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SHOOTING METHOD FOR NON-LINEAR VIBRATION AND THERMAL BUCKLING OF HEATED ORTHOTROPIC CIRCULAR PLATES

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1. INTRODUCTION

As circular/annular plates are extensively used in sensors and other elemental structures in engineering, the investigation on the behavior of mechanics such as deformation, vibration and buckling, etc., has been intensified by researchers and engineers (references [1–8], for example). Laura *et al.* [1–3], and Gupta *et al.* [4, 5] investigated the linear free vibrations and buckling of polar orthotropic circular and annular plates using analytical and/or numerical methods. Zheng and Zhou [6], and Zhou *et al.* [7] studied the characteristics of deflection and free vibration in the vicinity of the deflection configuration to the circular plates with geometrically non-linear deformation by means of the semianalytic method. Dumir *et al.* [9] discussed non-linear vibrations of orthotropic circular plates by an orthogonal point collocation method. Using the Kantorovich time-averaging method, Huang [10] analyzed the non-linear oscillations of an isotropic circular plate with a concentric rigid mass. Huang [11] employed the finite element method to get some results for a similar problem of hinged orthotropic circular plates. Although there are various investigations on the mechanical behaviors of different circular/annular plates in literature, almost no attention has been paid to the characteristic of large-amplitude vibration and thermal post-buckling of orthotropic circular/annular plates caused by an environment of changing temperatures.

The present investigation is concerned with the axisymmetrical non-linear vibrations and thermal buckling of a uniformly heated orthotropic circular plate with an edge which is fixed in the line displacements, and is elastically restrained against the rotation. First, by means of the Kantorovich time-averaging method [10, 12, 15], the time variable is eliminated and the dynamic governing equations are reduced to a non-linear eigenvalue problem, i.e., a set of non-linear ordinary differential equations dependent on the spatial variable. After that, we employ the shooting method [14] to solve the induced differential equations and then obtain the dynamic responses of non-linear vibration and thermal post-buckling of the plates. Finally, some numerical results on the mechanical behaviors of the plates varied with different parameters of rigidity ratio, temperature and elastic constraint are displayed in detail.

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2. GOVERNING EQUATIONS

Consider a thin polar orthotropic circular plate with radius a and thickness h. A cylindrical co-ordinate system (r, θ, z) is employed, and its original point is located at the center of middle plane of the plate. The edge of the plate is immovably simply supported, and its rotation about the tangential direction along the edge is elastically restrained by a rotation spring with stiffness k_ϕ. Assume that the plate is subjected to a steady rise of temperature, $T(r)$. According to the theory of von Karman's plates and Hamilton's principles [11, 13], when the axisymmetrical deformations of vibration and/or thermal buckling of the plate are taken into account, the non-linear governing equations of the problem can be written in the form

$$\frac{\partial^2 U}{\partial x^2} + \frac{1}{x}\frac{\partial U}{\partial x} - \frac{k}{x^2}U + \frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x^2} + \frac{1-\nu_\theta}{2x}\left(\frac{\partial W}{\partial x}\right)^2 = \frac{\lambda}{12\delta^2}\left[(1-\mu)\frac{\Theta}{x} + \frac{\mathrm{d}\Theta}{\mathrm{d}x}\right], \tag{1}$$

$$\frac{\partial^4 W}{\partial x^4} + \frac{2}{x}\frac{\partial^3 W}{\partial x^3} - \frac{k}{x^2}\frac{\partial^2 W}{\partial x^2} + \frac{k}{x^3}\frac{\partial W}{\partial x} + \lambda\Theta\left(\frac{\partial^2 W}{\partial x^2} + \frac{1}{x}\frac{\partial W}{\partial x}\right) + \lambda\frac{\mathrm{d}\Theta}{\mathrm{d}x}\frac{\partial W}{\partial x} + \frac{\partial^2 W}{\partial \tau^2}$$
$$= 12\delta^2\frac{1}{x}\frac{\partial}{\partial x}\left[x\left(\frac{\partial U}{\partial x} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2 + \frac{\nu_\theta}{x}U\right)\frac{\partial W}{\partial x}\right], \tag{2}$$

