Rayleigh who suggested an original approach to diffraction, which was named after him [1].

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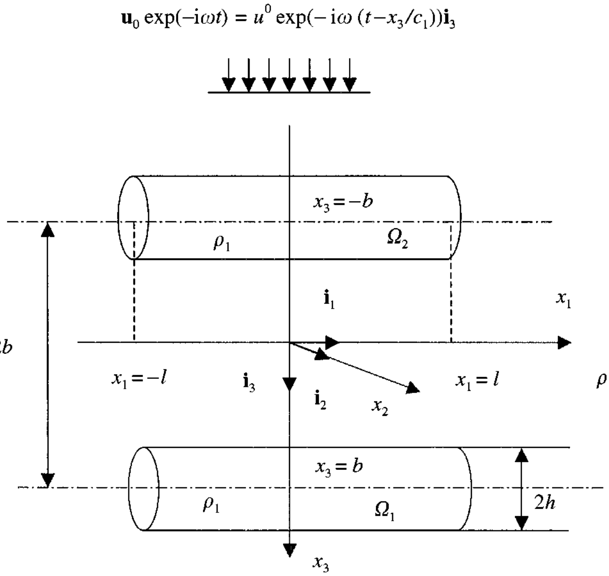

Figure 1. Configuration of the system.

The other approach to the scattering on the thin inclusion is to consider the two-dimensional (2-D) problem, assuming that the inclusion length is infinite. If the total vector of the external load is equal to zero, this 2-D approach allows one to find the displacements at the points of an infinite homogeneous elastic plane. In the general case, the displacement obtained within the framework of a 2-D problem does not have a finite static limit and tends to infinity as the logarithm of the frequency ω as $\omega \to 0$ [2]. For this reason the 2-D statement cannot be used at low frequencies. In the 3-D stationary problem (for a finite length of the inclusion) the displacement is finite, and it transforms into a finite solution of the corresponding 3-D static problem as $\omega \to 0$.

Another approach (engineering) is based on the concept that the inclusions are governed by beam theory equations, the medium is governed by the equations of Navier, and interface conditions complete the statement. In this case, the use of the beam theory equations is not quite correct from the mathematical point of view. The estimation of the error of the solution is a significant problem. In the case of inclusions whose properties are close to those of the medium, the beam approach leads to absurd results.

Direct numerical methods (such as FEM, BEM, BIEM, FDM, etc.) allow one to analyze diffraction successfully. But implementation of these methods requires voluminous calculations. In our case, when the aspect ratio of the inclusions is small, the degeneration of the domain occupied by inclusions into a set of segments creates the principal difficulty. This obstacle is associated with numerous reflections–refractions of elastic waves in a very slender region, and numerical solution should take into account all these phenomena. For this reason an asymptotic approach seems to be promising.

In the corresponding static problems for thin domains, the presence of a small geometrical parameter causes difficulties in the implementation of the numerical methods as well. However, in this case the numerical solutions have already been improved by the results of asymptotic analysis.

In order to list the significant results for thin domains, we should mention the asymptotic solutions of Geer [3], Fedoriuk [4, 5] and do Rego Silva [6], devoted to diffraction on thin inclusions in an acoustic medium, and the solutions of Zhdanova [7–9]. She considered diffraction on the thin straight cavity in both acoustic and elastic media, and used a special integral representation for the unknown displacement. She derived the integral equation for the unknown auxiliary function (density of potential).

We propose an original asymptotic approach to elastic wave scattering on thin inclusions. We simplify the original 3-D boundary value problem of elastodynamics for a nonhomogeneous medium, using the presence of the natural small parameters. In the results the original 3-D dynamic problem is decomposed to a combination of two problems, of reduced dimension. The first of them, (for unknown displacement averaged over the cross-section), is governed by the integral equation over the mid-line contour of the inclusions. It contains information about 3-D dynamic effects of the process. The second one is a 3-D quasi-static problem, which describes the stress and displacement distribution inside the inclusions at a found, averaged displacement. All equations obtained are suitable for further numerical solution, and in this paper the results of calculations of the inclusions' deflection and averaged axial displacement are presented. These results were partly presented at ICCE/7 [10].

We carry out the asymptotic analysis of some singular integrals we met in the same manner as it was originally done in the asymptotic theory of static contact, between elastic bodies at a slender area of interaction, developed by Kalker [11, 12]. Similar analysis was done in reference [13], where the results of Kalker were generalized to a dynamic case, (dynamic problem of the slender die, when the incident wavelength is comparable with the length of the contact area).

The paper is organized as follows. In the next section, the general mathematical formulation is presented. The Navier equations for homogeneous (with density ρ), and non-homogeneous media, and the fundamental solution (Green function), for the homogeneous medium are used to obtain the integral equation for the scattered field. It is shown that in the case when the inclusions and surrounding medium differ only in mass density, the scattered field is represented as a triple convolution over the domain occupied by the inclusions of an unknown "body force", and the fundamental solution for the homogeneous medium. The asymptotic analysis of this equation is carried out in the following section. The special representation of an arbitrary "body force" being made, the actions of the load, self-balanced in the cross-section of the inclusion, and of the linear load, applied at the mid-line contour of the inclusion, are considered separately (sections 3.2 and 3.3). The results of the asymptotic analysis are summarized in section 3.4. Section 4 is devoted to the asymptotic decomposition. The original 3-D problem of elastodynamics is reduced to two problems of decreased dimension. The two problems mentioned above are considered in detail. The asymptotic integral equations for the single inclusion are given in section 5. The results of the numerical analysis are presented in section 6 for the single inclusion (section 6.1) and for two inclusions (section 6.2). Some auxiliary asymptotic estimates are given in Appendix A.

2. GENERAL FORMULATION

Diffraction of the plane harmonic longitudinal wave, $\mathbf{u}_0 \exp(-\mathrm{i}\omega t)$ ($\mathbf{u}_0 = u^0 \exp(\mathrm{i}\omega x_3/c_1)\mathbf{i}_3$), on the system of two collinear cylindrical inclusions is under study (see Figure 1). Here u^0 is an amplitude of the displacement in the incident wave, x_1, x_2, x_3 are co-ordinates in the Cartesian basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, ω is a frequency of the incident wave, t is time, and c_1 is the speed of longitudinal waves in the medium. It is assumed that the length $2l$ of the inclusions, and the distance $2b$ between them, are comparable with the length

$\lambda = 2\pi c_1/\omega$ of the incident wave, and that the radius h of the inclusion is much smaller than this length. Thus, two small parameters of the same order are present in this problem: the geometrical parameter $\varepsilon = h/l$, and parameter $\varepsilon_1 = h/\lambda$ associated with time scale. Inclusions and the surrounding medium are homogeneous, isotropic, and linearly elastic. The present consideration is limited by the case of different mass densities; here ρ_1 is the density of the inclusions, ρ is the density of the medium.

The incident wave strikes the inclusions and creates the displacement field

$$\mathrm{e}^{-\mathrm{i}\omega t}\mathbf{u} = \mathrm{e}^{-\mathrm{i}\omega t}(u_1, u_2, u_3). \tag{1}$$

The time factor $\mathrm{e}^{-\mathrm{i}\omega t}$ will be omitted below.

The medium with inclusions can be considered as a non-homogeneous medium with density ρ

$$\rho_* = \rho + (\rho_1 - \rho)P_\Omega(\mathbf{x}),$$

where the factor $P_\Omega(\mathbf{x})$ is equal to unity in the region, $\Omega = \Omega_1 \cup \Omega_2$ inside the inclusions ($\mathbf{x} \in \Omega$) and equal to zero otherwise ($\mathbf{x} \notin \Omega$).

The displacement $\mathbf{u}$ in this non-homogeneous medium is governed by the equation of Navier,

$$\mu\left(\Delta\mathbf{u} + \frac{1}{1-2\nu}\,\mathbf{grad}\,\mathbf{div}\,\mathbf{u}\right) + (\rho + (\rho_1 - \rho)P_\Omega)\omega^2\mathbf{u} = 0, \tag{2a}$$

where Δ is the Laplacian, ν is the Poisson ratio, $\mu = \rho c_2^2$ is the shear modulus, and c_2 is the speed of shear waves in the medium ($c_1^2 = 2c_2^2(1-\nu)/(1-2\nu)$). Sommerfeld's radiation condition at infinity, and the condition of continuity of the displacement at the interface complete the statement of this problem.

The incident wave is the solution of the homogeneous equation of Navier for the homogeneous medium with the density ρ,

$$\mu\left(\Delta\mathbf{u}_0 + \frac{1}{1-2\nu}\,\mathbf{grad}\,\mathbf{div}\,\mathbf{u}_0\right) + \rho\omega^2\mathbf{u}_0 = 0. \tag{2b}$$

Subtracting equation (2b) from equation (2a), yields the equation for the scattered field $\mathbf{u} - \mathbf{u}_0$,

$$\mu\left(\Delta(\mathbf{u} - \mathbf{u}_0) + \frac{1}{1-2\nu}\,\mathbf{grad}\,\mathbf{div}(\mathbf{u} - \mathbf{u}_0)\right) + \rho\omega^2(\mathbf{u} - \mathbf{u}_0) = \omega^2(\rho - \rho_1)\,\mathbf{u}P_\Omega. \tag{2c}$$

Equation (2c), for $\mathbf{u} - \mathbf{u}_0$, looks like the non-homogeneous equation of Navier for a homogeneous medium (with constant density ρ), subjected to the (unknown) "body force" $\mathbf{f} = \omega^2(\rho_1 - \rho)\,\mathbf{u}P_\Omega$. The displacement $\mathbf{u} - \mathbf{u}_0$ can be expressed by the convolution of the fundamental solution for a homogeneous medium $\mathbf{G}^\omega(R)$ and this body force,

$$\mathbf{u}(\mathbf{x}) - \mathbf{u}_0(\mathbf{x}) = \iiint\limits_{\mathbf{R}^3} \mathbf{G}^\omega(|\mathbf{x} - \mathbf{x}'|)\cdot\mathbf{f}(\mathbf{x}')\,\mathrm{d}\Omega(\mathbf{x}') = (\rho_1 - \rho)\omega^2 \iiint\limits_{\Omega} \mathbf{G}^\omega(|\mathbf{x} - \mathbf{x}'|)\cdot\mathbf{u}(\mathbf{x}')\,\mathrm{d}\Omega(\mathbf{x}'), \tag{3}$$

where $\mathbf{R}_3$ denotes the infinite elastic medium.

