Journal of Sound and Vibration (2001) **248**(2), 371–378
doi:10.1006/jsvi.2001.3706, available online at http://www.idealibrary.com on IDEAL®

LETTERS TO THE EDITOR

EXACT SOLUTIONS FOR AXISYMMETRIC VIBRATIONS OF SOLID CIRCULAR AND ANNULAR MEMBRANES WITH CONTINUOUSLY VARYING DENSITY

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(*Received* 23 *February* 2001)

1. INTRODUCTION

The vibration of membrane is a subject of considerable scientific and practical interest as it is a popular element in nature and technology. Membranes are widely used as transducers that convert energy from one form to another [1].

Several investigators [1–11] have tackled the study of vibrating membranes with varying density. The studies are analytical [2, 7, 8, 10, 11] and numerical [1, 3–6, 9] in nature. The objective of this letter is to draw attention to two families of solutions for the traverse vibration of annular membranes with continuously varying densities. Transformations for obtaining solutions for these profiles are presented. The solutions are obtained in terms of special functions. Using these transformations, specific examples are worked out. The eigenvalues corresponding to the first two modes are presented for some cases, and their dependence on the density variation is discussed.

2. THE EQUATION OF MOTION

The current work deals with two situations: (1) a solid circular membrane of radius R, (2) an annular membrane of outer radius R and inner radius R_0.

The membrane density is assumed to be of the form

$$\rho(r) = \rho_0 f(r), \tag{1}$$

where $r = \tilde{r}/R$ and $\tilde{r}$ is the radius. For the axisymmetric modes of vibration of a solid circular membrane or an annular membrane of outer radius R and inner radius R_0, the governing differential equation for the displacement $W(r)$ [9, 12] is

$$\frac{\mathrm{d}^2 W}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}W}{\mathrm{d}r} + \Omega^2 f(r) W = 0, \quad 0 \leqslant r_0 \leqslant r \leqslant 1, \tag{2}$$

where $r_0 = R_0/R$, and the non-dimensional frequency $\Omega = \omega R\sqrt{\rho_0/S}$, S is the tension per unit length. The boundary conditions for the solid circular membrane is

$$W(1) = 0, \quad W'(0) = 0 \tag{3}$$

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and for the annular case is

$$W(r_0) = W(1) = 0. \tag{4}$$

Equation (2) has variable coefficients. Therefore, exact solutions of this equation for a general density variation $f(r)$ cannot be obtained. However, for certain specific density variations, exact solutions can be obtained. In the following sections, using appropriate transformations, equation (3) will be reduced to analytically solvable differential equations for two families of density profiles.

3. SOLUTIONS IN TERMS OF CONFLUENT HYPERGEOMETRIC FUNCTIONS

In this section, a general transformation for obtaining a family of solutions in the form of Kummer's hypergeometric functions is presented. Assuming a functional dependence for W of the form

$$W(r) = r^P e^{g(r)} F(r). \tag{5}$$

Equation (3) reduces to

$$rF''(r) + [(2P+1) + 2rg'(r)]F'(r) + \frac{\zeta(r)}{r}F(r) = 0, \tag{6}$$

where

$$\zeta(r) = r^2(g''(r) + g'^2(r)) + (2P+1)rg'(r) + P^2 + \Omega^2 r^2 f(r). \tag{7}$$

Assuming $g(r)$ to be of the form

$$g(r) = \frac{\phi r^n}{n} \tag{8}$$

equation (7) will reduce to

$$rF''(r) + [(2P+1) + 2\phi r^n]F'(r) + \frac{\zeta}{r}F(r) = 0. \tag{9}$$

Equation (10) can be further simplified by introducing β, k and l such that

$$\beta(r) = \frac{\zeta}{r^n}, \quad k = 2P+1 \quad \text{and} \quad l = 2\phi. \tag{10a}$$

This yields

$$rF''(r) + [k + lr^n]\,F'(r) + \beta r^{n-1}F(r) = 0. \tag{10b}$$

Using the transformation $s = qr^n$, equation (10b) can be reduced to the following Kummer's confluent hypergeometric equation [13, 14] when β (and therefore ε) is a constant:

$$sF''(s) + [\gamma - s]F'(s) - \varepsilon F(s) = 0, \tag{11}$$

where, $\gamma = 1 + (k-1)/n$, $\varepsilon = -\beta/n^2 q$ and $q = -l/n$.

