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FUNDAMENTAL FREQUENCY OF TRANSVERSE VIBRATION OF A RECTANGULAR, ANISOTROPIC PLATE OF DISCONTINUOUSLY VARYING THICKNESS

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1. INTRODUCTION

The present study undertakes the task of obtaining an approximate solution of the title problem when the plate edges are either simply supported to clamped; see Figure 1. Apparently, no solutions are available in the open literature for the problem under analysis [1–4] which, undoubtedly, possesses practical importance in several fields of engineering and applied science.

Since an exact solution appears to be out of question the classical Rayleigh–Ritz method is employed to determine the fundamental eigenvalue. Polynomial co-ordinate functions are used for both types of boundary conditions and in the case of simply supported edges an independent solution is found by expressing the displacement amplitude in terms of a truncated double Fourier series. Good engineering agreement is shown to exist between both solutions.

2. APPROXIMATE SOLUTION OF THE PROBLEM BY MEANS OF A VARIATIONAL APPROACH

In the case of normal modes of vibration the governing functional is [2]

$$J(W) = \iint_p [D_{11} W_{\bar{x}^2}^2 + 2D_{12} W_{\bar{x}^2} W_{\bar{y}^2} + D_{22} W_{\bar{y}^2}^2 + 4D_{66} W_{\overline{xy}}^2$$

$$+ 4(D_{16} W_{\bar{x}^2} + D_{26} W_{\bar{y}^2}) W_{\overline{xy}}]\, \mathrm{d}\bar{x}\, \mathrm{d}\bar{y} - \rho\omega^2 \iint_{\bar{p}} h W^2\, \mathrm{d}\bar{x}\, \mathrm{d}\bar{y}, \qquad (1)$$

where

$$h(\bar{x}, \bar{y}) = \eta h_0, \qquad \eta = \begin{cases} 1, & 0 \leqslant \bar{x} < c, \\ h_1/h_0, & c < \bar{x} \leqslant a. \end{cases}$$

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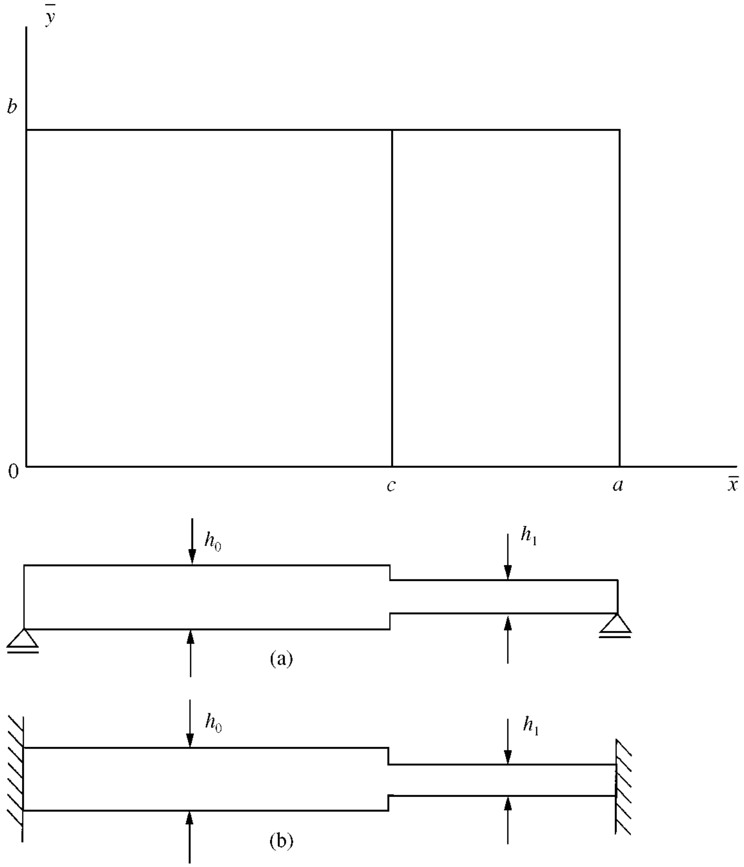

Figure 1. Anisotropic rectangular plate of discontinuously varying thickness executing transverse vibrations: (a) simply supported case, (b) clamped edges.

