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THE SYMPTOM OBSERVATION MATRIX FOR MONITORING AND DIAGNOSTICS

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Monitoring of complex technical systems in operation/service is a task concerning safety, reliability and risk management. Additionally, the monitoring and diagnosis of such systems is also recommended for the reduction of costs, and to increase the life of systems under consideration.

The fundamental quantities for monitoring and diagnosis are symptoms contained in measured data of a time-variant system. Symptoms are observable and sensitive with respect to faults or damage. Questions such as how to find symptoms, how they should be evaluated for information condensation in order to detect, and if possible to localize, faults or damage, are discussed, and, additionally, the choice of the measuring matrix and the test forces are treated in the context of the state space formulation of time-variant models.

For the purpose of information condensation and the evaluation of symptoms, the symptom observation matrix is introduced and manipulated by the singular value decomposition. The properties of the observation matrix with its singular values and singular vectors show their applicability for monitoring and diagnosis.

1. PROBLEM FORMULATION

A system in operation is understood here to be a technical system performing its designed purpose, such as production or service. The mechanical part undergoes ageing, wear, etc., and, briefly, its mechanical properties are time dependent, and tend mostly in the direction of lowering its operational/service capabilities, safety and reliability. Consequently, a holistic approach has to be applied, which means that the system life stages have to be considered starting with the elaboration of need right up to the phase-out, including recycling [1]. The related mathematical models are therefore time-variant models. For reasons of safety, operation and serviceability, to reduce costs and to increase the lifetime of the system under consideration, monitoring of the state condition and diagnosis of the results of monitoring are recommended, if not essential. Monitoring and diagnosis use symptoms which must be observable and sensitive to faults or damage as fundamental quantities. The question immediately arises: how should they be evaluated? This paper deals with this question and introduces a matrix with lifetime-dependent symptoms, which is manipulated by singular value decomposition.

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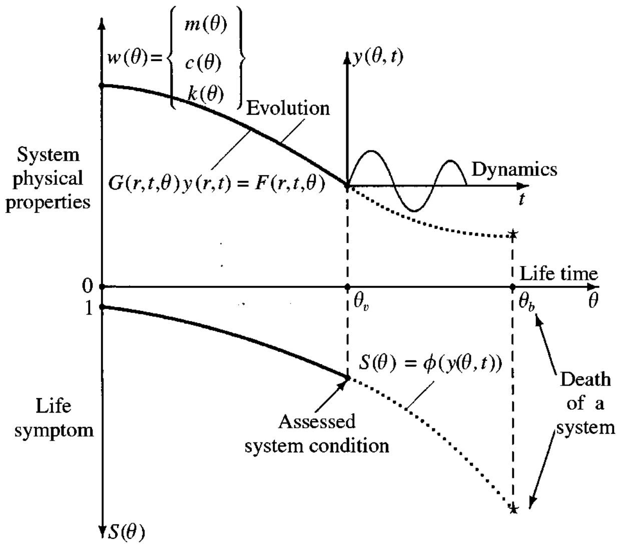

Figure 1. Three domains of evolution of the system in operation/service, dynamic reaction, and physical properties as observed by evolving symptoms.

Given a mechanical system with its time-variant description, the current dynamics and the system evolution obey different time scales. A slow time co-ordinate θ designating the system's evolution (e.g., in months, years) is thus introduced, and, in addition, the fast time co-ordinate t describing the current dynamics (e.g., in s, min). Figure 1 illustrates the situation.

The dynamics of the system in the state space at a (fixed) lifetime θ is described in the usual notation by

$$\dot{\mathbf{x}}_\theta(t) = \mathbf{A}_\theta \mathbf{x}_\theta(t) + \mathbf{B}_\theta \mathbf{f}(t), \tag{1}$$

assuming $2n$ components of the state vector $\mathbf{x}_\theta(t) := (\mathbf{u}'_\theta(t) \cdot \dot{\mathbf{u}}'_\theta(t))'$, where the transposition of the row vector (.) is designated by (.)$'$. The following notation is used: $\mathbf{u}_\theta(t)$ is the vector of displacements; $\mathbf{A}_\theta$ the system or state matrix, $\mathbf{B}_\theta$ the input matrix and $\mathbf{f}(t)$ the vector of external forces.

Dots indicate differentiation with respect to time. The orders of the related matrices are given accordingly. The excitation is assumed to be unchanged during the lifetime (lifespan) θ_b in a first approach.

The measuring equation for system observation is written as

$$\mathbf{y}_\theta(t) = \mathbf{H}\mathbf{x}_\theta(t): \tag{2}$$

m components of the state vector are measured. Consequently, the measuring matrix (or output matrix) $\mathbf{H}$ is an $(m, 2n)$-matrix. Here, it is also presumed that the measurements are made during the lifetime in the same manner.

Symptoms as sensitive quantities of a defect/fault/damage of the system with suitable properties are applied. In general, they indicate a modification of the system [1]. They are generally independent of the fast time co-ordinate through the application of an integral operator T_v over the time t, where the r various symptoms are designated by the subscript $v, v = 1, \ldots, r$. Therefore the (r, m)-matrix

$$\mathbf{T} = \begin{Bmatrix} T_1 \\ T_2 \\ \cdot \\ \cdot \\ \cdot \\ T_r \end{Bmatrix}$$

applied to the measurements $\mathbf{y}_\theta(t)$ results in the vector

$$\mathbf{s}(\theta) = \mathbf{s}_\theta := \mathbf{T}\mathbf{y}_\theta(t) = \mathbf{T}\mathbf{H}\mathbf{x}_\theta(t), \tag{3}$$

using equation (2), where **TH** is an $(r, 2n)$-matrix. Consequently, $\mathbf{s}(\theta)$ is a vector of r symptoms at the lifetime θ; the symptoms are assumed to be dimensionless and normalized.

The symptom observation matrix [2] of a continuously used system, in brief the observation matrix, is now defined as

$$\mathcal{O} = \mathcal{O}_{pr} := \begin{Bmatrix} \mathbf{s}'_{\theta_1} \\ \mathbf{s}'_{\theta_2} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{s}'_{\theta_p} \end{Bmatrix} = \begin{bmatrix} s_{\theta 1,1} & s_{\theta 1,2} & \cdots & s_{\theta 1,r} \\ s_{\theta 2,1} & s_{\theta 2,2} & \cdots & s_{\theta 2,r} \\ \cdots & \cdots & \cdots & \cdots \\ s_{\theta p,1} & s_{\theta p,2} & \cdots & s_{\theta p,r} \end{bmatrix}. \tag{4}$$

The observation matrix is a (p, r)-matrix. The various symptom values are ordered columnwise (symptom vectors of length p), and the rows (observation vectors $\mathbf{s}'(\theta)$ with r components) represent different lifetimes θ_i, $i = 1, \ldots, p$ of the symptoms. The symptom life-curves [1] are given by the various columns. The elements $s_{\theta i, v}$, $i = 1(1)p$, $v = 1(1)r$, of $\mathcal{O}$ are assumed to be real. However, there is no difficulty in principle in assuming them to be complex.

If the symptoms are to be defined in an image domain by a pre-defined integral transform, for example by the Fourier transform or by applying (stochastic) time series analysis ($\rightsquigarrow$ spectra), then the various time-dependent symptoms are substituted by the frequency-dependent quantities; for example, the spectral amplitudes taken at fixed lifetimes. It is also possible to mix the symptom values, which means combining the values of the different domains. In order to put additional information into the observation matrix one can add as a column, for instance, the lifetimes themselves taken as the symptoms [2].

The observation matrix can be a very large matrix. For trend forecasts, for example for turbine generators, it is $r \gg p$. This means a rectangular matrix with many more columns than rows. However, the reverse can also appear, especially when the condition monitoring system is already developed. The information in the various columns (of the various symptoms) may differ only slightly: there is redundant information, which can be avoided by eliminating the corresponding columns. This recommendation can only be given if one knows *a priori* how these symptoms will develop during the lifetime.

Problems arising with the evaluation of the observation matrix have to be seen within the context of monitoring and diagnostics.

These are the following.

(1) Which symptoms are most representative of defects/faults/damage in the sense of amplifying modifications in the state conditions? It is not only a question of discriminants and, consequently, of the distinguishability of faults, but it is a question of (selective) sensitivity and robustness: what is the relationship between symptom and damage? This question concerns the choice of $\mathbf{s}_\theta$ including T (see equation (3)).

(2) Then the question as to how modifications of the system can be measured has to be answered. It concerns the choice of the measuring matrix $\mathbf{H}$ ($\rightsquigarrow$ observability).

(3) If the operating forces cannot be taken for symptom generation, that means if test forces have to be applied for detecting modifications of the system under monitoring, then $\mathbf{B}_\theta\mathbf{f}(t)$ in equation (1) has to be chosen in such a way that modifications in $\mathbf{A}_\theta(t)$ are detectable in the state vector ($\rightsquigarrow$ controllability), with the restriction that the applied loading must not increase the fault.

(4) How has the spacing during the lifetime to be chosen for symptom readings?

(5) How can the elimination of redundant information (symptoms) be carried out?