The solution to equation (11) can be expressed in the form of Kummer's confluent hypergeometric function [13–15]

$$F(s) = C_1 {}_1F_1(\varepsilon; \gamma; s) + C_2 U(\varepsilon; \gamma; s). \quad (12)$$

The function ${}_1F_1$ is sometimes referred to as M. Substituting $g(r)$ (give by equation (8)) into the expression for $\zeta(r)$ (equation (7)) yields

$$\phi^2 r^{2n} + \phi r^n \left[2P + n - \frac{\beta}{\phi} \right] + P^2 + \Omega^2 r^2 f(r) = 0. \quad (13)$$

Assuming a functional dependence of $f(r)$ of the form

$$f(r) = \frac{A}{r^2} + Br^t + Cr^m \quad (14)$$

equation (13) will be satisfied, for a constant β, if

$$t = 2n - 2, \quad m = n - 2, \quad P = \mathrm{i}\Omega\sqrt{A}, \quad \phi = \mathrm{i}\Omega\sqrt{B} \quad \text{and} \quad \beta = \phi\left[2P + n + \frac{\Omega^2 C}{\phi}\right]. \quad (15)$$

Therefore, a function $f(r)$ of the following form yields a solution in the form of Kummer's confluent hypergeometric function.

$$f(r) = \frac{A}{r^2} + Br^{2n-2} + Cr^{n-2}, \quad (16)$$

where n can be any complex number in general.
The solution of equation (3) for an $f(r)$ of the form given by equation (16) can now be written as [13–15]

$$W(r) = r^{\mathrm{i}\Omega\sqrt{A}} \mathrm{e}^{(\mathrm{i}\Omega\sqrt{B}/n)r^n} \Bigg(C_1 {}_1F_1\left[-\frac{1}{2n}\left(\mathrm{i}\Omega \frac{C}{\sqrt{B}} - (2P + n) \right); 1 + \frac{\mathrm{i}2\Omega\sqrt{A}}{n}; -\frac{\mathrm{i}2\Omega\sqrt{B}}{n} r^n \right]$$

$$+ C_2 U\left[-\frac{1}{2n}\left(\mathrm{i}\Omega \frac{C}{\sqrt{B}} - (2P + n) \right); 1 + \frac{\mathrm{i}2\Omega\sqrt{A}}{n}; -\frac{\mathrm{i}2\Omega\sqrt{B}}{n} r^n \right] \Bigg). \quad (17)$$

Example 1. $f(r) = 1 + \alpha r^2$. This profile can be obtained by making $A = 0, n = 2, B = \alpha$ and $C = 1$ in equation (16). This results in the following transformation:

$$W = F(r)\mathrm{e}^{(ar^2/2)}, \quad \text{where} \quad a = \mathrm{i}\sqrt{\alpha\Omega^2} \quad \text{and} \quad s = -ar^2. \quad (18)$$

Equation (3) can be reduced to the confluent hypergeometric equation [13–15]

$$sF'' + F'[1 - s] - \frac{1}{2a}\left[a + \frac{\Omega^2}{2} \right] F = 0. \quad (19)$$

The solution for W can then be written in the form of equation (17) with

$$F(s) = C_1 {}_1F_1(\delta; 1; s) + C_2 U(\delta; 1; s), \quad (20)$$

TABLE 1

Fundamental frequency ($f(r) = 1 + \alpha r^2$)

	α					
R_0	0·5	1	1·5	2	2·5	3
0	2·2819	2·1736	2·0778	1·9925	1·9162	1·8474
0·1	3·0736	2·8754	2·7092	2·5678	2·4456	2·3389
0·2	3·4969	3·2431	3·0538	2·8627	2·7157	2·5887
0·3	3·9943	3·6730	3·4171	3·2073	3·0315	2·8014
0·4	4·6320	4·2233	3·9054	3·6493	3·4374	3·2584
0·5	5·5071	4·9782	4·5762	4·2572	3·9974	3·7794
0·6	6·8050	6·0985	5·5737	5·1642	4·8333	4·5587
0·7	8·9546	7·9558	7·2300	6·6722	6·2262	5·8591
0·8	13·2394	11·6616	10·5397	9·6895	9·0166	8·4668

TABLE 2

Second mode ($f(r) = 1 + \alpha r^2$)