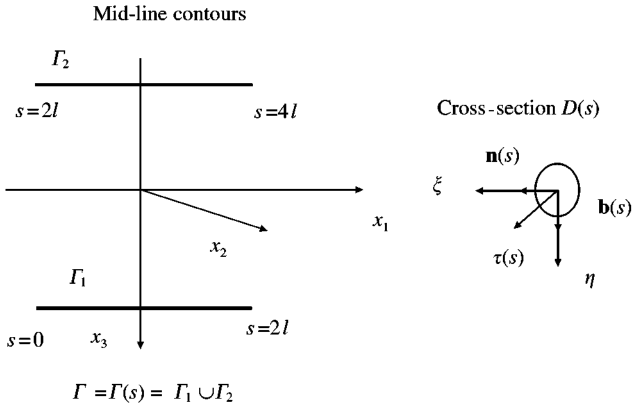

Figure 2. Local co-ordinate system.

The components of the tensor $\mathbf{G}^{\omega}(R)$ are the corresponding components of the displacement due to an oscillating point force applied to the infinite medium. In the Cartesian co-ordinate system they are [2]

$$\mathbf{G}^{\omega}_{jk}(R) = \frac{1}{4\pi\mu}\left[\delta_{jk}\left(\frac{1}{R}\mathrm{e}^{-\mathrm{i}\omega R/c_2} - \frac{c_2^2}{\omega^2}P(R)\right) - \frac{c_2^2}{\omega^2}x_k x_j Q(R)\right], \tag{4}$$

where

$$P(R) = \frac{1}{R^3}\left[\mathrm{e}^{\mathrm{i}\omega R/c_1}\left(\mathrm{i}\frac{\omega R}{c_1} - 1\right) - \mathrm{e}^{\mathrm{i}\omega R/c_2}\left(\mathrm{i}\frac{\omega R}{c_2} - 1\right)\right],$$

$$Q(R) = \frac{1}{R^5}\left[\mathrm{e}^{\mathrm{i}\omega R/c_1}\left(-\frac{\omega^2 R^2}{c_1^2} - 3\mathrm{i}\frac{\omega R}{c_1} + 3\right) - \mathrm{e}^{\mathrm{i}\omega R/c_2}\left(-\frac{\omega^2 R^2}{c_2^2} - 3\mathrm{i}\frac{\omega R}{c_2} + 3\right)\right],$$

$$R^2 = x_1^2 + x_2^2 + x_3^2.$$

Here δ_{jk} denotes the Kronecker delta, and R is the distance between the field point and source point.

To carry out the asymptotic analysis, we introduce the local co-ordinate system associated with the mid-line contour of the inclusions (see Figure 2). The general mid-line contour $\Gamma(s)$ with axial (mid-line) co-ordinate s, is a set of contours Γ_1 and Γ_2 and a natural trihedral $\boldsymbol{\tau}(s)$, $\mathbf{n}(s)$ and $\mathbf{b}(s)$ is associated with point s of this contour. In this case, the position vector $\mathbf{x}$ is represented as

$$\mathbf{x} = \mathbf{x}(s, \xi, \eta) = \mathbf{r}(s) + \xi\mathbf{n}(s) + \eta\mathbf{b}(s),$$

where ξ and η are local co-ordinates in the cross-section $D = D(s)$, and the vector $\mathbf{r}(s) = \mathbf{x}(s, 0, 0)$ is the position vector of the point belonging to contour Γ. In this particular example, when the inclusions are straight, the basic vectors of this local system are constant and they coincide with Cartesian basic vectors:

$$\boldsymbol{\tau}(s) = \mathbf{i}_1$$
$$\mathbf{n}(s) = \mathbf{i}_2$$
$$\mathbf{b}(s) = \mathbf{i}_3$$

and

$$\mathbf{r}(s) = (s - l)\,\boldsymbol{\tau}(s) + b\mathbf{n}(s) = x_1\mathbf{i}_1 + b\mathbf{i}_2, \quad s \in (0, 2l),$$
$$\mathbf{r}(s) = (s - 3l)\,\boldsymbol{\tau}(s) + b\mathbf{n}(s) = x_1\mathbf{i}_1 + b\mathbf{i}_2, \quad s \in (2l, 4l).$$

The introduction of this local system allows one to analyze the more complicated case of curvilinear inclusions on the basis of the same approach.

3. ASYMPTOTIC ANALYSIS OF INTEGRAL EQUATION

The displacement of the non-homogeneous medium (both inside and outside inclusions), is governed by equation (3) containing triple integrals. Direct numerical solution of equation (3) meets the principal obstacle, caused by the degeneration of the region Ω. However, in the case under study, the presence of small parameters allows us to simplify equation (3) by asymptotic methods. In this paper, we mainly take an interest in the displacement and stress field inside the inclusion, and the consequent asymptotic analysis is carried out below for internal points $\mathbf{x} \in \Omega$ of the inclusions.

3.1. REPRESENTATIONS FOR DISPLACEMENT AND LOAD

We are looking for the displacement in the following form:

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}\left(\frac{s}{l}, \frac{\eta}{h}, \frac{\xi}{h}\right).$$

Let us represent the unknown "body force" **f** by a sum of two terms. The first one is a linear load applied at the mid-line contour. It has the same total force vector **F** in the cross-section $s = const$ as the original "body force" **f**. The second term is a self-balanced load (in every cross-section $s = const$), which is equal to the difference between the two mentioned loads:

$$\mathbf{f}(\mathbf{x}) = \mathbf{F}(s)\delta(\mathbf{x} - \mathbf{r}(s)) + [\mathbf{f}(\mathbf{x}) - \mathbf{F}(s)\delta(\mathbf{x} - \mathbf{r}(s))],$$

$$\mathbf{F}(s) = \iint_D \mathbf{f}(\mathbf{x})\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

δ denotes the delta-function of Dirac. Since the system is linear, the contributions of two loads on the right-hand side to the displacement will be estimated separately.

Let the vector $\mathbf{U}(s) = (U_1, U_2, U_3)$ denote the average displacement over the cross-section D,

$$\mathbf{U}(s) = \iint_D \mathbf{u}(\mathbf{x})\,\mathrm{d}\xi\,\mathrm{d}\eta \Big/ \iint_D \mathrm{d}\xi\,\mathrm{d}\eta.$$

In our case of cylindrical inclusions we have

$$\mathbf{F}(s) = \omega^2(\rho_1 - \rho)\iint_D \mathbf{u}(\mathbf{x})\,\mathrm{d}\xi\,\mathrm{d}\eta = \omega^2(\rho_1 - \rho)\pi h^2\mathbf{U}(s).$$

For the reason of symmetry the components $U_2 = F_2 = 0$, (in the Cartesian basis).

Thus equation (3) is rewritten in the form

$$\mathbf{u}-\mathbf{u}_0 = \int_0^{4l}\iint_D \mathbf{G}^\omega(|\mathbf{x}(s,\xi,\eta)-\mathbf{x}(s',\xi',\eta')|)\cdot\mathbf{f}(s',\xi',\eta')\,\mathrm{d}\xi'\,\mathrm{d}\eta'\,\mathrm{d}s'$$

$$= \int_0^{4l}\mathbf{G}^\omega(|\mathbf{x}(s,\xi,\eta)-\mathbf{r}(s')|)\cdot\mathbf{F}(s')\,\mathrm{d}s' + \mathbf{G}^\omega{*}{\circ}{\circ}[\mathbf{f}(\mathbf{x})-\mathbf{F}(s)\delta(\mathbf{x}-\mathbf{r}(s))], \qquad (5)$$

where the convolution with respect to the axial co-ordinate s (integral over the contour Γ) is denoted by an asterisk,

$$\varphi_1 * \varphi_2 = \varphi_1(s) * \varphi_2(s) = \int_\Gamma \varphi_1(s') * \varphi_2(s-s')\,\mathrm{d}s',$$

and the convolutions with respect to "short" co-ordinates ξ and η (integral over the cross-section) are denoted by the circles:

$$\psi_1 \circ\circ \psi_2 = \psi_1(\xi,\eta)\circ\circ\psi_2(\xi,\eta) = \iint_D \psi_1(\xi',\eta')\psi_2(\xi-\xi',\eta-\eta')\,\mathrm{d}\xi'\,\mathrm{d}\eta'.$$

In the following sections we estimate the two terms on the right-hand side of equation (5) separately.