$$U = 0, \quad \frac{\partial W}{\partial x} = 0 \quad \text{at } x = 0 \quad \text{and} \quad \frac{\partial^3 W}{\partial x^3} + \frac{1}{x}\frac{\partial^2 W}{\partial x^2} = 0 \quad \text{when } x \to 0. \tag{3}$$

$$U = 0, \qquad W = 0, \qquad \frac{\partial^2 W}{\partial x^2} + (\nu_\theta + K_\phi)\frac{\partial W}{\partial x} = 0 \quad \text{at } x = 1 \tag{4}$$

in which the following non-dimensional quantities:

$$(x, U, W) = (r, u, w)/a, \quad \tau = (t/a^2)(D/\rho h)^{1/2}, \quad \lambda = 12(1+\nu_\theta\beta)\delta^2\alpha_r T_0, \quad \delta = a/h,$$
$$k = E_\theta/E_r = \nu_{\theta r}/\nu_{r\theta}, \quad \beta = \alpha_\theta/\alpha_r, \quad K_\phi = ak_\phi/D, \quad \mu = (\nu_\theta + \beta k)/(1+\beta\nu_\theta), \quad \nu_\theta = \nu_{\theta r} \tag{5}$$

are introduced. Here, $u(r, t)$, and $w(r, t)$ denote the radial and the transverse displacements of the plate on the middle plane respectively. t is the time variable, and ρ represents the density of mass. $D = E_r h^3/[12(1-\nu_{\theta r}\nu_{r\theta})]$ is the flexural rigidity of the plate; E, ν and α are the Young modulus, the Poisson ratio, and the coefficients of thermal expansion of the plate respectively. The subscripts "r" and "θ" indicate that the quantities correspond to r and θ directions respectively. For the axisymmetric case, the applied temperature field at the middle plane can be formulated by $T(r) = T_0\Theta(x)$ in which T_0 is a magnitude of the field, while $\Theta(x)$ denotes the profile function of the field. In this paper, we consider the profile function to be pre-known.

When the terms of inertia forces in the above governing equations are set to be zero, one gets those governing equations for thermal post-buckling of the heated plates. In this case, we have $U(x, \tau) = U(x)$ and $W(x, \tau) = W(x)$ in equations (1)–(4).

3. PROGRAM FOR SOLUTIONS

An exact solution of the problem defined by equations (1)–(4) is at present unknown. Herein, an approximate solution in the "assumed-time-mode" form [7, 12, 15] is to be found. For an unbuckled plate, considering the essential vibration with harmonic response, we take approximate solutions of the form

$$U(x,\tau) = \xi_0(x) + \xi(x)\cos^2\omega\tau, \quad W(x,\tau) = \eta(x)\cos\omega\tau, \tag{6}$$

where ω is a non-dimensional frequency, $\xi(x)$ and $\eta(x)$ are shape functions to be determined, and $\xi_0(x)$ is a solution of the displacement U to the static thermal stress problem of the heated plate, i.e., it satisfies

$$\xi_0'' + \frac{1}{x}\xi_0' - \frac{k}{x^2}\xi_0 = \frac{\lambda}{12\delta^2}\left[(1-\mu)\frac{\Theta}{x} + \Theta'\right], \quad \xi_0(0) = \xi_0(1) = 0. \tag{7}$$

where $(\ )'$ represents derivatives with respect to x. Substituting equation (6) of the assumed solutions into equations (1)–(4), and using the Kantorovich time-averaging method [11, 13, 15] and considering equation (7), we obtain the non-linear ordinary differential equations and the corresponding boundary conditions as follows:

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x} = \mathbf{H}(x, \mathbf{Y}; \lambda) \quad (0 < \Delta x \leqslant x < 1), \tag{8}$$

$$\mathbf{B}_1\mathbf{Y}(\Delta x) = \{0, A/\delta, 0, 0\}^{\mathrm{T}}, \qquad \mathbf{B}_2\mathbf{Y}(1) = \{0, 0, 0\}^{\mathrm{T}}, \tag{9a, b}$$

in which

$$\mathbf{H} = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}^{\mathrm{T}} = \{\xi, \xi', \eta, \eta', \eta'', \eta''', \omega^2\}^{\mathrm{T}}, \tag{10}$$

$$\mathbf{H} = \{y_2, \phi_1, y_4, y_5, y_6, \phi_2, 0\}^{\mathrm{T}}, \quad \phi_1 = -y_2/x + ky_1/x^2 - y_4y_5 - (1-v_\theta)y_4^2/(2x), \tag{11}$$

$$\phi_2 = -\frac{2}{x}y_6 + \frac{k}{x^2}y_5 - \frac{k}{x^3}y_4 + y_7y_3 - \lambda\Theta\left(y_5 + \frac{1}{x}y_4\right) - \lambda\Theta' y_4 + C_1\varphi + C_2\psi, \tag{12}$$

$$\varphi = (1+v_\theta)y_2y_4/x + y_2y_5 + \phi_1 y_4 + v_\theta y_1 y_5/x + 3y_4^2y_5/2 + y_4^3/(2x), \tag{13}$$

$$\psi = (1+v_\theta)\xi_0' y_4/x + \xi_0' y_5 + \xi_0'' y_4 + v_\theta \xi_0 y_5/x. \tag{14}$$

Here, $C_1 = 9\delta^2$, $C_2 = 12\delta^2$, Δx is a very small positive quantity, and $\mathbf{B}_1$ and $\mathbf{B}_2$ are matrixes of order 7×4 and 7×3 respectively. Their elements vanish except for

$$B_1(1,1) = B_1(2,3) = B_1(3,4) = B_1(4,6) = B_2(1,1) = B_2(2,3) = B_2(3,5) = 1, \tag{15}$$

$$B_1(4,5) = 1/\Delta x, \quad B_2(3,4) = v_\theta + K_\phi. \tag{16}$$

In order to assure a unique relationship between η and ω, we introduce a normalization condition in equation (9a) of the form

$$\eta(c) = A/\delta. \tag{17}$$

Then, it is found that $A = \delta\eta(\Delta x) = w(\Delta x, 0)/h$ is the non-dimensional amplitude of transverse deflection at the center of the plate. It can be shown that when $\omega = 0$, $C_1 = C_2$, and $y_7 = \lambda$, the resulting governing equations (8) and (9) for the vibration become those for the thermal post-buckling of the heated plate.

Next, we use the shooting method [14, 15] to obtain numerical solutions of equations (8) and (9). Corresponding to the boundary-value problem defined by equations (8) and (9a), we denote

$$\frac{d\mathbf{Z}}{dx} = \mathbf{H}(x, \mathbf{Z}; \lambda), \quad \mathbf{Z}(\Delta x) = \mathbf{I}(A, \mathbf{V}) \tag{18}$$

in which $\mathbf{Z} = \{z_1, z_2, z_3, z_4, z_5, z_6, z_7\}^T$, $\mathbf{I}(A, V) = \{0, v_1, A/\delta, 0, v_2, (-v_2/\Delta x), v_3\}^T$, and $\mathbf{V} = \{v_1, v_2, v_3\}^T$ which is an unknown vector related to the values of $\mathbf{Y}$ at $x = \Delta x$. Then, we have

$$\mathbf{Z}(x; A, \mathbf{V}, \lambda) = \mathbf{I}(A, \mathbf{V}) + \int_{\Delta x}^{x} \mathbf{H}(\zeta, \mathbf{Z}; \lambda)\, d\zeta. \tag{19}$$

For a prescribed value of A, the components of $\mathbf{V}$ can be sought such that the boundary condition (9b), or

$$\mathbf{B}_2 Z(1; A, \mathbf{V}^*, \lambda) = \{0, 0, 0\}^T \tag{20}$$

is satisfied. Up to now, a root of equation (20), denoted by $\mathbf{V} = \mathbf{V}^*$, is obtained. Further, the solution of the boundary-value problem of equations (8) and (9) is gained by the expression

$$\mathbf{Y}(x) = \mathbf{Z}(x; A, \mathbf{V}^*, \lambda). \tag{21}$$

Finally, the solution of the problem considered here is obtained by equation (6).