(6) If the singular value decomposition (SVD) is chosen for evaluation, what is the physical interpretation of the corresponding vectors?

(7) What is the role of observation matrix manipulations, such as symptom scaling, average value subtractions, etc.?

These questions will be discussed, if not answered, in the following.

2. CHOICE OF SYMPTOMS

The detection and diagnosis of failures/faults/damage, expressed as system modifications, are based on model-aided measurements. Measurements should be reduced to a minimum number of measures which are most sensitive and informative with respect to the expected system modifications. Measurable quantities which are sensitive to system modifications are called symptoms. Additionally, symptoms should be sensitive to damage evolution (related to θ), but should be insensitive to distortions.

A discriminant is a symptom which is sensitive to a particular fault, and therefore discriminants allow us to distinguish between various faults evolving in a system.

Features are special arrangements of symptoms which enable us to distinguish between several faults (with respect to a class). Patterns are established by features in order to characterize different system conditions.

Weak point analysis and sensitivity investigations are used to discover symptoms. A symptom should have the following properties (closely following [1], section 2.2):

- directly (e.g., strain) or indirectly measurable (e.g., stress as a model-based reconstruction);
- functional relationship to a damage measure;
- high sensitivity to a fault/damage as a local property, but robust towards unknown disturbances as a global property (contradictory requirements which need optimization);
- distinguishability of various terms in the fault model, which also includes fault separation;
- the absolute value is a non-decreasing function of time, unless the system is repaired, etc.;
- permit trend estimation.

2.1. SYMPTOMS OF LINEAR SYSTEMS DEPENDENT ON θ

The following system-related characteristics can serve as symptoms:

- constants (e.g., cross-section measures independent of the fast time co-ordinate t, but which can be dependent on the lifetime θ, e.g., due to corrosion;
- functionals (scalars like an eigenfrequency which, for example, can be expressed as a lifetime-dependent Rayleigh quotient);
- vectors (discretized function or an assembly of scalar symptoms);
- functions (like eigenfunctions but also lifetime-dependent);
- field descriptions (multi-dimensional, for example the velocity field of a continuum) by direct or indirect (model-supported) measured quantities.

Examples of scalars are:

- maximum response $\max_t x(\theta, t)$;
- input energy;
- r.m.s.-value $x_{r.m.s.}(\theta) := |\sqrt{(1/T)\int_0^T x^2(\theta, t)\,\mathrm{d}t}|$ for periodic signals;
- average of the absolute signal $x(\theta, t)$: $x_{av} := (1/T)\int_0^T |x(\theta, t)|\,\mathrm{d}t$;
- form (shape) factor $x_{r.m.s.}(\theta)/x_{av}(\theta)$;
- crest factor $x_{peak}(\theta)/x_{r.m.s.}(\theta)$;
- impulse factor $x_{peak}(\theta)/x_{av}(\theta)$;
- variance $\sigma_x^2(\theta)$ (total power, see reference [3]);
- 4th root of Kurtosis $\beta(\theta)$, with $\beta(\theta) := (1/T)\int_{-\infty}^{\infty} x^4(\theta, t)\,\mathrm{d}t/((1/T)\int_{-\infty}^{\infty} x^2(\theta, t)\,\mathrm{d}t)^2$;
- rice frequency $f_r(\theta) = \dot{x}_{r.m.s.}(\theta)/[2\pi x_{r.m.s.}(\theta)]$, where $\dot{x}_{r.m.s.}(\theta)$ stands for the r.m.s.-value of the velocity signal (it is connected with the power spectral density and not with the probability density function (pdf) as a variance, etc.).

We distinguish between global and local symptoms in terms of the spatial co-ordinate. Global symptoms are, for example, the above scalars as enumerated, and the norms of residuals [4] to be composed, i.e., the output, input and generalized residuals (see Figures 2–4), with respect to the dynamic responses. The components of the residual vectors generated from dynamic responses are also global residuals, because they implicitly contain

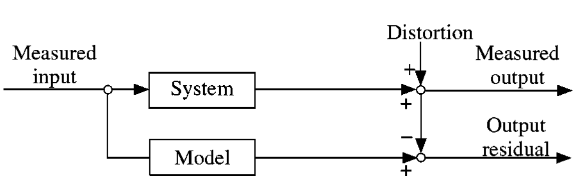

Figure 2. Output residual.

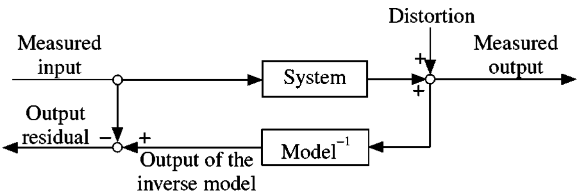

Figure 3. Input residual.

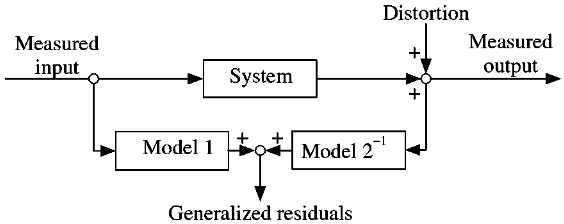

Figure 4. Generalized residual.

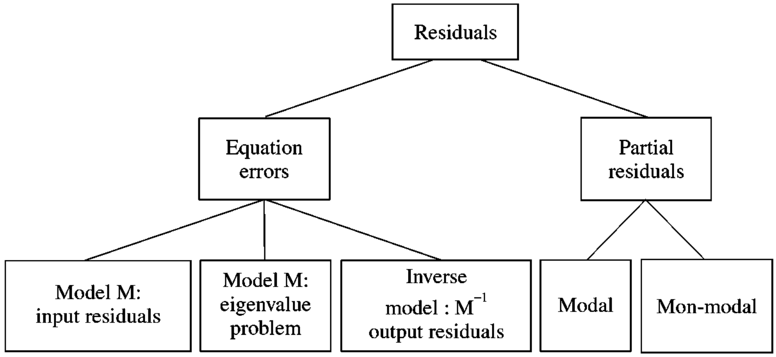

Figure 5. Classification of residuals.

only the information of local parameter modifications. Additionally, the equation error can be introduced, which in the case of dynamic response problems is equal to the input (force) residual. In the case of the eigenvalue problem, we can distinguish between the equation error (zero force residual) and partial residuals, such as the differences between the eigenfrequencies of the damaged and undamaged system, or such as the eigenfrequency residuals and the corresponding differences between the eigenvectors and, in addition, between the generalized masses. The fulfilment of the generalized orthogonality properties, etc., can also be taken into account as a possible symptom. Figure 5 gives an overview of possible residuals. The non-modal residuals are important in engineering, because the engineer is mostly interested in, for example, stress distributions and acoustic levels. One obtains local residuals, for example, by taking the elementwise differences of the stiffness matrices at lifetimes θ_i and θ_{i+1}. Flexibility and inertia matrices can be taken instead of stiffness matrices.

Of course, global residuals can be combined with local ones. One can also choose special indicators, like the MAC,† [5] etc. Local symptoms with their local information contents will serve simultaneously for fault detection and localization. The residuals of the modal vector components taken directly or transformed [6] can also be suitable symptoms for fault detection.

As already stated in the Introduction, equation (3), the symptom operator **T** transforms the measured t-dependent signal $\mathbf{y}_\theta(t)$ into the symptom space (see Figure 1), which means it is only θ-dependent: $\mathbf{s}(\theta)$.

†Modal assurance criterion (MAC): it is the cosine of the angle between, for example, a measured and a calculated eigenvector.

2.2. SYMPTOMS OF NON-LINEAR SYSTEMS DEPENDENT ON θ

Non-linear behaviour can result from modifications to a linear system. The symptoms discussed in the previous section can also be applied to non-linear systems. However, non-linear effects differ fundamentally from linear behaviour. Linearized models cannot describe the non-linear behaviour. Consequently, when compared with the corresponding quantities of a linear model, symptoms including non-linear effects lead to deviations from the linear behaviour, so they can serve for detection as violated assumptions of linearity. Therefore, as already stated, the detection of non-linear behaviour can be performed indirectly by tests, if one assumes a linear model and looks for violated assumptions. Here artificial harmonic excitation with different levels of response amplitudes is most effective. The principles and characteristics which can be chosen are

- superposition,
- reciprocity,
- parameter independence of sample (initial conditions, damping),
- Nyquist plots (their geometry).

The distortions are sometimes informative (pattern recognition) when one looks, for example, at distorted Nyquist plots. Non-linearities affect the dynamic response most in the resonance neighbourhood.

Direct methods for detection are based on such symptoms as

- signals due to special excitation and filtering,
- indicator functions (e.g., SIG-function),
- Hilbert transforms,
- high order correlation functions,
- multispectral density functions,
- dispersion functions,
- histogram measures (pattern classification),
- NARMA, NARMAX models,
- polynomial fits.

Details of the itemized methods and references can be found in reference [1, section 2.2.3]. Some problems should be mentioned in this context.