3.2. THE LOAD, SELF-BALANCED IN THE CROSS-SECTION

Let the body force $\mathbf{g}(\mathbf{x}) = \mathbf{g}(s/l, \xi/h, \eta/h)$ be applied to the region Ω, and let it be self-balanced over the cross-section $s = const$:

$$\iint_D \mathbf{g}(\mathbf{x})\,\mathrm{d}D = \iint_D \mathbf{g}(\mathbf{x})\,\mathrm{d}\xi\,\mathrm{d}\eta = \mathbf{g}(\mathbf{x})\circ\circ 1 = 0.$$

Then the following asymptotic estimations can be made:

$$\mathbf{G}^\omega * \circ\circ\, \mathbf{g}(\mathbf{x}) = (\mathbf{G}^\omega(\mathbf{x}) - \mathbf{G}^3(\mathbf{x})) * \circ\circ\, \mathbf{g}(\mathbf{x}) + \mathbf{G}^3(\mathbf{x}) * \circ\circ\, \mathbf{g}(\mathbf{x})$$

$$\underset{\varepsilon\to 0}{\to} (\mathbf{G}^\omega(\mathbf{r}) - \mathbf{G}^3(\mathbf{r})) * [1 \circ\circ\, \mathbf{g}(\mathbf{x})] + \mathbf{G}^3(\mathbf{x}) * \circ\circ\, \mathbf{g}(\mathbf{x}) = \mathbf{G}^3(\mathbf{x}) * \circ\circ\, \mathbf{g}(\mathbf{x}). \qquad (6)$$

Here $\mathbf{G}^2$ and $\mathbf{G}^3$ are the fundamental solutions of Kelvin for the two- and three-dimensional static problems respectively (the two-dimensional problem consists of plane strain and longitudinal shear problems). Since the dependence of the difference $\mathbf{G}^\omega - \mathbf{G}^3$ on the small parameter ε is uniformly continuous and since this difference is limited at $\varepsilon = 0$, it is possible to make an asymptotic replacement $\mathbf{G}^\omega(\mathbf{x}) - \mathbf{G}^3(\mathbf{x})$ by $\mathbf{G}^\omega(\mathbf{r}) - \mathbf{G}^3(\mathbf{r})$ at $\varepsilon \to 0$, and obtain the estimate (6). In the Cartesian basis the components of tensors $\mathbf{G}^2$ and $\mathbf{G}^3$ are [14]

$$G^2_{11}(x_2, x_3) = -\frac{1}{4\pi\mu}\ln(x_2^2 + x_3^2),$$

$$G_{ij}^2(x_2, x_3) = \frac{1}{16\pi\mu(1-\nu)}\left[-(3-4\nu)\delta_{ij}\ln(x_2^2+x_3^2) + \frac{2x_ix_j}{x_2^2+x_3^2}\right], \quad i,j = 2,3,$$

$$G_{12}^2 = G_{21}^2 = G_{13}^2 = G_{31}^2 = 0,$$

$$G_{ij}^3(x_1, x_2, x_3) = \frac{1}{16\pi\mu(1-\nu)}\left[\frac{(3-4\nu)}{R}\delta_{ij} + \frac{x_ix_j}{R^3}\right]. \tag{7}$$

The component ij means i-component of the displacement caused by the unit point force applied at the origin and oriented along the axis j.

It should be noted that the influence of the self-balanced load applied in the region Ω_2 on the displacement in the region Ω_1 (and *vice versa*) is asymptotically small. Thus instead of the integral (convolution) over the whole region Ω, we take into account only the integral over the region Ω_1 at $\mathbf{x} \in \Omega_1$ and the integral over Ω_2 at $\mathbf{x} \in \Omega_2$.

Then we consider only interior points $\mathbf{x}$ located far from the ends of the inclusions. The substitution of the explicit form of the fundamental solution $\mathbf{G}^3$ into equation (6), and subsequent analysis of the corresponding convolution $\mathbf{G}^3$ and self-balanced load $\mathbf{g}(\mathbf{x})$, allows one to derive the following estimate (see Appendix A, sections A.1 and A.2):

$$\mathbf{G}^3(\mathbf{x}) * \circ \circ\, \mathbf{g}(\mathbf{x}) \underset{\varepsilon\to 0}{\to} \mathbf{G}^2(\mathbf{x}-\mathbf{r}) \circ \circ\, \mathbf{g}(\mathbf{x}) = \iint_D \mathbf{G}^2(\xi-\xi', \eta-\eta') \cdot \mathbf{g}(s, \xi', \eta')\, \mathrm{d}\xi'\, \mathrm{d}\eta'. \tag{8}$$

In such a way the displacement (at the points far from the ends) due to self-balanced load can be determined within the framework of a 2-D quasi-static problem. Near the ends estimate (8) can be invalid and the corresponding 3-D quasi-static problem (formulated below) should be solved.

3.3. THE LINEAR LOAD, APPLIED AT THE MID-LINE CONTOUR

Let us now consider the convolution of the fundamental solution and the linear load applied at the mid-line contour of the inclusion. The contributions (to the displacement) of the load applied to the mid-line contours of the different inclusions are estimated separately. Suppose we are looking for the displacement of the first inclusion $\mathbf{x} \in \Omega_1$ ($s \in (0, 2l)$). Then, upon taking into account the fact that the function $\mathbf{G}^\omega(|\mathbf{x}(s, \xi, \eta) - \mathbf{r}(s')|)$ at $s \in (0, 2l) \cup s' \in (2l, 4l)$, and the difference $\mathbf{G}^\omega(|\mathbf{x}(s, \xi, \eta) - \mathbf{r}(s')|) - \mathbf{G}^3(|\mathbf{x}(s, \xi, \eta) - \mathbf{r}(s')|)$ at $s \in (0, 2l) \cup s' \in (0, 2l)$, are uniformly continuous functions of the small parameter ε and that they are limited at $\varepsilon = 0$, the following asymptotic estimate is obtained:

$$\begin{aligned}\int_0^{4l} \mathbf{G}^\omega(|\mathbf{x}(s, \xi, \eta) - \mathbf{r}(s')|) \cdot \mathbf{F}(s')\, \mathrm{d}s' &\underset{\varepsilon\to 0}{\to} \int_{2l}^{4l} \mathbf{G}^\omega(|\mathbf{r}(s) - \mathbf{r}(s')|) \cdot \mathbf{F}(s')\, \mathrm{d}s' \\ &+ \int_0^{2l} (\mathbf{G}^\omega(|\mathbf{r}(s) - \mathbf{r}(s')|) - \mathbf{G}^3(|\mathbf{r}(s) - \mathbf{r}(s')|)) \cdot \mathbf{F}(s')\, \mathrm{d}s \\ &+ \int_0^{2l} \mathbf{G}^3(|\mathbf{x}(s, \xi, \eta) - \mathbf{r}(s')|) \cdot \mathbf{F}(s')\, \mathrm{d}s'.\end{aligned} \tag{9}$$

The last term of the right-hand side of equation (9) still contains a singularity at $s = s'$, $\varepsilon = 0$. However, the asymptotic analysis allows one to isolate it. To derive the asymptotic estimate the explicit form of certain components of the fundamental solution should be substituted. The other terms of equation (9) are easy to calculate.

Analysis of the convolutions of the non-diagonal components of tensor $\mathbf{G}^3$ and corresponding components of vector $\mathbf{F}$ is presented in Appendix A, section A.3. The singular diagonal terms can be analyzed in the following way:

$$\begin{aligned}
&\int_0^{2l} G^3_{11}(|\mathbf{x}(s,\xi,\eta)-\mathbf{r}(s')|)\,F_1(s')\,\mathrm{d}s' \\
&\quad = \frac{1}{16\pi\mu(1-\nu)}\left[\int_0^{2l}\frac{4(1-\nu)}{|\mathbf{x}(s,\xi,\eta)-\mathbf{r}(s')|}F_1(s')\,\mathrm{d}s' - \int_0^{2l}\frac{(\xi^2+\eta^2)\,F_1(s')}{|\mathbf{x}(s,\xi,\eta)-\mathbf{r}(s')|^3}\mathrm{d}s'\right] \\
&\quad = \frac{1}{16\pi\mu(1-\nu)}\left[\int_0^{2l}\frac{4(1-\nu)\,F_1(s')}{\sqrt{(s-s')^2+\xi^2+\eta^2}}\mathrm{d}s' - \int_0^{2l}\frac{(\xi^2+\eta^2)\,F_1(s')}{\left(\sqrt{(s-s')^2+\xi^2+\eta^2}\right)^3}\mathrm{d}s'\right] \\
&\underset{\varepsilon\to 0}{\to} \frac{1}{4\pi\mu}\left[\int_0^{2l}\frac{F_1(s')-F_1(s)}{|s-s'|}\mathrm{d}s' + F_1(s)\left(\ln\frac{2sl-s^2}{\xi^2+\eta^2}+2\ln 2\right)\right] - \frac{F_1(s)}{8\pi\mu(1-\nu)} \\
&\quad = \frac{1}{4\pi\mu}\left[\int_0^{2l}\frac{F_1(s')-F_1(s)}{|s-s'|}\mathrm{d}s' + F_1(s)\left(\ln(2sl-s^2)+2\ln 2-\frac{1}{2(1-\nu)}\right)\right] + G^2_{11}F_1(s).
\end{aligned} \tag{10}$$

This is analogous to

$$\begin{aligned}
&\int_0^{2l} G^3_{33}(|\mathbf{x}(s,\xi,\eta)-\mathbf{r}(s')|)\,F_3(s')\,\mathrm{d}s' \\
&\quad = \frac{1}{16\pi\mu(1-\nu)}\left[\int_0^{2l}\frac{(3-4\nu)}{|\mathbf{x}(s,\xi,\eta)-\mathbf{r}(s')|}F_3(s')\,\mathrm{d}s' + \int_0^{2l}\frac{\eta^2 F_3(s')}{|\mathbf{x}(s,\xi,\eta)-\mathbf{r}(s')|^3}\mathrm{d}s'\right] \\
&\quad = \frac{1}{16\pi\mu(1-\nu)}\left[\int_0^{2l}\frac{(3-4\nu)\,F_3(s')}{\sqrt{(s-s')^2+\xi^2+\eta^2}}\mathrm{d}s' + \int_0^{2l}\frac{\eta^2 F_3(s')}{\left(\sqrt{(s-s')^2+\xi^2+\eta^2}\right)^3}\mathrm{d}s'\right] \\
&\underset{\varepsilon\to 0}{\to} \frac{(3-4\nu)}{16\pi\mu(1-\nu)}\left[\int_0^{2l}\frac{F_3(s')-F_3(s)}{|s-s'|}\mathrm{d}s' + F_3(s)\left(\ln\frac{2sl-s^2}{\xi^2+\eta^2}+2\ln 2\right)\right] + \frac{F_3(s)}{16\pi\mu(1-\nu)}\frac{2\eta^2}{\xi^2+\eta^2}
\end{aligned}$$

$$= \frac{(3-4\nu)}{16\pi\mu(1-\nu)}\left[\int_0^{2l}\frac{F_3(s')-F_3(s)}{|s-s'|}\mathrm{d}s' + F_3(s)(\ln(2sl-s^2)+2\ln 2)\right] + G^2_{33}F_3(s). \tag{11}$$

We do not consider the term containing G^3_{22} as $F_2 = 0$.