Introducing the dimensionless variables $\bar{x} = ax$, $\bar{y} = by$ and defining $\lambda = a/b$ and $\lambda_c = c/a$, one obtains, after substitution into equation (1),

$$\frac{a^2\lambda}{D_{110}} J(W) = \iint_p \eta^3 \left[W_{x^2}^2 + 2\frac{D_{120}}{D_{110}} \lambda^2 W_{x^2} W_{y^2} + \frac{D_{220}}{D_{110}} \lambda^4 W_{y^2}^2 + 4\frac{D_{660}}{D_{110}} \lambda^2 W_{xy}^2 + 4\left(\frac{D_{160}}{D_{110}} \lambda W_{x^2} + \frac{D_{260}}{D_{110}} \lambda^3 W_{y^2} \right) W_{xy} \right] \mathrm{d}x\,\mathrm{d}y - \Omega^2 \iint \eta W^2 \,\mathrm{d}x\,\mathrm{d}y, \qquad (2)$$

where $\Omega^2 = \rho h_0 a^4 \omega^2 / D_{110}$ and

$$h(x, y) = \eta h_0, \qquad \eta = \begin{cases} 1, & 0 \leqslant x \langle \lambda_c, \\ h_1/h_0, & \lambda_c \langle x \leqslant 1. \end{cases}$$

In the case of simply supported and clamped plates the following polynomial expression was used in order to approximate the fundamental mode shape:

$$W_a = \sum_{j=1}^{J} C_j \varphi_j(x, y) = \sum_{j=1}^{J} C_j (x^q + \alpha_{j3} x^3 + \alpha_{j2} x^2 + \alpha_{j1} x)(y^q + \beta_{j3} y^3 + \beta_{j2} y^2 + \beta_{j1} y),$$

$$q = p + j - 1. \qquad (3)$$

where "p" is Rayleigh's optimization exponent [5] and the coefficients $\alpha_{j3}, \ldots, \beta_{j1}$ are determined substituting each polynomial co-ordinate function in the essential boundary conditions corresponding to each mechanical configuration, i.e., simply supported or clamped edges.

Substituting equation (3) into equation (2) and applying Ritz' minimization condition one obtains

$$\frac{1}{2}\frac{a^2\lambda}{D_{110}}\frac{\partial J}{\partial C_i} = \sum_{j=1}^{J}\left\{\iint_p \eta^3\left[\varphi_{jx^2}\varphi_{ix^2} + \frac{D_{120}}{D_{110}}\lambda^2(\varphi_{jy^2}\varphi_{ix^2} + \varphi_{jx^2}\varphi_{iy^2})\right.\right.$$
$$+ \frac{D_{220}}{D_{110}}\lambda^4\varphi_{jy^2}\varphi_{iy^2} + 4\frac{D_{660}}{D_{110}}\lambda^2\varphi_{jxy}\varphi_{ixy}$$
$$+ 2\left[\frac{D_{160}}{D_{110}}\lambda(\varphi_{jxy}\varphi_{ix^2} + \varphi_{jx^2}\varphi_{ixy})\right.$$
$$\left.\left.+ \frac{D_{260}}{D_{110}}\lambda^3(\varphi_{jxy}\varphi_{iy^2} + \varphi_{jy^2}\varphi_{ixy})\right]\mathrm{d}x\,\mathrm{d}y$$
$$\left. - \Omega^2\iint_p \eta\varphi_j\varphi_i\,\mathrm{d}x\,\mathrm{d}y\right\}C_j = 0. \tag{4}$$

Equation (4) leads to a homogeneous linear system of equations in the C_j's which generates a determinantal equation in the eigenvalues Ω_j. Since

$$\Omega_1 = \Omega_1(p), \tag{5}$$

by minimizing it with respect to "p" one obtains an optimized value of Ω_1.
In the case of a simply supported plate one can also use

$$W_a = \sum_{n=1}^{N}\sum_{m=1}^{N} C_{nm}\varphi_{nm}(x, y) = \sum_{n=1}^{N}\sum_{m=1}^{M} C_{nm}\sin n\pi x\sin m\pi y. \tag{6}$$