- It is hard to distinguish between the bias in the process model and in the noise model.
- Does the detected non-linear behaviour require and also permit a non-linear model?
- Detection in general does not include structure (of the model) identification.
- The choice of inputs and outputs must reveal the structure patterns (complete set of information with respect to the initial conditions: sufficiently large deflection amplitudes without system destruction).

It is noted that the disadvantage of the Fourier transform is the lack of a localization property: local time modification of a signal changes the transformed signal everywhere. New developments in the description (detection) of linear and non-linear behaviour thus apply wavelets [7] and the Wigner distribution [8], which do not have this disadvantage.

2.3. SUMMARY

The choice of symptoms is essential for monitoring and diagnostics. It depends on the dynamic properties of the symptom under monitoring, *a priori* knowledge and simulations (weak point analysis) of faults/damage to be expected. Sensitivity here means both with respect to the possible faults/damage and to the lifetime. The measured quantities are dependent on t and θ, and the application of a suitable transformation generates the symptoms which are only lifetime-dependent.

3. CHOICE OF THE MEASURING MATRIX

Initially, modifications in the system matrix $\mathbf{A}$, denoted by $\mathbf{A} + \Delta\mathbf{A}$, are assumed to be contained in the state vector: $\mathbf{x}(t) + \Delta\mathbf{x}(t)$. Here the index θ for designating the lifetime-dependency is suppressed without introducing any confusion. Assuming no changes in the external forces applied, the resulting model is (see equation (1))

$$\dot{\mathbf{x}}(t) + \Delta\dot{\mathbf{x}}(t) = (\mathbf{A} + \Delta\mathbf{A})[\mathbf{x}(t) + \Delta\mathbf{x}(t)] + \mathbf{Bf}(t). \tag{5}$$

The measuring equation is as given in equation (2). Suppressing the time argument, it follows from equation (5) with equation (1) that

$$\Delta\dot{\mathbf{x}} = \Delta\mathbf{A}(\mathbf{x} + \Delta\mathbf{x}) + \mathbf{A}\Delta\mathbf{x}. \tag{6}$$

With the assumption $\Delta\dot{\mathbf{x}}(t) = 0$ (which has to be proven in every case), the above equation leads to

$$\begin{aligned} \Delta\mathbf{x} &\approx -(\mathbf{A} + \Delta\mathbf{A})^{-1}\Delta\mathbf{A}\mathbf{x} \\ &= -(\mathbf{I} + \mathbf{A}^{-1}\Delta\mathbf{A})^{-1}\mathbf{A}^{-1}\Delta\mathbf{A}\mathbf{x} \\ &\approx -(\mathbf{I} - \mathbf{A}^{-1}\Delta\mathbf{A})\mathbf{A}^{-1}\Delta\mathbf{A}\mathbf{x}, \end{aligned} \tag{7}$$

with $\mathbf{I}$ the unit matrix, and presuming that $\|\mathbf{A}^{-1}\Delta\mathbf{A}\| \ll 1$.

The measuring equation with the modification $\Delta\mathbf{x}$ results in

$$\Delta\mathbf{y} = \mathbf{H}\Delta\mathbf{x}. \tag{8}$$

By inserting equation (7) into the above equation it follows due to linearization of the modification that

$$\begin{aligned} \Delta\mathbf{y} &= -\mathbf{H}(\mathbf{A} + \Delta\mathbf{A})^{-1}\Delta\mathbf{A}\mathbf{x} \\ &\approx -\mathbf{H}\mathbf{A}^{-1}\Delta\mathbf{A}\mathbf{x}. \end{aligned} \tag{9}$$

The question now has to be answered: how does $\mathbf{H}$ have to be chosen so that the measured modified dynamic response $\Delta\mathbf{y}$ contains the modified state vector $\Delta\mathbf{x}$ (describing the system modification $\Delta\mathbf{A}$) to a non-negligible extent?

3.1. INVESTIGATION BY SENSITIVITY

This is defined by the Euclidian norm

$$\sigma(\Delta x_k) := \left\| \frac{\partial \Delta \mathbf{y}}{\partial \Delta x_k} \right\|^2, \quad k = 1, \ldots, 2n.$$

Equation (8) will be written with the matrix elements h_{ik} of $\mathbf{H}$:

$$\Delta \mathbf{y} = \sum_{k=1}^{2n} h_{ik} \Delta x_k.$$

It follows that

$$\frac{\partial y_i}{\partial \Delta x_k} = h_{ik},$$

$$\left\| \frac{\partial \Delta \mathbf{y}}{\partial \Delta x_k} \right\|^2 = \sum_{k=1}^{2n} h_{ik}^2.$$

This clearly means that all components of $\Delta \mathbf{x}$ unequal to zero should be taken into account by setting the corresponding h_{ik} equal to 1, and this complies directly with equation (8).

3.2. DEPENDENT ON ONE ELEMENT OF THE DYNAMIC STIFFNESS MATRIX

The classic second order formulation of the system is taken in the image domain using the Laplace transform with s the Laplacian variable,

$$(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K})\mathbf{U}(s) = \mathbf{NP}(s), \quad \text{with the dynamic stiffness matrix} \quad \mathbf{S}(s) := s^2\mathbf{M} + s\mathbf{C} + \mathbf{K} \tag{10}$$

of order n, and where the rectangular input matrix $\mathbf{N}$ specifies the components with non-zero external forces assembled in the vector $\mathbf{P} \cdot \mathbf{U}(s)$ and $\mathbf{P}(s)$ are the Laplace transforms of the displacement $\mathbf{u}(t)$ and of the external force $\mathbf{p}(t)$ respectively. $\mathbf{I}$ is the unity matrix. Let $\mathbf{e}_k$ be the unit vector of a proper dimension with the 1 in the kth component, and let Δ be a modification operator applied to $\mathbf{S}(s)$; this will result in a modification of the (k, k)th element $S_{kk}(s)$ of $\mathbf{S}(s)$ only:

$$(\mathbf{I} + \mathbf{e}_k\mathbf{e}_k'\Delta)\mathbf{S}(s) = \mathbf{S}(s) + \mathbf{e}_k\mathbf{e}_k'\Delta\mathbf{S}(s) = \mathbf{S}(s) + \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \Delta S_{kk}(s) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{11}$$

The modification of the dynamic stiffness matrix will lead to a modification of the response vector, $\mathbf{U}(s) + \Delta\mathbf{U}(s)$, assuming an unchanged excitation:

$$(\mathbf{I} + \mathbf{e}_k\mathbf{e}_k'\Delta)\mathbf{S}(s)[\mathbf{U}(s) + \Delta\mathbf{U}(s)] = \mathbf{N}\mathbf{P}(s). \tag{12}$$

With the unmodified model it follows that

$$[\mathbf{S}(s) + \mathbf{e}_k\mathbf{e}_k'\Delta\mathbf{S}(s)]\Delta\mathbf{U}(s) = -\mathbf{e}_k\mathbf{e}_k'\Delta\mathbf{S}(s)\mathbf{U}(s) = -\begin{Bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \Delta S_{kk}(s)U_k(s) \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix}. \tag{13}$$

The right-hand side of this equation shows that only the kth component is unequal to zero. The modification of the dynamic response, when suppressing the argument s and second order terms in the modification is

$$\begin{aligned} \Delta\mathbf{U} &= -(\mathbf{S} + \mathbf{e}_k\mathbf{e}_k'\Delta\mathbf{S})^{-1}\mathbf{e}_k\mathbf{e}_k'\Delta\mathbf{S}\mathbf{U} \\ &= -[\mathbf{S}(\mathbf{I} + \mathbf{S}^{-1}\mathbf{e}_k\mathbf{e}_k'\Delta\mathbf{S})]^{-1}\mathbf{e}_k\mathbf{e}_k'\Delta\mathbf{S}\mathbf{U} \\ &\approx -(\mathbf{I} - \mathbf{S}^{-1}\mathbf{e}_k\mathbf{e}_k'\Delta\mathbf{S})\mathbf{S}^{-1}\mathbf{e}_k\mathbf{e}_k'\Delta\mathbf{S}\mathbf{U} \\ &\approx -\mathbf{S}^{-1}\mathbf{e}_k\mathbf{e}_k'\Delta\mathbf{S}\mathbf{U} \\ &= -\mathbf{S}^{-1}\begin{Bmatrix} 0 \\ \cdot \\ 0 \\ \Delta S_{kk}U_k \\ 0 \\ \cdot \\ 0 \end{Bmatrix} \end{aligned} \tag{14}$$

with

$$\mathbf{H} = [\mathbf{H}_0, \mathbf{O}_n], \tag{15}$$

where only displacements are measured, and with

$$\mathbf{Z}(s) + \Delta\mathbf{Z}(s) := \mathbf{H}_0[\mathbf{U}(s) + \Delta\mathbf{U}(s)]: \tag{16}$$

the measuring equation gives

$$\Delta \mathbf{Z}(s) = \mathbf{H}_0 \Delta \mathbf{U}(s) = \mathbf{H}_0 \mathbf{S}^{-1}(s) \begin{Bmatrix} 0 \\ \cdot \\ 0 \\ \Delta S_{kk}(s) U_k(s) \\ 0 \\ \cdot \\ 0 \end{Bmatrix} = \Delta S_{kk}(s) U_k(s) \mathbf{H}_0 \begin{Bmatrix} S_{1k}^{-1}(s) \\ S_{2k}^{-1}(s) \\ \cdot \\ \cdot \\ \cdot \\ S_{nk}^{-1}(s) \end{Bmatrix} \tag{17}$$

with $S_{ik}^{-1}(s)$ the elements of $\mathbf{S}^{-1}(s) =: (S_{ik}^{-1}(s))$. The matrix $\mathbf{H}$ should be chosen so that the maximum magnitude of the element $S_{ik}^{-1}(s)\Delta S_{kk}(s)U_k(s)$ is contained.