In the same way, the displacement of the second inclusion $\mathbf{x}\in\Omega_2$ $(s\in(2l,4l))$ is estimated.

To describe the stationary motion of two inclusions by a single equation we introduce factors P_1 and P_2, and the auxiliary functions $\Phi(s,s')$ and $\Psi(s,s')$:

$$P_1(x) = \mathrm{H}(x)\mathrm{H}(2l-x), \qquad P_2(x) = \mathrm{H}(x-2l)\mathrm{H}(4l-x),$$

$$\Phi(s,s') = P_1(s)P_1(s') + P_2(s)P_2(s'), \qquad \Psi(s,s') = P_2(s)P_1(s') + P_1(s)P_2(s').$$

Here $\mathrm{H}(x)$ is the Heaviside step function.

The last term of equation (9) has a singularity only at $s \in (0, 2l) \cup s' \in (0, 2l)$ or $s \in (2l, 4l) \cup s' \in (2l, 4l)$, and we write down the corresponding estimate as

$$\int_0^{4l} \mathbf{G}^3(|\mathbf{x}(s,\xi,\eta) - \mathbf{r}(s')|) \cdot \mathbf{F}(s')\, \Phi(s,s')\, \mathrm{d}s' \underset{\varepsilon\to 0}{\to} \mathbf{T}(s) \cdot \mathbf{F}(s) + \mathbf{G}^2(|\mathbf{x}(s,\xi,\eta) - \mathbf{r}(s)|) \cdot \mathbf{F}(s)$$

$$+ \mathbf{N}(s) \cdot \int_0^{2l} \frac{\mathbf{F}(s') - \mathbf{F}(s)}{|s - s'|} \mathrm{d}s' + \mathbf{M}(s) \cdot \int_{2l}^{4l} \frac{\mathbf{F}(s') - \mathbf{F}(s)}{|s - s'|} \mathrm{d}s', \tag{12}$$

$$\mathbf{N}(s) = \frac{1}{4\pi\mu}\left(\mathbf{i}_1 \otimes \mathbf{i}_1 + \frac{3 - 4v}{4(1 - v)} \mathbf{i}_3 \otimes \mathbf{i}_3\right) P_1(s),$$

$$\mathbf{M}(s) = \frac{1}{4\pi\mu}\left(\mathbf{i}_1 \otimes \mathbf{i}_1 + \frac{3 - 4v}{4(1 - v)} \mathbf{i}_3 \otimes \mathbf{i}_3\right) P_2(s),$$

$$\mathbf{T}(s) = \frac{1}{4\pi\mu}\left\{\left[\ln(s(2l - s))P_1(s) + \ln((s - 2l)(4l - s))P_2(s) + 2\ln 2 - \frac{1}{2(1 - v)}\right]\mathbf{i}_1 \otimes \mathbf{i}_1\right.$$

$$\left. + \frac{3 - 4v}{4(1 - v)}[\ln(s(2l - s))P_1(s) + \ln((s - 2l)(4l - s))P_2(s) + 2\ln 2]\, \mathbf{i}_3 \otimes \mathbf{i}_3\right\}.$$

Here the sign $\otimes$ stands for the dyadic product of two vectors.

3.4. RESULTS OF ASYMPTOTIC ANALYSIS

We substitute the asymptotic estimates (8), (9) and (12) into equation (5) and finally obtain (at the points far from the ends)

$$\mathbf{u} - \mathbf{u}_0 \underset{\varepsilon\to 0}{\to} \int_0^{4l} \mathbf{K}(s,s') \cdot \mathbf{F}(s')\, \mathrm{d}s' + \mathbf{T}(s) \cdot \mathbf{F}(s) + \mathbf{G}^2(|\mathbf{x}(s,\xi,\eta) - \mathbf{r}(s)|) \cdot \mathbf{F}(s)$$

$$+ \mathbf{N}(s) \cdot \int_0^{2l} \frac{\mathbf{F}(s') - \mathbf{F}(s)}{|s - s'|} \mathrm{d}s' + \mathbf{M}(s) \cdot \int_{2l}^{4l} \frac{\mathbf{F}(s') - \mathbf{F}(s)}{|s - s'|} \mathrm{d}s'$$

$$+ \mathbf{G}^2(\mathbf{x} - \mathbf{r}) \circ\circ [\mathbf{f}(\mathbf{x}) - \mathbf{F}(s)\delta(\mathbf{x} - \mathbf{r}(s))], \tag{13}$$

$$\mathbf{K}(s,s') = \Phi(s,s')[\mathbf{G}^\omega(|\mathbf{r}(s) - \mathbf{r}(s')|) - \mathbf{G}^3(|\mathbf{r}(s) - \mathbf{r}(s')|)] + \Psi(s,s')\mathbf{G}^\omega(|\mathbf{r}(s) - \mathbf{r}(s')|).$$

As was expected, the relative displacement of two points belonging to the cross-section s (middle region) can be expressed as

$$\mathbf{u}(s,\xi_1,\eta_1) - \mathbf{u}(s,\xi_2,\eta_2) \underset{\varepsilon\to 0}{\to} \iint_D [\mathbf{G}^2(\xi_1 - \xi, \eta_1 - \eta) - \mathbf{G}^2(\xi_2 - \xi, \eta_2 - \eta)] \cdot \mathbf{f}(s,\xi,\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta.$$

The estimate (13) cannot be used in the vicinity of the ends. In this case, we have the asymptotic relation

$$\mathbf{u} - \mathbf{u}_0 \underset{\varepsilon\to 0}{\to} \int_0^{4l} \mathbf{K}(s,s') \cdot \mathbf{F}(s')\, \mathrm{d}s' + \iiint_\Omega \Phi(s,s')\mathbf{G}^3(|\mathbf{x}(s,\xi,\eta)$$

$$- \mathbf{x}(s',\xi',\eta')|) \cdot \mathbf{f}(s',\xi',\eta')\, \mathrm{d}s'\, \mathrm{d}\xi'\, \mathrm{d}\eta'. \tag{14}$$

4. ASYMPTOTIC DECOMPOSITION

The representation (13) for the scattered field contains terms associated with the linear load $\mathbf{F}(s)$, which depends only on the axial co-ordinate s, and the term associated with the self-balanced load $\mathbf{f}(\mathbf{x}) - \mathbf{F}(s)\delta(\mathbf{x}-\mathbf{r})$. If $\mathbf{F}$ were equal to zero, the displacement would be governed by the equation of the 2-D quasi-static problem,

$$\mathbf{u} - \mathbf{u}_0 = \mathbf{G}^2(\mathbf{x}-\mathbf{r}) \circ\circ \mathbf{f}(\mathbf{x}).$$

But in our case ($\mathbf{F}(s) = \omega^2(\rho_1 - \rho)\pi h^2 \mathbf{U}(s)$) vector $\mathbf{F}$ is unknown, and the dynamic process under study is essentially three-dimensional. We should pay attention to the structure of equation (13). It shows that these 3-D dynamic effects are formed only by the vector $\mathbf{F} \neq 0$: i.e., they have been asymptotically separated from the deformation process in the cross-section. Indeed the averaging equation (13) over the cross-section allows one to obtain the equation for the unknown averaged displacement $\mathbf{U}$. When the vector $\mathbf{U}$ is found, the displacement in every point of the cross-section (far from the ends) can be determined from the solution of the corresponding 2-D problem, and thus the original 3-D dynamic problem is asymptotically decomposed to the combination of two problems of reduced dimension. They are considered in detail below.

4.1. INTEGRAL EQUATION FOR AVERAGED DISPLACEMENT $\mathbf{U}(s)$

The equation for the averaged displacement $\mathbf{U}(s)$ is obtained by the averaging of equation (13) over the cross-section D. It is easy to work out that the integral (over the cross-section) of the non-diagonal components of fundamental solution $\mathbf{G}^2$ is equal to zero. The contribution of the integral over the cross-section from the last term of the right-hand side of equation (13) is equal to zero as well (the load is self-balanced in the cross-section). Thus, the averaged displacement $\mathbf{U}(s) = U_1(s)\mathbf{i}_1 + U_3(s)\mathbf{i}_3$ is governed by the equation

$$\mathbf{U}(s) - u_0(s)\mathbf{i}_3 = \omega^2(\rho_1-\rho)\pi h^2 \Bigg[\tilde{\mathbf{T}}(s)\cdot\mathbf{U}(s) + \int_0^{4l} \tilde{\mathbf{K}}(s,s')\cdot\mathbf{U}(s')\,\mathrm{d}s' + \mathbf{N}(s)\cdot\int_0^{2l} \frac{\mathbf{U}(s')-\mathbf{U}(s)}{|s-s'|}\mathrm{d}s' + \mathbf{M}(s)\cdot\int_{2l}^{4l} \frac{\mathbf{U}(s')-\mathbf{U}(s)}{|s-s'|}\mathrm{d}s' \Bigg], \qquad (15)$$

where

$$u_0(s) = u^0(\mathrm{e}^{\mathrm{i}\omega b/c_1} P_1(s) + \mathrm{e}^{-\mathrm{i}\omega b/c_1} P_2(s)),$$

$$\tilde{\mathbf{K}}(s,s') = \Phi(s,s')\{G^{\omega}_{jj}(|\mathbf{r}(s)-\mathbf{r}(s')|) - G^3_{jj}(|\mathbf{r}(s)-\mathbf{r}(s')|)\}\mathbf{i}_j\otimes\mathbf{i}_j + \Psi(s,s')\mathbf{G}^{\omega}(|\mathbf{r}(s)-\mathbf{r}(s')|), \left(\sum_j\right),$$

$$\tilde{\mathbf{T}}(s) = \mathbf{T}(s) + \frac{1}{4\pi\mu}\left\{(1-2\ln h)\mathbf{i}_1\otimes\mathbf{i}_1 + \left[-\frac{3-4\nu}{2(1-\nu)}\ln h + 1\right]\mathbf{i}_3\otimes\mathbf{i}_3\right\},$$

and the additional term to the tensor $\mathbf{T}$ is obtained by the integration of the product of the diagonal part of tensor $\mathbf{G}^2$ and vector $\mathbf{F}$ over the cross-section.