	α					
R_0	0·5	1	1·5	2	2·5	3
0	5·1412	4·8416	4·5969	4·3914	4·215	4·0608
0·1	6·3195	5·9011	5·5622	5·2791	5·0373	4·8271
0·2	7·1061	6·5875	6·173	5·8307	5·5411	5·2917
0·3	8·066	7·4189	6·9101	6·4955	6·1486	5·8526
0·4	9·3186	8·5003	7·8687	7·3613	6·9417	6·5869
0·5	11·0525	9·9958	9·1962	8·5634	8·0459	7·6126
0·6	13·6365	12·2255	11·1795	10·3638	9·7044	9·1569
0·7	17·9269	15·9312	14·4825	13·3691	12·4786	11·7454
0·8	26·4893	23·3353	21·0933	19·3943	18·0492	16·9502

where

$$\delta = \frac{1}{2a}\left[a + \frac{\Omega^2}{2}\right]. \tag{21}$$

A simple midpoint root finding scheme was used to obtain the roots of the characteristic equation. Tables 1 and 2 show the values of Ω (fundamental and second frequency coefficients, respectively) for solid circular and annular membranes with r_0 varying from 0 to 0·8 and α varying from 0·5 to 3·0. As r_0 increases, the natural frequency increases. The trend of the natural frequencies decreasing with increasing α can also be clearly seen from these results.

This problem has been studied numerically by Gutierrez *et al.* [9] using (1) differential quadrature method, (2) finite element method (3) an optimized and/or improved Rayleigh quotient method, and (4) a lower bound based on the Stodola–Vianello method. The natural frequencies given by them are in excellent agreement with the results presented above calculated using the closed-form solution.

Note that the hypergeometric U function tends to infinity when $r_0 = 0$. Therefore, the coefficient of this function was chosen to be zero, in order to make the displacement finite at $r = 0$. The derivative of ${}_1F_1$ when $r_0 = 0$ is zero, and therefore $W'(r)$ is zero. The eigenvalues obtained were in excellent agreement with the values given by Gutierrez *et al.* [8].

Example 2. $f(r) = 1 + \alpha/r$. This profile can be obtained by making $A = 0, n = 1, B = 1$ and $C = \alpha$ in equation (17). Equation (3) can then be transformed to equation (11) in the form

$$su'' + [1 - s]u' - \frac{[1 - i\alpha\Omega]}{2}u = 0, \tag{22}$$

where

$$W = u(r)e^{i\Omega r} \quad \text{and} \quad s = -i2\Omega r. \tag{23}$$

The solution to equation (22) is

$$u(s) = C_{11}F_1(\delta; 1; s) + C_{21}F_1U(\delta; 1; s), \tag{24}$$

where

$$\delta = \left[\frac{1 - i\alpha\Omega}{2}\right].$$

Tables 3 and 4 show the first and second eigenfrequencies for an annular membrane. As in the previous example, the natural frequencies increase with r_0 and decreases with α.

Example 3. $f(r) = \alpha r^N$, where N can be any real number (integer or non-integer). This case worked out in by De [2], can be obtained by setting $A = 0, B = 0$ and $n = N + 2$.

Using the transformation

$$r = bx^e, \quad b = \left(\frac{1}{\alpha e^2\Omega^2}\right)^{1/N+2} \quad \text{and} \quad e = \frac{2}{2 + N},$$

TABLE 3

Fundamental frequency ($f(r) = 1 + \alpha/r$)

	α					
R_0	0·5	1	1·5	2	2·5	3
0·1	2·3129	1·8762	1·6189	1·4472	1·3168	1·2177
0·2	2·7646	2·2735	1·9755	1·7704	1·6181	1·4994
0·3	3·2813	2·726	2·3814	2·1412	1·9615	1·8205
0·4	3·934	3·2955	2·892	2·6076	2·3934	2·2245
0·5	4·8212	4·0678	3·5839	3·2396	2·9786	2·7719
0·6	6·1297	5·2045	4·6016	4·169	3·8392	3·577
0·7	8·2888	7·0776	6·2794	5·6996	5·2564	4·9027
0·8	12·5823	10·7988	9·6072	8·7392	8·0706	7·5352

TABLE 4

Second mode ($f(r) = 1 + \alpha/r$)