3.3. RELATION WITH THE OBSERVABILITY

The observability of usual mechanical systems in their discrete formulations concerns the information within the measurements $\mathbf{y}(t)$ (suppressing the index θ). "Completely observable" means that the transfer matrix of the system,

$$\mathscr{H}(s) := \mathbf{H}(s\mathbf{I}_{2n} - \mathbf{A})^{-1}\mathbf{B}, \tag{18}$$

can be completely constructed with the chosen measuring matrix $\mathbf{H}$. The symbol s is, as usual, the Laplace variable, and $\mathbf{\Lambda}$ is the diagonal matrix of the eigenvalues. With the spectral decomposition of the system matrix,

$$\mathbf{A} = \mathscr{Q}\mathbf{\Lambda}\mathscr{Q}^{-1}, \tag{19}$$

and with

$$\mathbf{B} = \mathscr{Q}\mathscr{Q}' \begin{bmatrix} \mathbf{N} \\ \mathbf{0}_n \end{bmatrix}, \tag{20}$$

where the modal matrix $\mathscr{Q}$ consists of

$$\mathscr{Q} = \begin{bmatrix} \mathbf{Q} \\ \mathbf{Q}\mathbf{\Lambda} \end{bmatrix} \tag{21}$$

with the $(n, 2n)$-modal matrix

$$\mathbf{Q} = [\mathbf{Q}_0, \bar{\mathbf{Q}}_0]$$

and the modal matrix $\mathbf{Q}_0$ and its conjugate complex $\bar{\mathbf{Q}}_0$ of the corresponding second order differential equations of the damped system. The transfer matrix follows with these definitions:

$$\begin{aligned} \mathscr{H}(s) &= \mathbf{H}(s\mathbf{I}_{2n} - \mathbf{A})^{-1}\mathbf{B} \\ &= \mathbf{H}(s\mathbf{I}_{2n} - \mathscr{Q}\mathbf{\Lambda}\mathscr{Q}^{-1})^{-1}\mathscr{Q}\mathscr{Q}' \begin{bmatrix} \mathbf{N} \\ \mathbf{0} \end{bmatrix} \\ &= \mathbf{H}\mathscr{Q}(s\mathbf{I}_{2n} - \mathbf{\Lambda})^{-1}\mathscr{Q}' \begin{bmatrix} \mathbf{N} \\ \mathbf{0} \end{bmatrix}. \end{aligned} \tag{22}$$

If the following is chosen (see equation (15))

$$\mathbf{H} = [\mathbf{H}_1, \mathbf{0}],$$

for example, if only displacements are measured, $\mathbf{H}_1 = \mathbf{H}_0$, then the transfer matrix can be calculated through

$$\mathscr{H}(s) = \mathbf{H}_1\mathbf{Q}(s\mathbf{I}_n - \mathbf{\Lambda})^{-1}\mathbf{Q}'. \tag{23}$$

Complete observability is given if no modal vector $\mathbf{q}_i$, $i = 1(1)n$, exists with

$$\mathbf{H}_1\mathbf{q}_i = \mathbf{0}. \tag{24}$$

This requirement is to be seen in the context of modifications to systems.

For additional clarity, $\mathscr{H}(s)$ can be decomposed by the modal vectors $\mathbf{q}_i$. If equation (24) is fulfilled in equation (23), then the information about $\mathbf{q}_i$ is missing, and the transfer matrix cannot be determined completely.

3.4. SUMMARY

The elements of the $(m, 2n)$ measuring matrix

$$\mathbf{H} = [\mathbf{H}_1, \mathbf{0}] =: [h_{ik}], \quad \begin{cases} i = 1(1)m, \\ k = 1(1)2n \end{cases}$$

have to be chosen so that

- for general modifications represented as modifications of Δx_k they are contained in $\mathbf{\Delta} y = \mathbf{H}\Delta\mathbf{x}$,
- in the case of a modification $\Delta S_{kk}(s)$ the maximum magnitude of the element $S_{ik}^{-1}(s)\Delta S_{kk}(s)U_k(s)$ is contained in the measurements, and
- with the modal vectors $\mathbf{q}_{i,}\, i = 1(1)_n$ of the eigenvalue problem corresponding to equation (10) no $\mathbf{q}_i$ exists, fulfilling equation (24).

4. CHOICE OF TEST SIGNALS

If the operating/service forces of the system, which cause realistic stresses, cannot be taken for monitoring, then test forces have to be applied in such a way that existing faults will not increase. Specially favoured test forces are impulse, harmonic, periodic and pseudo-random forces; broadband random forces are seldom applied [3]. The choice generally depends on the type of fault/damage expected, on its frequency content, on possibilities concerning the mounting of forcing devices, and the available space for mounting. In brief, excitation with artificial signals requires an optimum test design. This task also requires the choice of $\mathbf{B}_\theta\mathbf{f}(t)$ so that the effect of a fault/damage will be excited directly and can therefore be measured. However, the force has to be restricted in its effect with respect to the second requirement of not increasing the fault/damage.

The first requirement is known as controllability, and it is closely connected with the observability of the previous section. For classic mechanical systems they are designated as completely controllable if no eigenvector exists with

$$\mathbf{q}'_i\mathbf{N} = 0, \quad i = 1(1)n.$$

Assuming that the fault/damage is detected by monitoring, the next steps consist of the localization and investigation of the fault effects with respect to the use/service of the system, including the permissible stresses and, therefore, forces. If a mathematical model of the system is available, this topic requires adjustment to the most recent state. This adjustment is outside the scope of these considerations, but it is extensively described in reference [1]. Here the relationships between the measuring matrix, the input matrix and the observability matrix are of interest.

If no mathematical model of the system or its part with the detected modification is available, reliability investigations [9] have to be performed based on the symptoms due to the test forces.

5. PROPERTIES OF THE OBSERVATION MATRIX

The requirements for choosing symptoms have been discussed in the previous sections, including those for the input matrix and the measurement matrix. The observation matrix defined in equation (4) can now be made. It is, in general, a huge matrix which has to be handled in such a way that maximum information about the fault/damage is obtained. In order to answer the questions posed in the Introduction, first handling using the SVD currently recommended is discussed. The problems with respect to handling huge matrices with measured data, redundant information, and lifetime spacings will then be outlined.

5.1. THE SINGULAR VALUE DECOMPOSITION

In condition monitoring, the symptoms assembled in the observation matrix (4) are essential. The SVD applied to the observation matrix is discussed with some examples in reference [2]. In order to understand and extend these results the following must first be repeated.

5.1.1. *Singular values and singular vectors as characteristics of the system life*

It must be stressed that the observation matrix (see equation (4)) $\mathcal{O}$ is a (p, r)-matrix: $\mathcal{O}_{pr}$. Each of its rows is the observation vector with r symptom values as its components for each measured lifetime. The symptom vectors with components dependent on the lifetime are contained in the observation matrix as column vectors for p lifetimes.

The SVD of equation (4) reads [10] as

$$\mathbf{U}'_{pp}\mathcal{O}_{pr}\mathbf{V}_{rr} = \mathbf{\Sigma}_{pr} \tag{25}$$

with $\mathbf{U}_{pp}$ the orthogonal matrix of the left singular vectors $\mathbf{u}_i, i = 1(1)p$, and $\mathbf{V}_{rr}$ the orthogonal matrix of the right singular vectors $\mathbf{v}_v, v = 1(1)r$, and the "subdiagonal" matrix $\mathbf{\Sigma}_{pr}$ of the singular values (SV) $\sigma_i \neq 0$, if n is the rank of $\mathcal{O}_{pr}$, $n \leqslant l = \min(p, r)$. The observation matrix (4) can therefore be decomposed as

$$\mathcal{O}_{pr} = \mathbf{U}_n \mathbf{\Sigma}_n \mathbf{V}_n'$$
$$= \Sigma_{i=1}^{n} \sigma_i \cdot (\mathbf{u}_i \mathbf{v}_i') =: \Sigma_{i=1}^{n} (\mathcal{O}_{pr})_i \text{ with } \operatorname{rank}(\mathcal{O}_{pr}) = n, \tag{26}$$

and with the SVs

$$\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n > \sigma_{n+1} = \cdots = \sigma_l = 0.$$

The second line of the observation matrix (26) is a fault-orientated decomposition. As can be seen from equation (26), if one fault is characterized by the index J within the decomposition ($\rightsquigarrow$ discriminant [1]), then the observation matrix contains this information in the superposition of all SVs and corresponding dyadic products of the singular vectors.