The obtained integral equation (15) is suitable for numerical solution. Instead of three convolutions over the region Ω there is only a single convolution over the mid-line contour. The other advantage of this equation is that the kernel (tensor $\tilde{\mathbf{K}}$) does not contain the singularity (in comparison with the original equation (3)) as it was already isolated analytically. The kernel $\tilde{\mathbf{K}}$ is the combination of the components of the fundamental solutions of dynamic stationary 3-D and static 3-D problems. Comparison of two terms in the kernel $\tilde{\mathbf{K}}$ (integrals over the contours Γ_1 and Γ_2) shows that the contributions of two parts of contour Γ (Γ_1 and Γ_2) to the displacement have different structure. Although here all tensors are represented in the Cartesian basis, they can be easily rewritten in the corresponding local system. The latter allows one to analyze a curvilinear contour.

4.2. THE DISPLACEMENT FIELD INSIDE INCLUSIONS

With equation (15) being solved and the average displacement $\mathbf{U}(s)$ being found, the displacement inside inclusions (far from the ends) can then be obtained through solving the corresponding 2-D quasi-static problem. By subtracting equation (15) from equation (13), the governing equation of this problem is derived (the corresponding integrals over the mid-line contour are calculated as the averaged displacement $\mathbf{U}$ is assumed to have been found):

$$\mathbf{u} - \omega^2(\rho_1 - \rho)\mathbf{G}^2(\mathbf{x} - \mathbf{r}) \circ\circ \mathbf{u} = \mathbf{u}_0 - u_0(s)\mathbf{i}_3 + \mathbf{U}(s)$$
$$+ \omega^2(\rho_1 - \rho)\pi h^2 \left\{ [\mathbf{T}(s) - \tilde{\mathbf{T}}(s)] \cdot \mathbf{U}(s) + \int_0^{4l} [\mathbf{K}(s, s') - \tilde{\mathbf{K}}(s, s')] \cdot U(s')\,\mathrm{d}s' \right\}. \tag{16}$$

The analysis of equation (16) shows that the contributions of the difference $\mathbf{u}_0 - u_0(s)\mathbf{i}_3$ and the last term on the right-hand side of this equation to the displacement are asymptotically small, and we finally obtain

$$\mathbf{u} - \omega^2(\rho_1 - \rho)\mathbf{G}^2(\mathbf{x} - \mathbf{r}) \circ\circ \mathbf{u} = \mathbf{U}(s). \tag{17}$$

It should be noted that analysis of the displacement and stress concentration near the ends requires us to solve the corresponding 3-D quasi-static problem, (equation (14) for unknown displacement $\mathbf{u}$), since the averaged displacement $\mathbf{U}(s)$ has been obtained.

5. SINGLE INCLUSION

In the case of the single inclusion $s \in (0, 2l)$ [15], the averaged displacement $\mathbf{U}(s)$ has only one non-zero component—the deflection $U_3(s)$, which can be determined from the equation

$$U_3(s) - u^0 = -\left(1 - \frac{\rho_1}{\rho}\right)\left(\frac{\omega h}{2c_2}\right)^2 \left\{ \int_0^{2l} K(|s - s'|)U_3(s')\,\mathrm{d}s' + \frac{3 - 4\nu}{4(1 - \nu)} \right.$$
$$\left. \times \left[\int_0^{2l} \frac{U_3(s') - U_3(s)}{|s - s'|}\mathrm{d}s + U_3\left(2\ln 2 - 2\ln h + \ln(2ls - s^2) + \frac{4(1 - \nu)}{3 - 4\nu}\right)\right]\right\}, \tag{18}$$

$$K(r) = \frac{1}{r}\left\{ \mathrm{e}^{\mathrm{i}\omega r/c_2} - \frac{3 - 4\nu}{4(1 - \nu)} - \left(\frac{c_2}{\omega r}\right)^2 \left[\mathrm{e}^{\mathrm{i}\omega r/c_1}\left(\mathrm{i}\frac{\omega r}{c_1} - 1\right) - \mathrm{e}^{-\mathrm{i}\omega r/c_2}\left(\mathrm{i}\frac{\omega r}{c_2} - 1\right)\right]\right\},$$

where the kernel $K(r)$ is limited at $r = 0$:

$$K(0) = \frac{\mathrm{i}\omega}{3c_2}(2 + c_2^3/c_1^3).$$

The other two components vanish: $U_1(s) = U_2(s) = 0$.

Equations of the corresponding 2-D quasi-static problem for the cross-section D take the form (the component u_1 is asymptotically small and cannot be determined within the framework of the present consideration):

$$\begin{aligned} u_2 - \omega^2(\rho_1 - \rho)(G^2_{22}(\mathbf{x} - \mathbf{r}) \circ \circ u_2 + G^2_{23}(\mathbf{x} - \mathbf{r}) \circ \circ u_3) &= 0, \\ u_3 - \omega^2(\rho_1 - \rho)(G^2_{32}(\mathbf{x} - \mathbf{r}) \circ \circ u_2 + G^2_{33}(\mathbf{x} - \mathbf{r}) \circ \circ u_3) &= U_3. \end{aligned} \tag{19}$$

In fact we obtain

$$u_2(s, \xi, \eta) = U_3(s) v_2(\xi, \eta),$$

$$u_3(s, \xi, \eta) = U_3(s) v_3(\xi, \eta).$$

Thus, the asymptotic decomposition is fulfilled and leads to the separation of variables.

6. NUMERICAL EXAMPLES

6.1. SINGLE INCLUSION

Equation (18) was solved numerically, and the results of the corresponding calculations are represented in the Figures 3–6.

The dependencies of the amplitude and phase of the dimensionless deflection U_3/u^0 on the density ratio $q = \rho_1/\rho$ are shown in Figures 3 and 4. If the density of the inclusion is smaller than the density of the medium ($q < 1$), then the deflection amplitude (for the inclusion) is smaller than the displacement amplitude in the incident wave, and the deflection curve is convex downwards (Figure 3). Otherwise, if the density of the inclusion is larger than the density of the medium ($q > 1$), the deflection amplitude is larger than the amplitude of the incident wave and the corresponding curve is convex upwards (Figure 3). The larger the inclusion density is, the larger the amplitude of the inclusion deflection is. Moreover, the difference in the densities of the medium and inclusion creates a phase shift between the incident wave and the deflection of the inclusion. It is positive when the density of

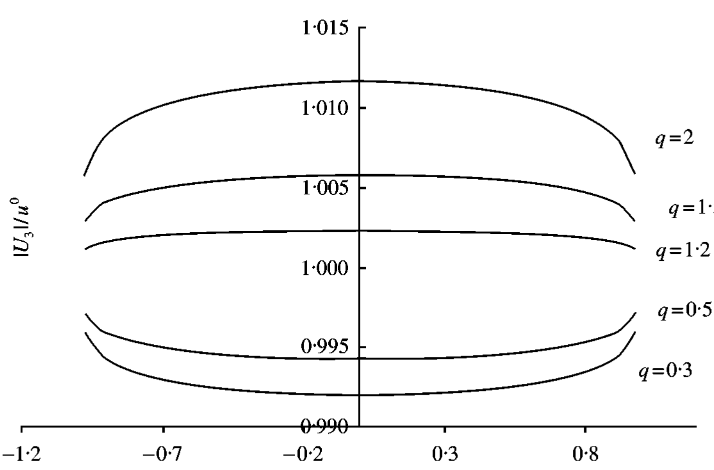

Figure 3. Influence of the density ratio $q = \rho_1/\rho$ on the amplitude of the displacement U_3/u^0 at given parameters $\varepsilon = 0{\cdot}1$, $p = 1$.

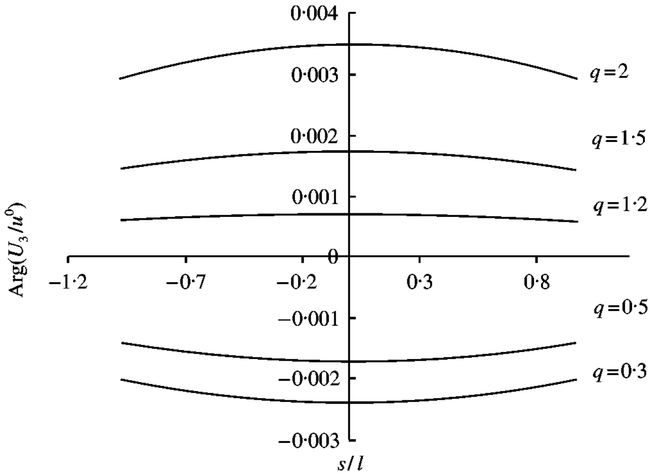

Figure 4. Influence of the density ratio $q = \rho_1/\rho$ on the argument of the displacement U_3/u^0 at given parameters $\varepsilon = 0{\cdot}1$, $p = 1$.