Obviously, the same general procedure previously described leads to

$$\frac{1}{2}\frac{a^2\lambda}{D_{110}}\frac{\partial J}{\partial C_{pq}} = \sum_{n=1}^{N}\sum_{m=1}^{M}\left\{\pi^4\iint_p \eta^3\left\{\left[n^2p^2 + \lambda^2\frac{D_{120}}{D_{110}}(m^2p^2 + n^2q^2) + \frac{D_{220}}{D_{110}}\lambda^4 m^2q^2\right]\varphi_{nm}\varphi_{pq}\right.\right.$$
$$+ 4\frac{D_{660}}{D_{110}}\lambda^2 nmpq\varphi_{nm}\varphi_{pq} - 2\left[\left(\frac{D_{160}}{D_{110}}\lambda p^2 + \frac{D_{260}}{D_{110}}\lambda^3 q^2\right)nm\varphi_{nm}\varphi_{pq}\right.$$
$$\left.\left.+ \left(\frac{D_{160}}{D_{110}}\lambda n^2 + \frac{D_{260}}{D_{110}}\lambda^3 m^2\right)pq\varphi_{nm}\varphi_{pq}\right]\right\}\mathrm{d}x\,\mathrm{d}y$$
$$\left. - \Omega^2\iint_p \eta\varphi_{nm}\varphi_{pq}\,\mathrm{d}x\,\mathrm{d}y\right\}C_{nm} = 0, \tag{7}$$

where $\varphi_{nm} = \cos n\pi x\cos m\pi y$ and $(p, q) = (1, 1)\cdots(N, M)$, and the classical algorithmic scheme finally leads to the fundamental eigenvalue of the vibrating system.

3. NUMERICAL RESULTS

Calculations were performed for a structural system with the following constitutive properties:

$$\frac{D_{120}}{D_{110}} = \frac{D_{220}}{D_{110}} = \frac{D_{660}}{D_{110}} = \frac{1}{2}, \qquad \frac{D_{160}}{D_{110}} = \frac{D_{260}}{D_{110}} = \frac{1}{3}.$$

On the other hand, the geometric parameters a/b, c/a and h_1/h_0 were conveniently chosen as $\lambda = 3/2$, 1, 2/3; $\lambda_c = 1/4$, 1/2, 3/4; $\eta = 0{\cdot}8$, 0·6 respectively.

Finally, when using the polynomial co-ordinate functions J was taken equal to 4 while $N = M = 8$ when performing the determination of the fundamental frequency coefficient for the simply supported case.

Table 1 depicts values of Ω_1 in the case of a simply supported, anisotropic plate of discontinuously varying thickness. In general, good agreement is found between the results obtained using polynomial and sinusoidal co-ordinate functions, the latter being considered as more accurate since a considerably larger number of terms is used.

Table 2 deals with the clamped case. In view of the reasonable agreement found between the eigenvalues contained in Table 1 one can hope for acceptable engineering accuracy in the case of the results shown in Table 2.

TABLE 1

Fundamental frequency coefficient of a simply supported anisotropic rectangular plate of discontinuously varying thickness: (1) *polynomial co-ordinate functions*, (2) *truncated double Fourier series*

		(1)			(2)		
λ_c	η	$\lambda = 3/2$	$\lambda = 1$	$\lambda = 2/3$	$\lambda = 3/2$	$\lambda = 1$	$\lambda = 2/3$
0·25	0·8	25·84	17·27	12·70	25·06	16·80	12·43
	0·6	21·54	14·57	10·61	20·78	14·12	10·35
0·50	0·8	27·03	17·97	13·28	26·22	17·48	13·00
	0·6	23·76	15·73	11·46	22·79	15·09	11·04
0·75	0·8	28·47	18·94	14·15	27·64	18·44	13·86
	0·6	26·92	17·92	13·48	25·52	16·97	12·80

TABLE 2

Fundamental frequency coefficient of a clamped anisotropic rectangular plate of discontinuously varying thickness

λ_c	η	$\lambda = 3/2$	1	$\lambda = 2/3$
0·25	0·8	44·03	29·63	23·97
	0·6	37·11	25·64	21·02
0·50	0·8	46·81	30·65	24·36
	0·6	43·18	27·80	21·55
0·75	0·8	49·79	31·94	25.10
	0·6	48·50	30·31	23·22

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REFERENCES

1. A. W. LEISSA 1969 *NASA SP* 160. Vibration of plates.
2. S. G. LEKHNITSKII 1968 *Anisotropic Plates*. New York, NY: Gordon and Breach Science Publishers.
3. J. VINSON 1999 *The Behavior of Sandwich Structures of Isotropic and Composite Materials*. Lancaster, PA: Technomic Publishing Company.
4. K. F. CHERNYKH 1998 *An Introduction to Modern Anisotropic Elasticity*. New York, NY: Begell House Inc. Publishers.
5. P. A. A. LAURA 1995 *Ocean Engineering* **22**, 235–250. Optimization of variational methods.