Assuming a fault/damage characterized by the index J of the SVs, it follows [10] that

$$\mathcal{O}_{pr} \mathbf{v}_J = \boldsymbol{\sigma}_J \mathbf{u}_J \quad \text{and} \quad \mathcal{O}' = \mathcal{O}'_{rp} \mathbf{u}_J = \boldsymbol{\sigma}_J \mathbf{v}_J,\ J \in \{1, \ldots, n\}. \tag{27}$$

Before these equations are interpreted, the left-hand side of the first equation of (27) is fully written as:

$$\mathcal{O}_{pr} \mathbf{v}_J = \begin{Bmatrix} s_{\theta 1,1} v_{J,1} + s_{\theta 1,2} v_{J,2} + \cdots + s_{\theta 1,J} v_{J,j} + \cdots + s_{\theta 1,r} v_{J,r} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ s_{\theta p,1} v_{J,1} + s_{\theta p,2} v_{J,2} + \cdots + s_{\theta p,J} v_{J,j} + \cdots + s_{\theta p,r} v_{J,r} \end{Bmatrix} = \boldsymbol{\sigma}_J \mathbf{u}_J. \tag{28}$$

The interpretation of these equations is that all the information about the fault/damage is contained in the observation matrix with SVs unequal to zero. In more detail:

- Equation (28) shows a weighting of the row elements of $\mathcal{O}_{pr}$ through the components $v_{J,v}$ of the vector $\mathbf{v}_J$, which means a weighting of the symptom values and their summation for each lifetime $\theta_i = const.$, $i = 1(1)p$ (including $v = J$). The vector $\mathbf{u}_J$ multiplied by the corresponding SV σ_J is representative of the weighted sum of the symptoms taken at $\theta_i =$ constants the components of $\mathbf{u}_J$ characterize the modification of the system dependent on θ_i.
- The second equation in (27) represents a weighting of the columns of $\mathcal{O}_{pr}$, which means a weighted (weighted by the components of $\mathbf{u}_J$) summation of the values of one symptom over all lifetimes $\theta_i, i = 1(1)p$. The singular vector $\mathbf{v}_J$ (multiplied by the SV σ_J) is representative of the lifetime-dependent columns (each column designating one symptom) of the observation matrix. It gives the sensitivity of the symptoms with respect to system modifications, and it will help to check the choice of the symptoms chosen.

Consequently, the singular vectors and the SVs can be taken separately for assessment of the modifications of the system dependent on the lifetime.

If the eigenvalue problems of the corresponding matrices are considered,

$$\mathcal{O}'_{rp} \mathcal{O}_{pr} \mathbf{x} = \lambda \mathbf{x},$$
$$\mathcal{O}_{pr} \mathcal{O}'_{rp} \mathbf{y} = \lambda \mathbf{y}, \tag{29}$$

$$\mathbf{W}_1 := \mathcal{O}'_{rp} \mathcal{O}_{pr}, \quad \mathbf{W}_2 := \mathcal{O}_{pr} \mathcal{O}'_{rp}. \tag{30}$$

Starting with relations (27) with running indices and multiplication the first one with $\mathcal{O}'_{rp}$ and the second relation with $\mathcal{O}_{pr}$, then the resulting right-hand sides can be substituted by the relations themselves, obtaining

$$\begin{aligned} \mathbf{W}_1\mathbf{v}_\nu &= \mathcal{O}'_{rp}\mathcal{O}_{pr}\mathbf{v}_\nu = \sigma_\nu^2\mathbf{v}_\nu,\ \nu = 1(1)r, \\ \mathbf{W}_2\mathbf{u}_i &= \mathcal{O}_{pr}\mathcal{O}'_{rp}\mathbf{u}_i = \sigma_i^2\mathbf{u}_i,\ i = 1(1)p. \end{aligned} \tag{31}$$

For the solutions i comparison of equation (31) with equation (29) yields

$$\lambda_i = \sigma_i^2,\ \ \sigma_i > 0,\ \ x_i = \mathbf{v}_i,\ \ y_i = \mathbf{u}_i,\ \ i = 1(1)n. \tag{32}$$

The following can be stated.

- The SVs and the corresponding singular vectors can, as is known, be computed with the eigenvalue problems given above.
- The matrix $\mathbf{W}_2$ can be decomposed by the vectors $\mathbf{u}_i$, which means it is a relationship between the symptoms and the vectors $\mathbf{u}_i$. In more detail, with the use of equation (4), $\mathbf{W}_2$ can be written in the form

$$\mathbf{W}_2 = \begin{bmatrix} s'_{\theta 1}s_{\theta 1} & s'_{\theta 1}s_{\theta 2} & \cdots & s'_{\theta 1}s_{\theta p} \\ s'_{\theta 2}s_{\theta 1} & s'_{\theta 2}s_{\theta 2} & \cdots & s'_{\theta 2}s_{\theta p} \\ \cdots & \cdots & \cdots & \cdots \\ s'_{\theta p}s_{\theta 1} & s'_{\theta p}s_{\theta 2} & \cdots & s'_{\theta p}s_{\theta p} \end{bmatrix} = \mathbf{W}'_2. \tag{33}$$

The matrix $\mathbf{W}_2$ is similar to the correlation/covariance matrix in stochastics: it is equivalent if the mean values of the rows are subtracted from their elements and if the symptoms can be interpreted as random variables. The elements of $\mathbf{W}_2$ are also similar to the modal assurance criterion (MAC) [5][‡] of the row vectors of $\mathcal{O}_{pr}$ if $\mathbf{W}_2$ is divided by (elementwise) $(diag\mathbf{W}_2 \cdot diag\mathbf{W}'_2)$ as normalization. Through its values, it characterizes the orthogonality between the lifetime-dependent symptom vectors. Mathematically, it can be taken for orthonormalization of the vectors. Additionally, it tells us about the quality of the choice of the vectors with respect to the lifetime modifications: large values of the out of main-diagonal elements compared with the main-diagonal values indicate linear dependency, which means redundant information with respect to θ.

- The non-zero SVs (see equation (25)) can be used as a measure of fault/damage intensity, as discussed in reference [11]. Starting with the eigenvector $\mathbf{u}_J$ corresponding to the largest eigenvalue $\sigma_J = \max_i(\sigma_i)$, and choosing the eigenvectors corresponding to the next smaller eigenvalues will provide a good assessment of the fault/damage intensity and priority: if rows are equal, which is the case for an unmodified system with respect to the measured symptoms at corresponding lifetimes, then they produce SVs equal to zero (in addition to the non-zero SV of the observation matrix of rank 1 corresponding to the healthy system), providing that no rows are proportional which also yield SVs equal to zero. Consequently, SVs unequal to zero indicate faults/damage, and, if ordered by their

[‡]It is the cosine of the angle between the corresponding vectors.

magnitudes and, if considered the SV of the observation matrix in the healthy state, they are ranking indices.

- For detection of the system modifications and for assessment of their intensities, one can take the determinant value of the matrix of the SVs in its "economy" size [12], which is the diagonal matrix of the non-zero SVs: linearly dependent rows of the observation matrix (redundant information: nothing has changed in this lifetime step) which are cancelled, and additional linearly independent rows will increase its value (see next paragraph).
- The fault-orientated decomposition of the observation matrix (see equation (26)) permits the calculation of another fault discriminant, which is similar to equation (27) but with much larger dynamics. According to reference [2] this discriminant is defined for the fault J as

$$\mathbf{FD}_J := \sigma_J \sum_{i=1}^{p} \left(\sum_{v=1}^{r} u_{J,i} v_{J,v} \right).$$

where the components of the Jth singular vectors are $\mathbf{u}_J =:\{u_{J,i}\}$ and $\mathbf{v}_J =:\{v_{J,v}\}$.

5.1.2. *Noise considerations*

Starting with equation (26) but with complex elements,

$$\mathcal{O} \equiv \mathcal{O}_{pr} = \mathbf{U\Sigma V}^* \Rightarrow \mathbf{W}_2 = \mathcal{O}\mathcal{O}^* = \mathbf{U\Sigma}^2\mathbf{U}^*. \tag{34}$$

The SVs of $\mathcal{O}$ are the square roots of the eigenvalues of $\mathbf{W}_2$, while the left singular vectors are the eigenvectors of $\mathbf{W}_2$, as already stated above. Now a perturbed matrix $\tilde{\mathcal{O}} := \mathcal{O} + \mathbf{N}$ is assumed. If $\mathbf{N}$ describes uncorrelated white noise with zero mean, the variance $\mathrm{E}\{\mathbf{NN}^*\}$ is asymptotically (for $r \to \infty$, r according to the number of symptoms) given by

$$\mathrm{E}\{\mathbf{NN}^*/r\} = \sigma^2\mathbf{I},$$

σ^2 the variance of the noise process. This means that the noise is uniformly spread in the observation space. It follows from the above that

$$\mathrm{E}\{\tilde{\mathcal{O}}\tilde{\mathcal{O}}^*/r\} = \mathrm{E}\{\mathcal{O}\mathcal{O}^*/r\} + \sigma^2\mathbf{I}.$$

For large r the SVD is [12]

$$\tilde{\mathcal{O}} \approx \mathbf{U}(\mathbf{\Sigma}^2 + r\sigma^2\mathbf{I})^{1/2}\tilde{\mathbf{V}}^*$$

for some disturbed unitary matrix $\tilde{\mathbf{V}}$. It follows that in a first approximation the left singular vectors remain the same, but the SV matrix increases with $\sigma\sqrt{r}$. $\tilde{\mathcal{O}}$ is now of full rank and all SVs are unequal to zero.