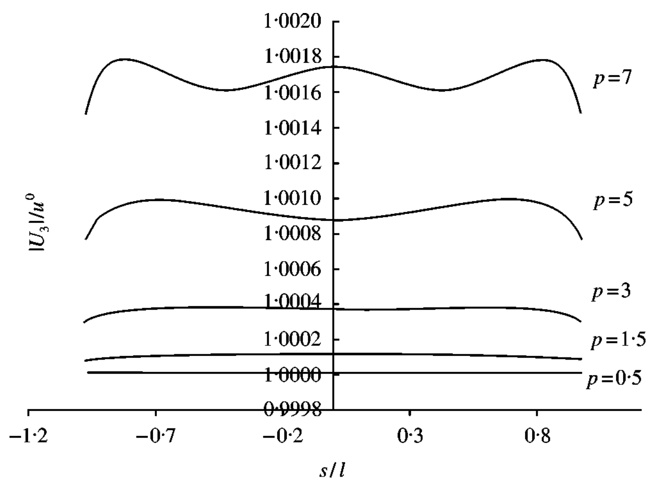

Figure 5. Influence of the frequency $p = \omega l/c_2$ on the amplitude of the displacement U_3/u^0 at given parameters $\varepsilon = 0{\cdot}005$, $q = 2$.

the inclusion is larger than the density of the medium, and negative otherwise. The larger the difference between the densities is, the larger the phase shift is (Figure 4).

The influence of the dimensionless frequency $p = \omega l/c_2$ on the deflection U_3/u^0 is demonstrated in Figures 5 and 6. They show that the vibration mode of the inclusion strongly depends on the frequency ω, and that it looks like some vibration eigenmodes for a free elastic beam with free ends (no surrounding medium).

6.2. TWO INCLUSIONS

In this case, the averaged displacement $\mathbf{U}(s)$ has two components—the axial component $U_1(s)$ and the deflection $U_3(s)$. The symmetry results in $U_2 = 0$ (Figure 1). Here the distance

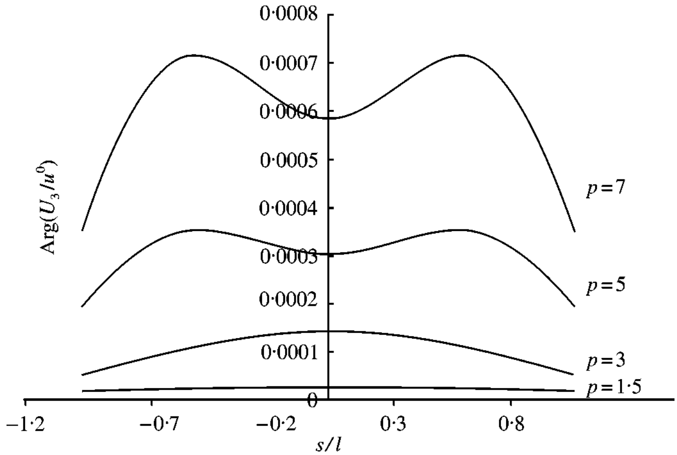

Figure 6. Influence of the frequency $p = \omega l/c_2$ on the argument of the displacement U_3/u^0 at given parameters $\varepsilon = 0{\cdot}005$, $q = 2$.

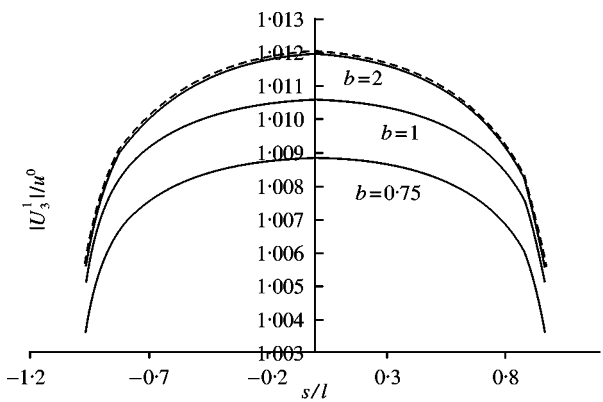

Figure 7. Influence of the distance b/l on the deflection of the lower inclusion (solid) and single inclusion (dashed) at given parameters $\varepsilon = 0{\cdot}1$, $q = 1{\cdot}5$, $p = 1{\cdot}5$.

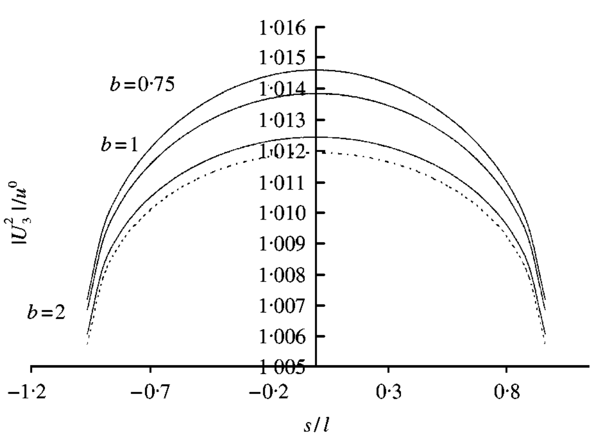

Figure 8. Influence of the distance b/l on the deflection of the upper inclusion (solid) and single inclusion (dashed) at given parameters $\varepsilon = 0{\cdot}1$, $q = 1{\cdot}5$, $p = 1{\cdot}5$.

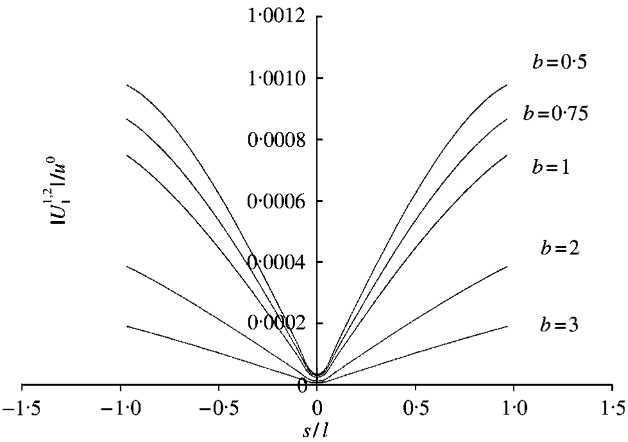

Figure 9. Influence of the distance b/l on the axial displacement $U_1^1 = U_1^2 = U_1^{1,2}$ of the upper and lower inclusions at given parameters $\varepsilon = 0·1$, $q = 1·5$, $p = 1·5$.

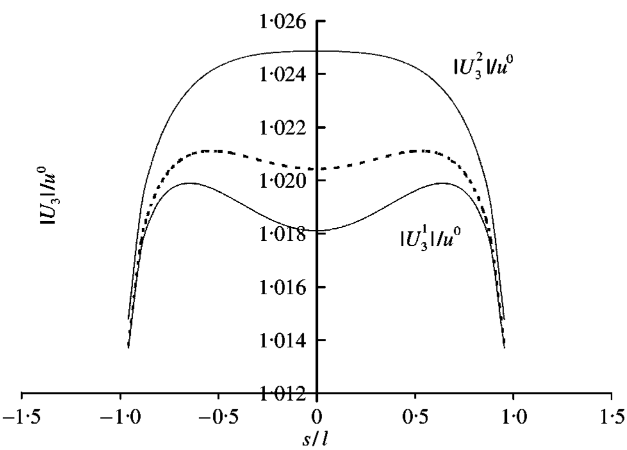

Figure 10. Deflections of upper and lower inclusions (solid) and single inclusion (dashed) versus axial co-ordinate s/l at given parameters $\varepsilon = 0·05$, $q = 2$, $p = 3$, $b = 0·5$.

between inclusions is an additional key parameter. Equation (15) has been solved numerically, and the vector $\mathbf{U}(s)$ has been found. Dependencies of the amplitude of the vibration for upper and lower inclusions on the distance b/l between them are represented in Figures 7–9. As expected, the larger the distance between the inclusions is, the closer the deflections $U_3^1(s)$ and $U_3^2(s)$ are to the deflection of a single inclusion (the upper index is associated with upper (1) and lower (2) inclusions), and the smaller the axial displacement $U_1(s)$ (amplitudes of axial displacements of both the inclusions are equal according to the symmetry of configuration).

Within the wide range of parameters, dependencies of amplitude and phase of deflection (for both inclusions) on mass density ratio and dimensionless frequency have the same character as in the case of single inclusion. However, if the frequency of the incident wave is high enough, the motions of the upper and lower inclusions might differ significantly from each other and from the motion of single inclusion (Figure 10).

At a short distance $2b$ between inclusions, a different asymptotic solution should be constructed. To derive the corresponding integral equation, we consider two inclusions as the single inclusion with combined cross-section.

7. CONCLUSIONS

The asymptotic decomposition in the problem of scattering of plane harmonic wave on the system of rod-like inclusions was fulfilled. In the result, it was obtained that 3-D dynamic effects of the process are described by the integral equation over the mid-line contour for an unknown deflection, and averaged axial displacement of the inclusions. Displacement and stress inside inclusions can be found within the framework of a 3-D quasi-static problem. However, at the points far from the ends they can be determined through solving a 2-D quasi-static problem for the cross-section. It should be noted that all of the equations derived are suitable for the numerical solution.

Although only the simple case of mass inclusions has been considered, the approach proposed is very promising. It allows one to analyze much more complicated cases, when (1) the shape of the cross-section is arbitrary; (2) the number of the inclusions is arbitrary but finite (including the periodical structure, and for example, lattice); (3) the contours of the inclusions are smooth curvilinear (the size of the cross-section is assumed to be much smaller than the radius of curvature); (4) the inclusions are not identical (they can differ in length, cross-section shape, Young's modulus, and mass density); (5) the incidence of the harmonic wave is oblique; (6) the orientation and position of the the inclusions are arbitrary; (7) the problem contains small parameters of different order (the case of tape-like inclusion).

One of the important advantages of the approach proposed is that it allows one to consider a large number of inclusions, without additional simplifying physical hypotheses.

