Journal of Sound and Vibration (2001) **248**(4), 725–744
doi:10.1006/jsvi.2001.3810, available online at http://www.idealibrary.com on IDEAL®

PHASE CONTROL OF SELF-SUSTAINED VIBRATION

I. J. SOKOLOV AND V. I. BABITSKY

Wolfson School of Mechanical and Manufacturing Engineering, Loughborough University, Leicestershire, LE11 3TU, England. E-mail: i.j.sokolov@lboro.ac.uk

(*Received* 29 *January* 2001, *and in final form* 18 *May* 2001)

Some new vibratory machines have recently been developed using the concept of *autoresonant systems*. These are electro-mechanical self-sustained vibrating systems with phase shifting feedback. The use of this concept for vibratory machine design overcomes the difficulties in excitation and maintaining the resonant regimes in systems with non-linearities and variable loads. The design of autoresonant systems relies entirely on the amplitude–phase characteristics of vibrating systems. This paper is devoted to the analysis of these characteristics and linear vibrating systems with one and two degrees of freedom and non-linear systems are considered. The main purpose is to investigate the main properties of these characteristics near resonance. It is also shown that amplitude–phase curves allow robust and reliable vibration control due to their flatness and single-valuedness.

1. INTRODUCTION

Traditionally, independent *external* periodical action was used to excite oscillation of vibratory equipment. Since most of this equipment has resonant properties, the relationship between the excitation frequency and natural frequencies of the vibratory system determines the regime of oscillation. The response of traditional vibratory machines with forced excitation is largely determined by the shape of their amplitude–frequency (resonance) curves $X(\omega)$, where X is the amplitude and ω is the frequency of vibration. The following methods are used in this case to control the regime and parameters of forced vibration.

(1) If an actuator of the synchronous type is used for forced excitation (electromagnet or synchronous motor, for example), both amplitude F and frequency ω of the exciting force can be varied independently (Figure 1(a, b)).

(2) When a synchronous-type actuator is used for kinematic excitation (synchronous motor, for example), both amplitude and frequency ω of displacement of the point of excitation can be varied independently.

(3) When an actuator of the asynchronous type is used for excitation (induction motor or DC motor, for example), the drive torque–speed curve $L(\omega)$ can be varied to control the amplitude and frequency of vibration (Figure 1(c)). In this figure, $N(\omega)$ is the average power of the driver; $N(\omega) = \omega L(\omega)$ and the resonance curve shows the dissipation power here [1].

The forced excitation (1) is often implemented through a kinematic one (2) using elastic or viscous intermediate; the clear boundary between forced and kinematic excitation does not exist. For many designs of exciters, frequency ω can be varied only simultaneously with an amplitude F; the type of function $F(\omega)$ depends on the exciter design. The first two methods

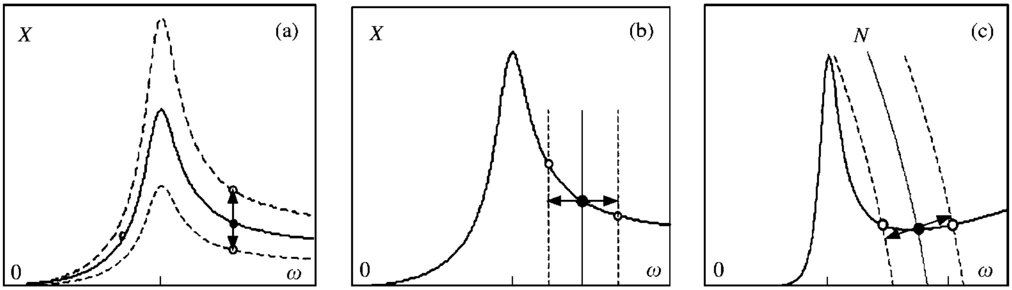

Figure 1. Traditional methods of forced vibration control.

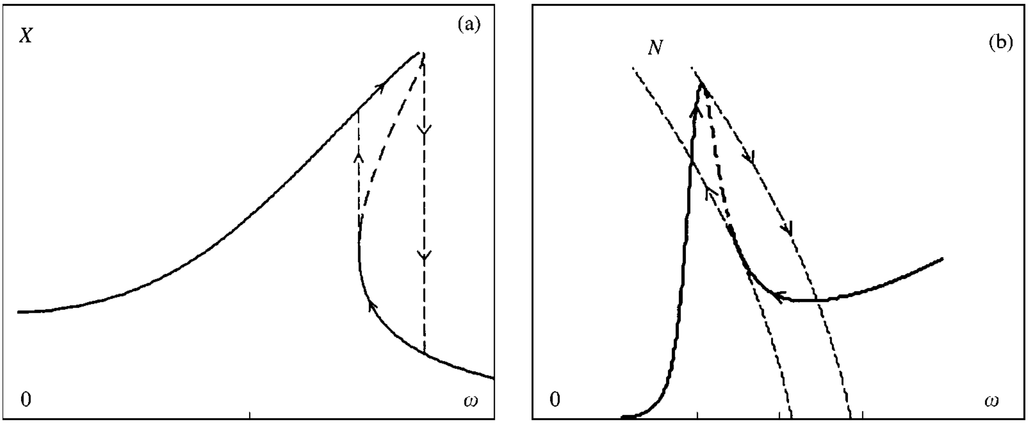

Figure 2. Jump phenomenon and Sommerfeld effect.