More generally, with regard to $\mathbf{N}$ it can be stated that the SVs increase by the order of $\|\mathbf{N}\|$, that is the largest SV of $\mathbf{N}$. All the singular vectors are also perturbed by the order of $\|\mathbf{N}\|$. The effect on the singular vectors can be much larger if the corresponding SVs are close to each other [13].

Finally, an example will illustrate the outstanding role of the smallest SV and the corresponding left singular vector [12]. Two columns (symptom vectors) are assumed which are nearly aligned (nearly common directions). Then the first singular vector u_1 is in the direction of the sum of the symptom vectors ς_1 and ς_2. The SV is $\sigma_1 = \|\varsigma_1 + \varsigma_2\|/\sqrt{2}$.

Additionally, the second singular vector depends on the difference between the observation vectors, with the SV $\sigma_2 = \|\varsigma_2 - \varsigma_1\|/\sqrt{2}$. If the angle between both symptom vectors becomes smaller, then σ_2 will become smaller and the observation matrix tends to a singular matrix. Consequently, u_2 shows the most sensitive direction for perturbations on the symptom vectors. The smallest SV is affected by a perturbation according to

$$\sigma_2^2 \leqslant \tilde{\sigma}_2^2 \leqslant \sigma_2^2 + \|\mathbf{N}\|^2.$$

5.1.3. *Perturbations seen as effects due to faults/damage*

If the perturbations are seen as effects of faults/damage which increase with the lifetimes θ_i, these effects can be used for detection (and localization in the case of discriminants) of faults/damage. As can be derived from the example presented above, the smallest SV depends strongly on the length of the observation vectors. This effect is more severe than the effect of an increasing number of observation vectors. Additionally, the singular vector u_2 corresponding to the smallest SV has the most sensitive direction with respect to the perturbation of the columns of $\mathcal{O}$.

The conclusions drawn from the perturbation investigations interpreted as effects of faults/damage are crucial. The largest SVs characterize the faults/damage; in brief, the system modifications in the sense of ranking indices. However, the small SVs are most sensitive with respect to lifetime-dependent system modifications. The right singular vectors are more sensitive to changes of the observation matrix than the left singular vectors. And, as stated above, the length of the observation vector is more serious than an increasing number of observation vectors.

5.1.4. *Example*

A cantilever will serve for illustration. It can be interpreted with other dimensions, for instance as a platform leg which suffers from corrosion at the sea surface. The cantilever length is 1·0 m, its cross-section $0{\cdot}001190639 \times 0{\cdot}001190639\ \mathrm{m}^2$ with Young's modulus of $E = 2{\cdot}1 \times 10^{11}\ \mathrm{N/m}^2$ and density $\rho = 7850\ \mathrm{kg/m}^3$. The cantilever is clamped at $x = 0$ (index 0), index 1 designates a point at $x = 0{\cdot}3$ m, index 2 designates the point $x = 0{\cdot}4$ m and index 3 characterizes $x = l = 1$ m. It is assumed that between $x = 0{\cdot}49$ and $0{\cdot}50$ m the beam's stiffness is reduced by 5% in each lifetime step $\theta_i, i = 1(1)10$. This means that at time θ_{10} in the interval described the stiffness is reduced by 50%. The lifetime θ_0 designates the time without stiffness reduction. Table 1 gives the chosen (simulated) measurements which will

TABLE 1

Symptom measurements

	$y(\theta_i)_{max,1}$	$y(\theta_i)_{m,2}$	$y(\theta_i)_{eff,3}$	$M(\theta_i)_{eff,0}$	$f(\theta_i)_1$	$f(\theta_i)_2$
θ_0	0·00389558	0·004216798	0·02029036	0·002508829	0·994798	6·2343
θ_1	0·00389569	0·004217112	0·02029867	0·002508901	0·994730	6·2326
θ_2	0·00389582	0·004217469	0·02030787	0·002508986	0·994648	6·2306
θ_3	0·00389597	0·004217877	0·02031811	0·002509087	0·994552	6·2282
θ_4	0·00389615	0·004218344	0·02032964	0·002509207	0·994439	6·2254
θ_5	0·00389636	0·004218881	0·02034267	0·002509349	0·994304	6·2220
θ_6	0·00389661	0·004219501	0·02035749	0·002509519	0·994143	6·2180
θ_7	0·00389691	0·004220222	0·02037459	0·002509722	0·993949	6·2132
θ_8	0·00389730	0·004221061	0·02039446	0·002509968	0·993715	6·2073
θ_9	0·00389778	0·004222051	0·02041791	0·002510268	0·993429	6·2002
θ_{10}	0·00389841	0·004223223	0·02044598	0·002510636	0·993074	6·1914

serve as symptoms: they are sometimes insensitive with respect to the system modification, but they are chosen intentionally for discussion. The loading was a sinusoidal impulse excitation with 1 Hz and an amplitude of 0.001 N. The observed symptoms are: $y_{max,1}$, the maximum deflection at point 1, $y_{m,2}$, the mean of the absolute values of the deflection, $y_{eff,3}$, the r.m.s.-value of the deflection at point 3, $M_{eff,0}$, the r.m.s.-value of the bending moment at the clamped end and f_1, f_2 are the first two eigenfrequencies in Hz.

The **T**-operation (see equation (3)) has already been performed here. In order to normalize the measurements and to make them dimensionless, the following transforms are introduced:

$$s_{\theta i,1} := \frac{y(\theta_i)_{max,1} - 0{\cdot}00389\ [\mathrm{m}]}{1\ [\mathrm{m}]}\, 10^5,$$

$$s_{\theta i,2} := \frac{y(\theta_i)_{m,2} - 0{\cdot}00421\ [\mathrm{m}]}{1\ [\mathrm{m}]}\, 10^5,$$

$$s_{\theta i,3} := \frac{y(\theta_i)_{eff,3} - 0{\cdot}0202\ [\mathrm{m}]}{1\ [\mathrm{m}]}\, 10^4,$$

$$s_{\theta i,4} := \frac{M(\theta_i)_{eff,0} - 0{\cdot}002508\ [\mathrm{N\,m}]}{1\ [\mathrm{N\,m}]}\, 10^6,$$

$$s_{\theta i,5} := \frac{f(\theta_i)_1 - 0{\cdot}993\ [\mathrm{Hz}]}{1\ [\mathrm{Hz}]}\, 10^3,$$

$$s_{\theta i,6} := \frac{f(\theta_i)_2 - 6{\cdot}20\ [\mathrm{Hz}]}{1\ [\mathrm{Hz}]}\, 10^2.$$

The symptom observation matrix then follows to

$$\mathcal{O}_{11,6} = \begin{bmatrix} 0{\cdot}558 & 0{\cdot}680 & 0{\cdot}904 & 0{\cdot}829 & 1{\cdot}798 & 3{\cdot}43 \\ 0{\cdot}569 & 0{\cdot}711 & 0{\cdot}987 & 0{\cdot}901 & 1{\cdot}730 & 3{\cdot}26 \\ 0{\cdot}582 & 0{\cdot}747 & 1{\cdot}079 & 0{\cdot}986 & 1{\cdot}648 & 3{\cdot}06 \\ 0{\cdot}597 & 0{\cdot}788 & 1{\cdot}181 & 1{\cdot}087 & 1{\cdot}552 & 2{\cdot}82 \\ 0{\cdot}615 & 0{\cdot}834 & 1{\cdot}296 & 1{\cdot}207 & 1{\cdot}439 & 2{\cdot}54 \\ 0{\cdot}636 & 0{\cdot}888 & 1{\cdot}427 & 1{\cdot}349 & 1{\cdot}304 & 2{\cdot}20 \\ 0{\cdot}661 & 0{\cdot}950 & 1{\cdot}575 & 1{\cdot}519 & 1{\cdot}143 & 1{\cdot}80 \\ 0{\cdot}691 & 1{\cdot}022 & 1{\cdot}746 & 1{\cdot}722 & 0{\cdot}949 & 1{\cdot}32 \\ 0{\cdot}730 & 1{\cdot}106 & 1{\cdot}945 & 1{\cdot}968 & 0{\cdot}715 & 0{\cdot}73 \\ 0{\cdot}778 & 1{\cdot}205 & 2{\cdot}179 & 2{\cdot}268 & 0{\cdot}429 & 0{\cdot}02 \\ 0{\cdot}841 & 1{\cdot}322 & 2{\cdot}460 & 2{\cdot}636 & 0{\cdot}074 & -0{\cdot}01 \end{bmatrix}.$$

The columns of the symptom observation matrix (symptom vectors) are illustrated in Figure 6.