Asymptotic decomposition technology can be successfully applied to various dynamic processes of interaction between an elastic solid body and an external elastic medium in the presence of natural small parameters (spatial and/or temporal). Strip foundation, railway, and tunnel are the examples of thin objects interacting with the medium.

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APPENDIX A: AUXILIARY ASYMPTOTIC ESTIMATES

A.1. ESTIMATE OF CONVOLUTION

Let $f(x/l)$ be a continuous bounded function and $g(x/h)$ be a summable function ($\int_{-\infty}^{+\infty} g(x'/h)\,\mathrm{d}x' = A < \infty$) of a real variable x, and l and h be real positive ($l > 0, h > 0$) parameters (scale factors). The smaller the ratio $\varepsilon = h/l$ is, the slower the function $f(x/l)$ changes in comparison with the function $g(x/h)$. The theorem of Lebesgue concerning the passage to the limit under the integral sign leads to the following asymptotic estimate:

$$f\left(\frac{x}{l}\right) * g\left(\frac{x}{h}\right) = \int_{-\infty}^{+\infty} f\left(\frac{x-x'}{l}\right) g\left(\frac{x'}{h}\right) \mathrm{d}x' \underset{\varepsilon\to 0}{\to} f\left(\frac{x}{l}\right) \int_{-\infty}^{+\infty} g\left(\frac{x'}{h}\right) \mathrm{d}x'. \qquad \text{(A.1)}$$

If $f(x/l) \neq 0, \int_{-\infty}^{+\infty} g(x'/h)\,\mathrm{d}x' \neq 0$, this limit transition can be used for asymptotic estimations. In the case, where the functions $f(x/l)$ and $g(x/h)$ vanish outside the interval $|x| < L$ (we assume that $l/L = O(1)$) at the point x located far from the ends $x = \pm L$ of interval (i.e., $h \ll L - |x|$), and therefore the following asymptotic estimate is valid:

$$f\left(\frac{x}{l}\right) * g\left(\frac{x}{h}\right) = \int_{-L}^{+L} f\left(\frac{x-x'}{l}\right) g\left(\frac{x'}{h}\right) \mathrm{d}x = h \int_{-(L-x)/h}^{(L+x)/h} f\left(\frac{x}{l} - \varepsilon s\right) g(s)\,\mathrm{d}s$$

$$\underset{\varepsilon\to 0}{\to} h \int_{-\infty}^{+\infty} f\left(\frac{x}{l} - \varepsilon s\right) g(s)\,\mathrm{d}s \underset{\varepsilon\to 0}{\to} f\left(\frac{x}{l}\right) \int_{-\infty}^{+\infty} g\left(\frac{x'}{h}\right) \mathrm{d}x'. \qquad \text{(A.2)}$$

A.2. DISPLACEMENT DUE TO SELF-BALANCED LOAD

Let us consider the triple convolution over the region Ω of the fundamental solution $\mathbf{G}^3$ and the function $\mathbf{g}$ given in Ω and self-balanced over the cross-section D:

$$\mathbf{y}(\mathbf{x}) = \mathbf{G}^3 * \circ \circ \mathbf{g}(\mathbf{x}) = \int_{-L}^{L} \iint_{D} \mathbf{G}^3(s - s', \xi - \xi', \eta - \eta') \cdot \mathbf{g}(s', \xi', \eta') \, \mathrm{d}\xi' \, \mathrm{d}\eta' \, \mathrm{d}s'. \quad \text{(A.3)}$$

For the first component of the vector $\mathbf{y}$ we have

$$y_1(\mathbf{x}) = \frac{1}{4\pi\mu} \iint_{D(s)} \int_{-L}^{L} \left[\frac{1}{R} - \frac{1}{R_h} - \frac{1}{4(1-\nu)} \frac{r^2}{R^3} \right] g_1\left(\frac{s'}{l}, \frac{\xi'}{h}, \frac{\eta'}{h} \right) \mathrm{d}s' \, \mathrm{d}\xi' \, \mathrm{d}\eta'$$

$$+ \frac{1}{16\pi\mu(1-\nu)} \iint_{D(s)} \left[(\xi - \xi') \int_{-L}^{L} \frac{s - s'}{R^3} g_2\left(\frac{s'}{l}, \frac{\xi'}{h}, \frac{\eta'}{h} \right) \mathrm{d}s' \right.$$

$$\left. + (\eta - \eta') \int_{-L}^{L} \frac{s - s'}{R^3} g_3\left(\frac{s'}{l}, \frac{\xi'}{h}, \frac{\eta'}{h} \right) \mathrm{d}s' \right] \mathrm{d}\xi' \, \mathrm{d}\eta', \quad \text{(A.4)}$$

$$R^2 = (s - s')^2 + (\xi - \xi')^2 + (\eta - \eta')^2, \quad R_h^2 = (s - s')^2 + h^2, \quad r^2 = (\xi - \xi')^2 + (\eta - \eta')^2.$$

We used the fact that the function $\mathbf{g}$ is self-balanced in the cross-section D and the function R_h does not depend on ξ' and η'; therefore, the addition of the function $-\mathbf{i}_1 \otimes \mathbf{i}_1 / R_h$ to the fundamental solution $\mathbf{G}^3$ does not change the value of the convolution.

The first argument of the function $\mathbf{g}$ has the scale factor l; arguments of the kernel have the scale factor h. As the ratio ε is small, the estimate (A.2) can be used for the evaluation of the convolution over the "long" co-ordinate s in the first term of the right-hand side of equation (A.4), taking into account that these kernels are integrated with respect to the co-ordinate s:

$$\int_{-L}^{L} \left(\frac{1}{R} - \frac{1}{R_h} \right) \mathrm{d}s' = \int_{-L+s}^{L+s} \left(\frac{1}{\sqrt{z^2 + r^2}} - \frac{1}{\sqrt{z^2 + h^2}} \right) \mathrm{d}z = \ln \left(\frac{z + \sqrt{z^2 + r^2}}{z + \sqrt{z^2 + h^2}} \right) \Bigg|_{-L+s}^{L+s}$$

$$\underset{\varepsilon \to 0}{\to} 2 \ln \frac{h}{r}, \quad \int_{-L}^{L} \frac{h^2}{R^3} \mathrm{d}s' = \int_{-L+s}^{L+s} \frac{h^2}{(\sqrt{z^2 + r^2})^3} \mathrm{d}z = \frac{z h^2}{r^2 \sqrt{z^2 + r^2}} \Bigg|_{-L+s}^{L+s} \underset{\varepsilon \to 0}{\to} 2 \frac{h^2}{r^2}. \quad \text{(A.5)}$$

The second term should be estimated separately, as for this term, the second condition of the estimate equation (A1) is not satisfied; therefore,

$$\int_{-L}^{L} g_i\left(\frac{s'}{l}, \frac{\xi'}{h}, \frac{\eta'}{h} \right) \frac{(s - s')h}{R^3} \mathrm{d}s' = \varepsilon \left[g_i\left(\frac{s}{l}, \frac{\xi'}{h}, \frac{\eta'}{h} \right) \int_{-L}^{L} \frac{(s - s')L}{R^3} \mathrm{d}s' \right.$$

$$\left. + \dot{g}_i\left(\frac{s}{l}, \frac{\xi'}{h}, \frac{\eta'}{h} \right) \int_{-L}^{L} \frac{(s - s')^2}{R^3} \mathrm{d}s' + \ddot{g}_i\left(\frac{s}{l}, \frac{\xi'}{h}, \frac{\eta'}{h} \right) \int_{-L}^{L} \frac{(s - s')^3}{LR^3} \mathrm{d}s' + \cdots \right]$$

$$= \varepsilon\left[g_i\left(\frac{s}{l},\frac{\xi'}{h},\frac{\eta'}{h}\right)\int_{-L+s}^{L+s}\frac{Lz}{\left(\sqrt{z^2+r^2}\right)^3}\mathrm{d}z + \dot{g}_i\left(\frac{s}{l},\frac{\xi'}{h},\frac{\eta'}{h}\right)\int_{-L+s}^{L+s}\frac{z^2}{\left(\sqrt{z^2+r^2}\right)}\mathrm{d}z \right.$$

$$\left. + \ddot{g}_i\left(\frac{s}{l},\frac{\xi'}{h},\frac{\eta'}{h}\right)\int_{-L+s}^{L+s}\frac{z^3}{L\left(\sqrt{z^2+r^2}\right)^3}\mathrm{d}z + \cdots \right], \tag{A.6}$$

where $i = 1, 2$, and the derivation with respect to the first argument s/L is denoted by an upper dot. The corresponding integrals on the right-hand side of this expression can be easily estimated:

$$\int_{-L+s}^{L+s}\frac{Lz}{\left(\sqrt{z^2+r^2}\right)^3}\mathrm{d}z = -\frac{L}{\sqrt{z^2+r^2}}\Bigg|_{-L+s}^{L+s} \underset{\varepsilon\to 0}{\to} \frac{2(s/L)}{1-(s/L)^2},$$

$$\int_{-L+s}^{L+s}\frac{z^2}{\left(\sqrt{z^2+r^2}\right)^3}\mathrm{d}z = \left(-\frac{z}{\sqrt{z^2+r^2}} + \ln\left(z+\sqrt{z^2+r^2}\right)\right)\Bigg|_{-L+s}^{L+s} \underset{\varepsilon\to 0}{\to} -2\ln\varepsilon$$

$$-2 - 2\ln\frac{r}{h} + \ln(1-(s/L)^2),$$

$$\frac{1}{L}\int_{-L+s}^{L+s}\frac{z^3}{\left(\sqrt{z^2+r^2}\right)^3}\mathrm{d}z = \frac{1}{L}\left(\sqrt{z+r^2} + \frac{r^2}{\sqrt{z^2+r^2}}\right)\Bigg|_{-L+s}^{L+s} \underset{\varepsilon\to 0}{\to} \frac{2s}{L}. \tag{A.7}$$