4. NUMERICAL RESULTS AND DISCUSSIONS

Due to the limit of space, in this paper, we only display the results of the problems considered here for the plates subjected to a field of rising uniform temperature. For this case, we have $T(r) = T_0$ or $\Theta(x) \equiv 1$ and there is no difficulty in finding the solution of equation (7) in the form

$$\xi_0(x) = \begin{cases} \dfrac{B}{(1-k)}\left(\dfrac{(c - c^{-\sqrt{k}})x^{\sqrt{k}} - (c - c^{\sqrt{k}})x^{-\sqrt{k}}}{c^{-\sqrt{k}} - c^{\sqrt{k}}} + x\right) & \text{for } k \neq 1, \\ 0 & \text{for } k = 1 \end{cases} \tag{22}$$

for $x > \Delta x > 0$. Here, $c = \Delta x$, $B = (1-\mu)\lambda/(12\delta^2)$. Applying the Rung–Kutta method to equation (19) and the Newton–Raphson method to equation (20) for finding the root V^*, we get the numerical solutions of equations (8) and (9) [14, 15]. After that, an A-dependent family of solutions of equations (8) and (9) will be get by the method of analytic continuation when A increases with a small step [7, 15].

Here, we choose the geometric parameters $\delta = a/h = 30$, and the Poisson ratio $v_\theta = 0{\cdot}3$. An error limit, $\varepsilon = 10^{-5}$, is taken in the numerical calculation. Since a singularity will exist when parameter c tends to be zero in the numerical computation, we set $c = \Delta x = 0{\cdot}0001$ approximately to take the place of the solid circular plate. In order to show the reliability of the numerical technique employed here, we firstly give some numerical tests for the linear vibration of an unheated circular plate. A comparison of the results obtained from the numerical method proposed in this paper with those from the Ritz method [5] for the fundamental frequency of the plate is presented in Table 1, which exhibits that they are in excellent agreement.

For the small-amplitude vibration, e.g., $A = 10^{-4}$, of uniformly heated circular plates with different values of rigidity ratio k, and parameter β, characteristic curves of the square

TABLE 1

Comparison of the linear frequency ω *with that in reference* [5] ($\lambda = 0{\cdot}0$, $\nu_\theta = 0{\cdot}3$ *and* $A = 10^{-4}$)

K_ϕ	$k = 0{\cdot}75$		$k = 1{\cdot}0$		$k = 5{\cdot}0$	$k = 10{\cdot}0$	
0	4·5421	4·5418†	4·9351	4·9351†	6·1456	11·2858	11·2858†
10	8·3877	8·3877†	8·7519	8·7519†	9·8843	14·7563	14·7563†
100	9·6168	9·6167†	10·0193	10·0192†	11·265	16·5434	16·5433†
∞	9·8057	9·8056†	10·2158	10·2158†	11·485	16·8616	16·8616†

†The results of reference [5].