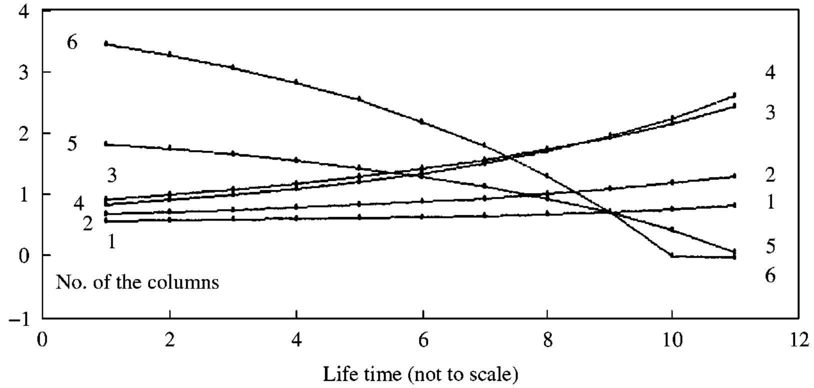

Figure 6. The observation vectors (columns of the symptom observation matrix).

The SVD of the above matrix results in the non-zero SVs

$$\{\boldsymbol{\sigma}_i\}' = (11{\cdot}0349, 4{\cdot}8899, 0{\cdot}2858, 0{\cdot}0527, 0{\cdot}0029, 0{\cdot}0006)$$

with their product $1{\cdot}4141 \times 10^{-6}$. The corresponding matrices of the left and right singular vectors are contained in Appendix A. The truncated symptom observation matrix (last row eliminated, which means one less lifetime measurement) gives the non-zero SVs

$$\{\boldsymbol{\sigma}_i\}' = (10{\cdot}7179, 3{\cdot}9278, 0{\cdot}0528, 0{\cdot}0029, 0{\cdot}0024, 0{\cdot}0006)$$

with the product $9{\cdot}2823 \times 10^{-9}$; as already confirmed by theory and examples, the products of the resulting SVs grow with increasing damage measured through symptoms dependent on an additional lifetime. The corresponding matrices of the singular vectors are also presented in Appendix A. The comparison between the SVs of the full observation matrix and the truncated one is shown in Figure 7. As can be seen, the system's evolution causes growing SVs.

It is noted that two SVs do actually exist which are essentially different from zero. This fact is due to the undamaged state and the damage introduced, as already stated in the paragraph singular values and singular vectors as characteristics of the system life.

The symmetric correlation matrix according to $\mathbf{W}_2$ is given by

$$\begin{bmatrix}
1{\cdot}000 & 0{\cdot}999 & 0{\cdot}996 & 0{\cdot}986 & 0{\cdot}959 & 0{\cdot}877 & 0{\cdot}623 & 0{\cdot}059 & -0{\cdot}448 & -0{\cdot}681 & -0{\cdot}656 \\
 & 1{\cdot}000 & 0{\cdot}999 & 0{\cdot}992 & 0{\cdot}969 & 0{\cdot}895 & 0{\cdot}652 & 0{\cdot}096 & -0{\cdot}414 & -0{\cdot}653 & -0{\cdot}628 \\
 & & 1{\cdot}000 & 0{\cdot}997 & 0{\cdot}981 & 0{\cdot}917 & 0{\cdot}690 & 0{\cdot}148 & -0{\cdot}366 & -0{\cdot}613 & -0{\cdot}628 \\
 & & & 1{\cdot}000 & 0{\cdot}993 & 0{\cdot}977 & 0{\cdot}819 & 0{\cdot}223 & -0{\cdot}294 & -0{\cdot}551 & -0{\cdot}525 \\
 & & & & 1{\cdot}000 & 0{\cdot}977 & 0{\cdot}819 & 0{\cdot}339 & -0{\cdot}177 & -0{\cdot}441 & -0{\cdot}420 \\
 & & & & & 1{\cdot}000 & 0{\cdot}922 & 0{\cdot}530 & 0{\cdot}036 & -0{\cdot}247 & -0{\cdot}220 \\
 & & & & & & 1{\cdot}000 & 0{\cdot}818 & 0{\cdot}421 & 0{\cdot}148 & 0{\cdot}172 \\
 & & & & & & & 1{\cdot}000 & 0{\cdot}866 & 0{\cdot}691 & 0{\cdot}703 \\
 & & & & & & & & 1{\cdot}000 & 0{\cdot}960 & 0{\cdot}959 \\
 & & & & & & & & & 1{\cdot}000 & 0{\cdot}992 \\
 & & & & & & & & & & 1{\cdot}000
\end{bmatrix}.$$

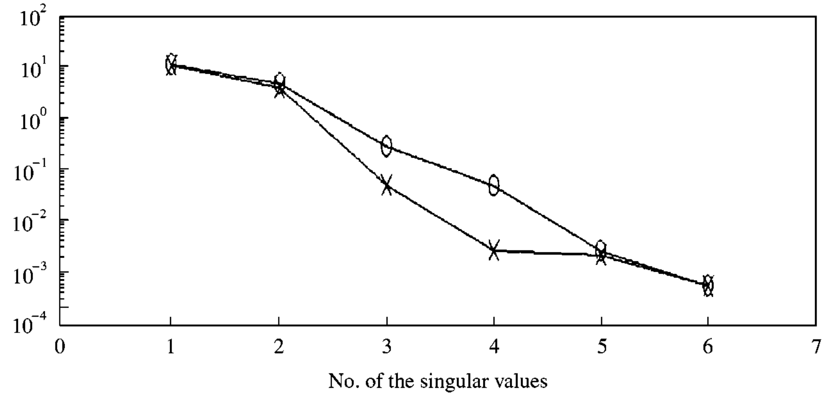

Figure 7. The SVs of the full (○) and the truncated (×) observation matrix.

It shows a strong correlation between neighbouring observation vectors and between the first symptoms. Reversal correlation is also noted.

The MAC values from matrix $\mathbf{W}_2$ are assembled in the following symmetric matrix:

$$\begin{bmatrix}
1{\cdot}000 & 0{\cdot}999 & 0{\cdot}995 & 0{\cdot}987 & 0{\cdot}970 & 0{\cdot}940 & 0{\cdot}878 & 0{\cdot}778 & 0{\cdot}620 & 0{\cdot}411 & 0{\cdot}360 \\
 & 1{\cdot}000 & 0{\cdot}999 & 0{\cdot}993 & 0{\cdot}979 & 0{\cdot}951 & 0{\cdot}898 & 0{\cdot}804 & 0{\cdot}653 & 0{\cdot}449 & 0{\cdot}399 \\
 & & 1{\cdot}000 & 0{\cdot}998 & 0{\cdot}989 & 0{\cdot}966 & 0{\cdot}920 & 0{\cdot}835 & 0{\cdot}692 & 0{\cdot}495 & 0{\cdot}446 \\
 & & & 1{\cdot}000 & 0{\cdot}996 & 0{\cdot}981 & 0{\cdot}944 & 0{\cdot}869 & 0{\cdot}739 & 0{\cdot}552 & 0{\cdot}505 \\
 & & & & 1{\cdot}000 & 0{\cdot}994 & 0{\cdot}969 & 0{\cdot}908 & 0{\cdot}793 & 0{\cdot}621 & 0{\cdot}575 \\
 & & & & & 1{\cdot}000 & 0{\cdot}990 & 0{\cdot}949 & 0{\cdot}855 & 0{\cdot}703 & 0{\cdot}661 \\
 & & & & & & 1{\cdot}000 & 0{\cdot}984 & 0{\cdot}920 & 0{\cdot}796 & 0{\cdot}760 \\
 & & & & & & & 1{\cdot}000 & 0{\cdot}975 & 0{\cdot}092 & 0{\cdot}063 \\
 & & & & & & & & 1{\cdot}000 & 0{\cdot}970 & 0{\cdot}951 \\
 & & & & & & & & & 1{\cdot}000 & 0{\cdot}994 \\
 & & & & & & & & & & 1{\cdot}000
\end{bmatrix}.$$

It can be stated from the above two matrices that with growing lifetime measurements the "neighbourhood" of the vectors decreases.

A comparison of the left singular vectors of the full observation matrix and the truncated one is shown in Figure 8.

This academic example illustrates some of the statements made above, but it is not a substitute for a real application or practical experience.