	α					
R_0	0·5	1	1·5	2	2·5	3
0·1	4·6807	3·7871	3·2666	2·9153	2·6575	2·458
0·2	5·569	4·5703	3·9695	3·557	3·2513	3·013
0·3	6·5924	5·496	4·776	4·2938	3·9334	3·6508
0·4	7·8907	6·6039	5·7937	5·2235	4·7943	4·456
0·5	9·6597	8·1453	7·1749	6·4852	5·9625	5·5488
0·6	12·2722	10·4161	9·2084	8·3423	7·6822	7·1575
0·7	16·5866	14·1603	12·5594	11·4022	10·5154	9·8078

equation (3) can be transformed to the following Bessel's differential equation:

$$W''(x) + \frac{1}{x}W'(x) + W(x) = 0. \tag{25}$$

The solution to this equation can be expressed in terms of Bessel and Neumann functions as

$$W(x) = c_1 \mathrm{J}_0(x) + c_2 Y_0(x). \tag{26}$$

This relation can be obtained from equation (17) by using the identities [15]

$$\lim_{l\to\infty} {}_1F_1\left(l, m, -\frac{z}{l}\right) = z^{1/2-(1/2)m}\mathrm{J}_{m-1}(2\sqrt{z}) \quad \text{and}$$

$$\lim_{l\to\infty} U\left(l, m, -\frac{z}{l}\right) = z^{1/2-(1/2)m}Y_{m-1}(2\sqrt{z}). \tag{27}$$

4. FAMILY OF SOLUTIONS OF $f(r) = [1 + \alpha \ln(r)]^{\sigma}/r^2$

Another family of exact solutions exists for a non-homogenous annular membrane when the density profile is of the form

$$f(r) = \frac{[1 + \alpha \ln(r)]^{\sigma}}{r^2}, \quad \alpha \neq 0. \tag{28}$$

By using the following functional transformations:

$$\eta = \ln(r), \quad \theta = (1 + \alpha\eta)^{1/2\nu}, \quad W = \theta^{\nu}Z, \quad \nu = \frac{1}{\sigma + 2}. \tag{29}$$

Equation (3) solution reduces to a Bessel equation

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}\theta^2} + \frac{1}{\theta}\frac{\mathrm{d}Z}{\mathrm{d}\theta} + \left(\gamma^2 - \frac{\nu^2}{\theta^2}\right)Z = 0, \tag{30}$$

where $\gamma = 2\nu\Omega/|\alpha|, \alpha \neq 0$.

The solution to equation (30) is

$$Z = C_1 \mathrm{J}_\nu(\gamma\theta) + C_2 \mathrm{J}_{-\nu}(\gamma\theta) \quad \text{where } \nu \text{ is a non-integer,} \tag{31a}$$

$$Z = C_1 \mathrm{J}_\nu(\gamma\theta) + C_2 Y_\nu(\gamma\theta) \quad \text{where } \nu \text{ is a integer.} \tag{31b}$$

It is interesting to note that the solution for the special case of $f(r) = 1/r^2$, given by Wang [8] can be obtained by letting $\sigma = 0$ in equation (28). The solution can then be written using equation (31a) as

$$Z = C_1 \mathrm{J}_{1/2}(\gamma\theta) + C_2 \mathrm{J}_{-1/2}(\gamma\theta) = C_1 \frac{1}{\sqrt{\theta}} \sin(\gamma\theta) + C_2 \frac{1}{\sqrt{\theta}} \cos(\gamma\theta), \tag{32}$$

$$W = \sqrt{\theta}\left[C_1 \frac{1}{\sqrt{\theta}} \sin(\gamma\theta) + C_2 \frac{1}{\sqrt{\theta}} \cos(\gamma\theta) \right], \tag{33}$$

where $\zeta = (1 + \alpha\eta)$. (34)

Simplifying, and using the above expression of θ reduces the solution to the following form given in reference [8]:

$$W(r) = C_1 \sin(\gamma(1 + \alpha \ln(r))) + C_2 \cos(\gamma(1 + \alpha \ln(r))). \tag{35}$$

5. CONCLUSIONS

Exact analytical solutions describing the axisymmetric vibrations of solid circular and annular membranes with continuously varying density were obtained by transforming the equation of motion to standard differential equations that are analytically solvable in terms of special functions. An approach for obtaining solutions for families of density profiles is presented. The solutions are obtained in terms of Kummer's confluent hypergeometric and Bessel functions. It is shown that the natural frequency increases with the inner radius of the annulus and decreases with the inhomogeneity parameter α. The natural frequencies calculated for the density profile $f(r) = 1 + \alpha r^2$ are in excellent agreement with the numerical solutions presented in a previous study. The expressions presented in this paper are in terms of special functions that can easily be evaluated. The closed-form expressions presented herein can also be used as benchmarks for checking the results obtained from numerical or approximate methods.

ACKNOWLEDGMENT

The authors wish to thank Prof. Dewey Hodges, Georgia Institute of Technology for introducing them to this problem.

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