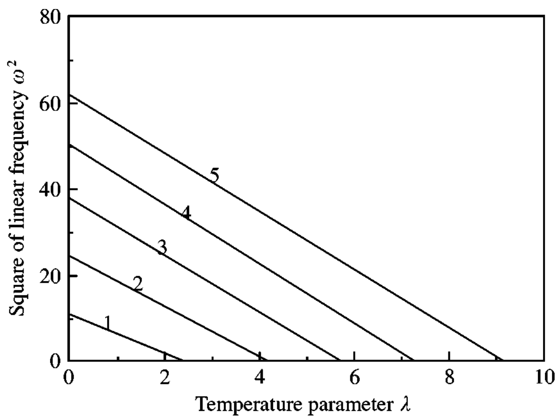

Figure 1. Characteristic curves of the square of the fundamental frequency ω^2 versus the temperature parameter λ for the plate with $K_\phi = 0{\cdot}0$ and $A = 0{\cdot}0001$: (1) $k = 0{\cdot}2$, $\beta = 2{\cdot}0$; (2) $k = 1{\cdot}0$, $\beta = 1{\cdot}0$; (3) $k = 2{\cdot}0$, $\beta = 2{\cdot}0$; (4) $k = 3{\cdot}0$, $\beta = 3{\cdot}0$; (5) $k = 4{\cdot}0$, $\beta = 4{\cdot}0$.

of fundamental frequency, ω^2, with respect to the temperature parameter λ are plotted in Figure 1. It shows that the square of the linear fundamental frequency, ω^2, decreases monotonically and approximately linearly with the increment of the temperature parameter, λ, and especially we have $\omega^2 = 0$ when $\lambda = \lambda_{cr}$. It is obvious that λ_{cr} is a critical temperature parameter over which the plate will be in buckled states.

Non-linear characteristic relationships between the fundamental frequency ω and the amplitude A are obtained by the method of analytical continuation. In Figure 2, we plot the characteristic curves of amplitude–frequency of the heated circular plate with different values of parameters, λ, k, β, and K_ϕ. It is found that the non-linear fundamental frequency decreases as the temperature parameter λ increases, and the frequency increases with increment of the dimensionless amplitude A for all cases. The effect of the temperature parameter λ on the frequency ω when A is small is more significant than that when A is large. Furthermore, these frequency–amplitude curves of the heated circular plate exhibit a similar behavior to that of the free oscillation of a single-degree-of-freedom hard-spring Duffing system [7, 8, 15].

In the case of study of thermal post-buckling of the heated circular plate, we take $C_1 = C_2 = 12\delta^2$, $\omega = 0$ and $y_7 = \lambda$ in the computational program. Figure 3 displays the secondary equilibrium paths of post-buckling state of the heated circular plates with specified values of parameters k and β. From these curves, we find that the fundamental or initial equilibrium states bifurcate at the point $(\lambda, A) = (\lambda_{cr}, 0)$ of each curve.