5.2. SOME REMARKS ON LARGE-SIZED OBSERVATION MATRICES

The integral operators T_v with respect to the fast time co-ordinate t and applied to the dynamic responses yield the symptoms dependent on the lifetime θ. The integral operators can also be chosen as Fourier or Laplace transform operators producing symptoms which are dependent on the frequency ω and on the lifetime θ: $s(\theta, \omega)$. Taking them, for example, as

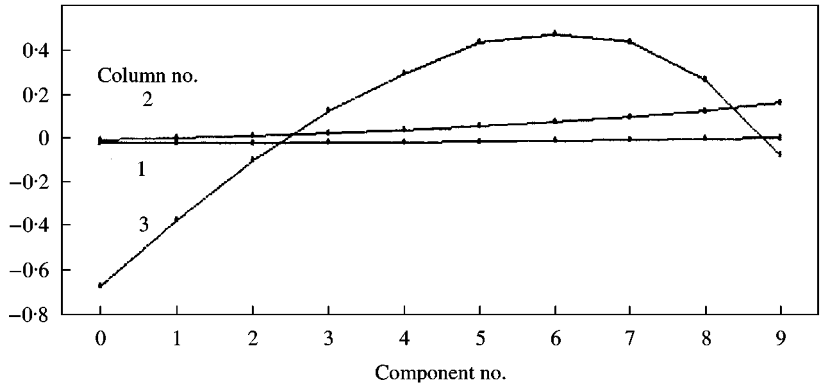

Figure 8. Difference between the first three left singular vectors of the full and the truncated observation matrix.

pre-chosen discrete frequencies $\omega_k, k = 1(1)K$, will give additional rows in the observation matrix which make the matrix very large.

Redundancy in the data may cause numerical difficulties. These data will not give any additional information and they are therefore superfluous. In more detail, if some rows are linearly dependent, then the interpretation is

(1) assuming the data are symptom values (this means that they are sensitive due to the fault/damage sought), nothing has changed during the lifetimes considered;
(2) assuming the data are not sensitive enough with respect to the fault/damage expected, then they are also superfluous. Depending on the interpretation of the data, classical rank investigations will lead to finding redundant data. Taking into account noise in the measurements, investigations on close linear dependency will give the result for the elimination of redundant vectors in the observation matrix.[§]

Another well-known tool is the subspace solution, which can be seen in the context of SVD (just for rank investigations). The SVD is initial information condensation.

In order to reduce the size of the observation matrix at the very outset of the monitoring design, one should think in terms of sub-systems [14]. This means that it is sufficient to measure in the near vicinity of the expected fault/damage, which requires a large amount of prior knowledge, if one is interested in small local system modifications. The near vicinity is defined as a sub-system which is related to one observation matrix with a relatively small size compared with that of the total system.

5.3. SOME REMARKS ON THE LIFETIME SPACING

The choice of symptoms in the context of observability and controllability has already been discussed in the previous sections. The question remains: when should the lifetime-dependent symptoms be measured? Of course, this question cannot be answered generally, but the answer depends on the system and its behaviour with regard to the loadings considered. First, rapid changes must be registered in an interval that is not too large. Then, for continuous symptom curves (see Figure 1), the lifetime spacings depend

[§]See the statement regarding the determinant of the "economy" size of the diagonal matrix of SVs. Redundancy of primary symptoms can be seen in the correlation/covariance matrix [2].

on the curvature of the corresponding symptom curve (if known); mathematically, the reconstruction of the corresponding curves should be possible through the chosen discrete measurements. Physically, the fault/damage intensities should not increase to an unacceptable extent. Here again it is seen that knowledge of a relationship between symptom and fault/damage measure is important [1].

5.4. APPLICATION TO MONITORING AND DIAGNOSTICS: SUMMARY

The SVD of the symptom observation matrix (symptom subspace considerations) is a powerful tool in the monitoring and diagnostics of technical systems in operation/service. The symptom observation matrix can be a very large matrix during the lifetime of a system, so the SVD is a step towards information condensation of the monitored data and an available diagnostic tool. In some detail:

- the observation matrix contains the lifetime-dependent symptom vectors, which permits information condensation via the SVD,
- the SVD results in generalized fault/damage indicators,
- in order to perform, for example, reliability investigations, one can relate one limit value [9] to each generalized fault/damage instead of relating a limit value to each symptom as in other techniques for monitoring and diagnosis,
- the matrix elements of $\mathbf{W}_2$ (equation (33)) serve as symptom assurance criteria (SAC),
- the non-zero SVs, ordered by their magnitudes, are fault/damage ranking indices,
- the magnitudes of the SVs describe fault/damage intensities,
- the small SVs are most sensitive to system modifications (and to measuring distortions),
- the (directions of the) left singular vectors characterize the system modifications during lifetime of the system. However, the right singular vectors are more sensitive than the left singular vectors with respect to modifications.

6. CONCLUSIONS AND OUTLOOK

Monitoring and diagnosis of complex technical systems is recommended or necessary for several reasons such as safety, reliability, risk management and economics, as itemized in section 5.4.

Suitably chosen lifetime-dependent symptoms, discriminants and patterns of the system in operation/service considered can be assembled in a (rectangular) symptom observation matrix. The SVD of the observation matrix determines the dimension of the fault/damage space of systems in operation/service, and it permits the redundancy of symptoms to be reduced. Additionally, it yields generalized discriminants in order to trace the fault/damage evolution.

The singular values and singular vectors are informative quantities/indicators for fault/damage detection, evaluation, and possibly for their localization. However, the system-dependent relationship between the SVs and/or the singular vectors and the physical fault/damage including their locations is lacking.

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APPENDIX A

Left and right singular matrices of the full observation matrix:

$\mathbf{U}_{11,11}$ =

0·3463	0·3342	−0·1416	−0·5354	−0·4004	−0·2754	0·0694	−0·0494	−0·2242	−0·3521	−0·2097
0·3415	0·2915	−0·1070	−0·2666	0·0112	0·6379	−0·0927	−0·1939	0·2016	0·1677	0·4402
0·3356	0·2417	−0·0740	−0·0252	0·2489	−0·1129	0·0001	0·1679	0·2684	0·5743	−0·5640
0·3282	0·1830	−0·0291	0·1540	0·2375	−0·5729	−0·0131	0·1929	−0·1105	0·1232	0·6204
0·3193	0·1147	0·0173	0·2821	0·3314	0·2901	0·1182	0·4585	0·0480	−0·6034	−0·1426
0·3081	0·0332	0·0805	0·3577	0·1588	−0·0497	−0·4920	−0·6319	−0·2100	−0·1469	−0·1866
0·2947	−0·0625	0·1441	0·3329	−0·2122	0·0768	0·7728	−0·2942	−0·1507	0·1422	−0·0086
0·2783	−0·1766	0·2167	0·2245	−0·5758	0·1697	−0·3591	0·4334	−0·2547	0·2281	0·0034
0·2575	−0·3150	0·3185	−0·0509	−0·2074	−0·2257	−0·0479	−0·1008	0·7617	−0·2004	0·0609
0·2323	−0·4816	0·4229	−0·4964	0·4082	0·0629	0·0442	0·0230	−0·3327	0·0690	−0·0151
0·2482	−0·5753	−0·7794	0·0076	−0·0001	−0·0009	−0·0001	−0·0052	0·0031	−0·0014	0·0017

$\mathbf{V}_{6,6}$ =

0·1935	−0·1141	0·0439	−0·2650	0·8093	−0·4716
0·2705	−0·2136	0·0882	0·0589	−0·5282	−0·7687
0·4369	−0·4542	0·0646	0·7195	0·1879	0·2134
0·4258	−0·5046	−0·2288	−0·6113	−0·1673	0·3319
0·3661	0·2623	0·8566	−0·1726	−0·0510	0·1760
0·6190	0·6416	−0·4472	0·0717	−0·0097	0·0004

.

Left and right singular matrices of the truncated observation matrix:

$\mathbf{U}_{10,10}$ =

0·3673	0·3396	0·5367	−0·4002	−0·0138	0·2790	−0·0320	−0·2083	−0·3072	−0·2912
0·3609	0·2876	0·2671	0·0038	0·2297	−0·6813	0·1035	0·3625	0·1917	0·1412
0·3530	0·2270	0·0265	0·2571	−0·2545	0·1550	−0·2229	−0·4120	0·2503	0·6214
0·3433	0·1557	−0·1538	0·2370	0·0210	0·5775	0·2895	0·5657	0·1860	−0·0748
0·3319	0·0728	−0·2816	0·3406	−0·2834	−0·2480	−0·1123	−0·2567	0·1484	−0·6685
0·3175	−0·0261	−0·3594	0·1458	0·4110	−0·0129	−0·4213	0·0870	−0·6178	0·1112
0·3003	−0·1420	−0·3345	−0·2154	0·0993	−0·0941	0·7355	−0·3587	−0·1552	0·1374
0·2794	−0·2801	−0·2254	−0·5585	−0·5529	−0·0884	−0·2321	0·3200	0·0110	0·1043
0·2532	−0·4476	0·0466	−0·2244	0·5312	0·1453	−0·2329	−0·1636	0·5343	−0·1154
0·2215	−0·6490	0·4928	0·4143	−0·1871	−0·0319	0·1248	0·0642	−0·2414	0·0347

$\mathbf{V}_{6,6}$ =

0·1843	−0·1305	0·2626	0·8063	0·1010	0·4688
0·2535	−0·2376	−0·0629	−0·5314	0·0835	0·7655
0·4006	−0·4890	−0·7208	0·1879	0·0023	−0·2135
0·3842	−0·5256	0·6237	−0·1583	−0·2406	−0·3228
0·3896	0·1995	0·1269	−0·0843	0·8617	−0·2104
0·6648	0·6107	−0·0493	0·0068	−0·4271	0·0166

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