Thus, we have evaluated the second term of the right-hand side of equation (A.4) as a small quantity of order $\varepsilon\ln\varepsilon$ and thus it can be neglected in comparison with the first term. Then taking into account that g_1 is self-equilibrated over the cross-section, we obtain

$$y_1(\mathbf{x}) \underset{\varepsilon\to 0}{\to} \frac{1}{4\pi\mu}\iint_{D(s)} g_1\left(\frac{s}{l},\frac{\xi'}{h},\frac{\eta'}{h}\right)\left[-2\ln((\xi-\xi')^2+(\eta-\eta')^2) + 2\ln h - \frac{1}{2(1-\nu)}\right]\mathrm{d}s'\,\mathrm{d}\xi'\,\mathrm{d}\eta'$$

$$= -\frac{1}{2\pi\mu}\iint_{D(s)} g_1\left(\frac{s}{l},\frac{\xi'}{h},\frac{\eta'}{h}\right)\ln[(\xi-\xi')^2+(\eta-\eta')^2]\,\mathrm{d}s'\,\mathrm{d}\xi'\,\mathrm{d}\eta' = G_{11}^2\circ\circ g_1. \tag{A.8}$$

Analogously,

$$y_2(\mathbf{x}) = \frac{1}{16\pi\mu(1-\nu)}\left\{\iiint_{D(s)}\int_{-L}^{L}\left[(3-4\nu)\left(\frac{1}{R}-\frac{1}{R_h}\right) + \frac{(\xi-\xi')^2}{R^3}\right]g_2\left(\frac{s'}{l},\frac{\xi'}{h},\frac{\eta'}{h}\right)\mathrm{d}s'\,\mathrm{d}\xi'\,\mathrm{d}\eta'\right.$$

$$+\iint_{D(s)}\left[(\xi-\xi')\int_{-L}^{L}\frac{s-s'}{R^3}g_1\left(\frac{s'}{l},\frac{\xi'}{h},\frac{\eta'}{h}\right)\mathrm{d}s'\right.$$

$$\left.\left. + (\xi-\xi')(\eta-\eta')\int_{-L}^{L}\frac{1}{R^3}g_3\left(\frac{s'}{l},\frac{\xi'}{h},\frac{\eta'}{h}\right)\mathrm{d}s'\right]\mathrm{d}\xi'\,\mathrm{d}\eta'\right\}$$

$$\underset{\varepsilon\to 0}{\to} \frac{1}{16\pi\mu(1-\nu)} \left\{ \iint\limits_{D(s)} (3-4\nu)\left[-\ln((\xi-\xi')^2+(\eta-\eta')^2) + 2\ln h + \frac{2(\xi-\xi')^2}{(\xi-\xi')^2+(\eta-\eta')^2} \right] g_2\left(\frac{s'}{l},\frac{\xi'}{h},\frac{\eta'}{h}\right) \mathrm{d}\xi'\,\mathrm{d}\eta' + \iint\limits_{D(s)} \frac{2(\xi-\xi')(\eta-\eta')}{(\xi-\xi')^2+(\eta-\eta')^2} g_3\left(\frac{s'}{l},\frac{\xi'}{h},\frac{\eta'}{h}\right) \mathrm{d}\xi'\,\mathrm{d}\eta' \right\}$$

$$= G^2_{22} \circ\circ g_2 + G^2_{23} \circ\circ g_3, \tag{A.9}$$

and

$$y_3(\mathbf{x}) = \frac{1}{16\pi\mu(1-\nu)} \left\{ \iint\limits_{D(s)} \int_{-L}^{L} \left[(3-4\nu)\left(\frac{1}{R}-\frac{1}{R_h}\right) + \frac{(\eta-\eta')^2}{R^3} \right] g_3\left(\frac{s'}{l},\frac{\xi'}{h},\frac{\eta'}{h}\right) \mathrm{d}s'\,\mathrm{d}\xi'\,\mathrm{d}\eta' + \iint\limits_{D(s)} (\xi-\xi')(\eta-\eta') \int_{-L}^{L} \frac{1}{R^3} g_2\left(\frac{s'}{l},\frac{\xi'}{h},\frac{\eta'}{h}\right) \mathrm{d}s' + (\eta-\eta') \int_{-L}^{L} \frac{s-s'}{R^3} g_1\left(\frac{s'}{l},\frac{\xi'}{h},\frac{\eta'}{h}\right) \mathrm{d}s' \Big] \mathrm{d}\xi'\,\mathrm{d}\eta' \right\}$$

$$\underset{\varepsilon\to 0}{\to} \frac{1}{16\pi\mu(1-\nu)} \left\{ \iint\limits_{D(s)} (3-4\nu)\left[-\ln((\xi-\xi')^2+(\eta-\eta')^2) + 2\ln h + \frac{2(\eta-\eta')^2}{(\xi-\xi')^2+(\eta-\eta')^2} \right] g_3\left(\frac{s'}{l},\frac{\xi'}{h},\frac{\eta'}{h}\right) \mathrm{d}\xi'\,\mathrm{d}\eta' + \iint\limits_{D(s)} \frac{2(\xi-\xi')(\eta-\eta')}{(\xi-\xi')^2+(\eta-\eta')^2} g_2\left(\frac{s'}{l},\frac{\xi'}{h},\frac{\eta'}{h}\right) \mathrm{d}\xi'\,\mathrm{d}\eta' \right\}$$

$$= G^2_{32} \circ\circ g_2 + G^2_{33} \circ\circ g_3. \tag{A.10}$$

Finally, the following asymptotic estimate for the convolution over the region Ω for the self-equilibrated load $\mathbf{g}(\mathbf{x})$ is obtained:

$$\mathbf{y}(\mathbf{x}) = \mathbf{G}^3 * \circ\circ \mathbf{g} \underset{\varepsilon\to 0}{\to} \mathbf{G}^2 \circ\circ \mathbf{g},$$

$$(G^2_{12} = G^2_{21} = G^2_{13} = G^2_{31} = 0). \tag{A.11}$$

A.3. DISPLACEMENT DUE TO LINEAR LOAD

As vector **F** (linear load) is the function of the slow variable s/L and the tensor $\mathbf{G}^3$ can be represented as a function of the fast variable s/h, the convolution (with respect to axial co-ordinate s) of non-diagonal components of tensor $\mathbf{G}^3$, and corresponding components of vector **F** is analyzed by using the estimate of section A.1. In order to estimate the convolution for components $G^3_{12}, G^3_{13}, G^3_{21}, G^3_{31}$ the following expression is considered:

$$\int_{-L}^{L} F_i\left(\frac{s'}{l}\right)\frac{(s-s')h}{\left(\sqrt{(s-s')^2+\xi^2+\eta^2}\right)^3}\mathrm{d}s' = \varepsilon\left[F_i\left(\frac{s}{l}\right)\int_{-L}^{L}\frac{(s-s')L}{\left(\sqrt{(s-s')^2+\xi^2+\eta^2}\right)^3}\mathrm{d}s'\right.$$

$$+\dot{F}_i\left(\frac{s}{l}\right)\int_{-L}^{L}\frac{(s-s')^2}{\left(\sqrt{(s-s')^2+\xi^2+\eta^2}\right)^3}\mathrm{d}s'$$

$$\left.+\ddot{F}_i\left(\frac{s}{l}\right)\int_{-L}^{L}\frac{(s-s')^3}{L\left(\sqrt{(s-s')^2+\xi^2+\eta^2}\right)^3}\mathrm{d}s'+\cdots\right]$$

$$=\varepsilon\left[F_i\left(\frac{s}{l}\right)\int_{-L+s}^{L+s}\frac{Lz}{\left(\sqrt{z^2+r^2}\right)^3}\mathrm{d}z+\dot{F}_i\left(\frac{s}{l}\right)\int_{-L+s}^{L+s}\frac{z^2}{\left(\sqrt{z^2+r^2}\right)^3}\mathrm{d}z\right.$$

$$\left.+\ddot{F}_i\left(\frac{s}{l}\right)\int_{-L+s}^{L+s}\frac{z^3}{L\left(\sqrt{z^2+r^2}\right)^3}\mathrm{d}z+\cdots\right]$$

$$=\varepsilon\left[F_i\left(\frac{s}{l}\right)\frac{2(s/L)}{1-(s/L)^2}+\dot{F}_i\left(\frac{s}{l}\right)\left(-2\ln\varepsilon-2-2\ln\frac{r}{h}+\ln\left(1-(s/L)^2\right)\right)\right.$$

$$\left.+\ddot{F}_i\left(\frac{s}{l}\right)\frac{2s}{L}\right]+o(\varepsilon\ln\varepsilon).$$

$$r^2=\xi^2+\eta^2.$$

Here $i = 1, 3$, and the derivation with respect to the first argument s/L is denoted by an upper dot. Thus these terms are asymptotically small.

Convolutions of the other components (G^3_{23}, G^2_{32}) are also estimated on the basis of section A.1, and they are finite:

$$\frac{1}{16\pi\mu(1-\nu)}\int_{-L}^{L}F_i\left(\frac{s'}{l}\right)\frac{\xi\eta}{\left(\sqrt{(s-s')^2+\xi^2+\eta^2}\right)^3}\mathrm{d}s'$$

$$\underset{\varepsilon\to 0}{\to}\frac{1}{16\pi\mu(1-\nu)}F_i\left(\frac{s}{l}\right)\int_{-L}^{L}\frac{\xi\eta}{\left(\sqrt{(s-s')^2+\xi^2+\eta^2}\right)^3}\mathrm{d}s'$$

$$\underset{\varepsilon\to 0}{\to}\frac{1}{16\pi\mu(1-\nu)}F_i\left(\frac{s}{l}\right)\frac{2\xi\eta}{\xi^2+\eta^2}=G^2_{ij}F_i\left(\frac{s}{l}\right),\quad i,j=2,3,\quad i\neq j.$$