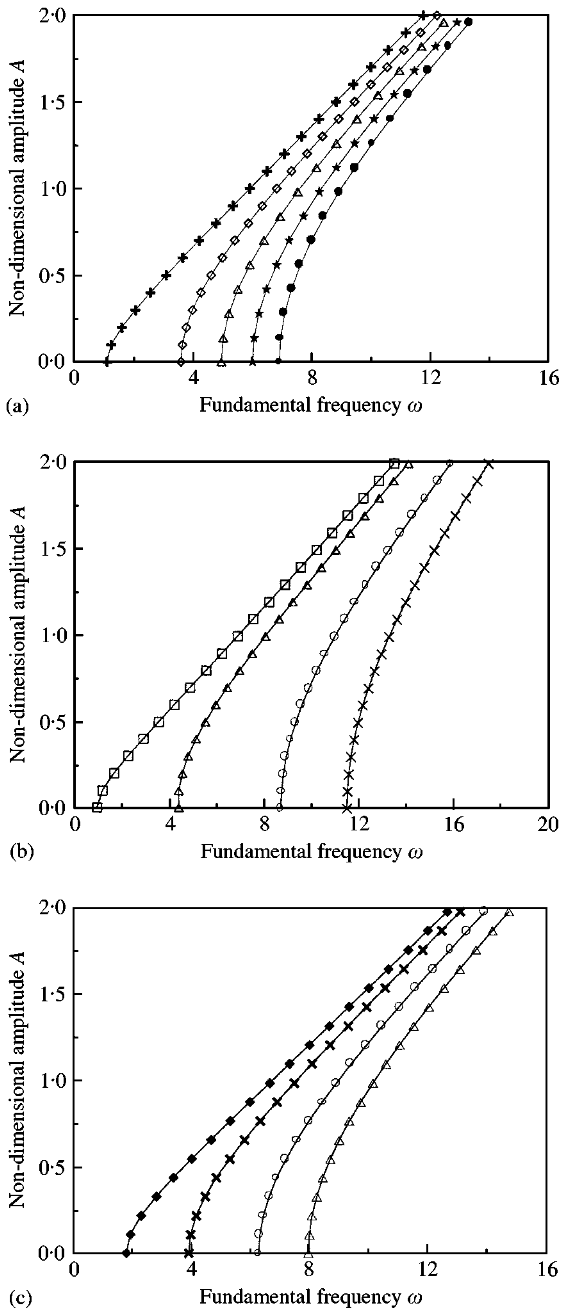

Figure 2. Characteristic curves of amplitude–frequency of the heated circular plates: (a) $k = 2{\cdot}0$, $\beta = 0{\cdot}75$ and $K_\phi = 0$: -+-, $\lambda = 4$; --◇--, $\lambda = 2$; -△-, $\lambda = 0$; --*--, $\lambda = -2$; —●—, $\lambda = -4$; (b) $k = 2{\cdot}0$, $\beta = 0{\cdot}75$ and $K_\phi = \infty$: —□—, $\lambda = 18{\cdot}5$; —△—, $\lambda = 16{\cdot}0$; —○—, $\lambda = 8{\cdot}0$; —×—, $\lambda = 0{\cdot}0$; (c) $k = 2{\cdot}0$, $\beta = 0{\cdot}5$ and $K_\phi = 0{\cdot}1$: —◆—, $\lambda = 6{\cdot}0$; —×—, $\lambda = 4{\cdot}0$; —○—, $\lambda = 0{\cdot}0$; —△—, $\lambda = -4{\cdot}0$.

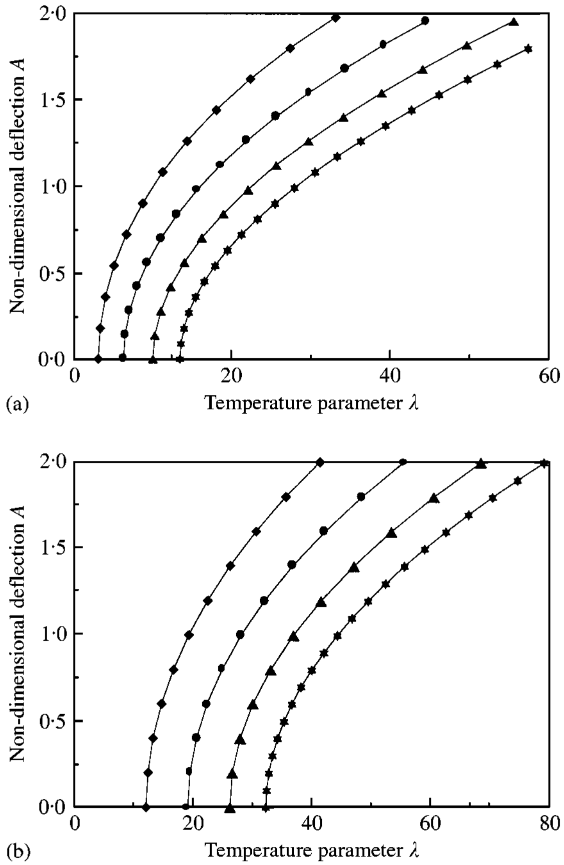

Figure 3. Secondary equilibrium paths of the thermal post-buckled plate: (a) Immovably simply supported edge, $\beta = 1/k$, $K_\phi = 0{\cdot}0$: —∗—, $k = 6{\cdot}0$; —▲—, $k = 7{\cdot}0$; —●—, $k = 2{\cdot}0$; —◆—, $k = 0{\cdot}5$; (b) Clamped edge, $\beta = 1/k$, $K_\phi = \infty$: —◆—, $k = 0{\cdot}5$; —●—, $k = 2{\cdot}0$; —▲—, $k = 4{\cdot}0$; —∗—, $k = 6{\cdot}0$.

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