(1) and (2) are also possible when using the asynchronous drive with unlimited ($\mathrm{d}\,N/\mathrm{d}\omega = -\infty$) power.

Thus, the independent or simultaneous changing of frequency, amplitude and power of excitation is used in traditional vibrating systems to control the regime and parameters of forced oscillation. The above methods do not directly control the phase shift between the exciting action and the resulting vibration. This shift is the dynamic characteristic of interaction between the driver and the system being excited.

The above methods of control of regime and parameters of forced vibration will be called *frequency control*. In this case, the driver determines the frequency of vibration directly or through its power-speed characteristic, which reflects the frequency dependence of the driver power.

Although the resonant regime of vibratory equipment is, in general, the most effective [2], it is usually avoided because of difficulties of regime maintenance especially in systems with a high Q-factor and varying parameters, as well as in systems with non-linearities or limited power excitation. The jump phenomenon of the non-linear systems (Figure 2(a)) [1, 3, 4] and the Sommerfeld effect in systems with limited power excitation (Figure 2(b)) [1, 3] are the most common difficulties. The resonant curve in the last figure represents the average energy dissipation due to vibration.

The new approach was developed to design the resonant vibratory equipment as a *self-sustained* oscillating system with electronic and electro-mechanical feedback and an

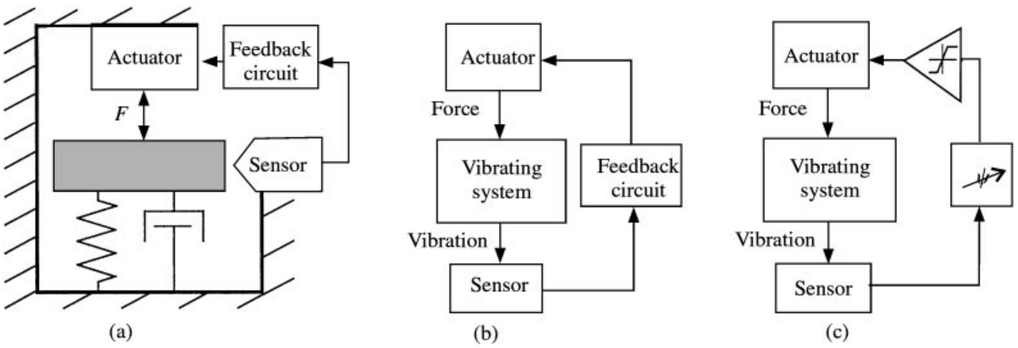

Figure 3. Self-sustained oscillating systems with feedback.

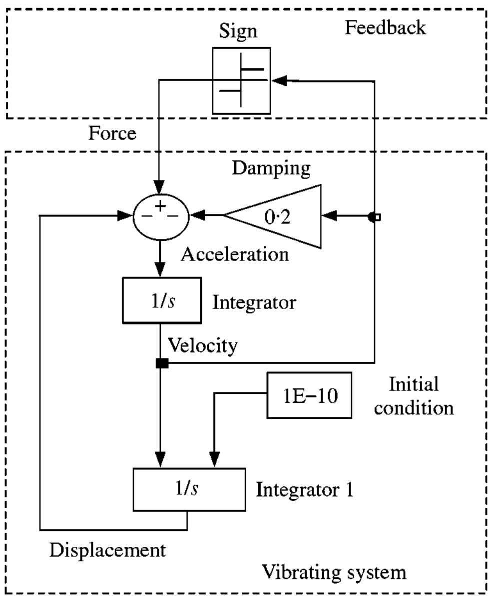

Figure 4. Elementary computer model of self-sustaining system.

actuator of the synchronous type (Figure 3(a)). The feedback produces the exciting force F by means of transformation and amplification of the displacement (velocity, acceleration) signal [2, 5]. A more general diagram is presented in Figure 3(b). The feedback in its simplest form (Figure 3(c)) shifts the phase of the vibration signal from the sensor and amplifies its power (with limitation as a rule). This powerful signal feeds the actuator, which transforms it to the exciting force. The elementary Matlab-Simulink computer model of such a system is shown in Figure 4. This model uses the velocity signal in the feedback circuit and does not need any special phase-shifting elements to get the resonant vibration. Figure 5 illustrates this model's behaviour.

The frequency (and amplitude) of self-sustained oscillation can be controlled by means of changing the phase shift between the exciting force and the vibrating system response. The amplitude can be varied independently by changing the limitation level in the feedback circuit. Phase shift and amplitude of excitation can be varied independently, when the drive has an essential reserve of power, otherwise they cannot be varied independently. This

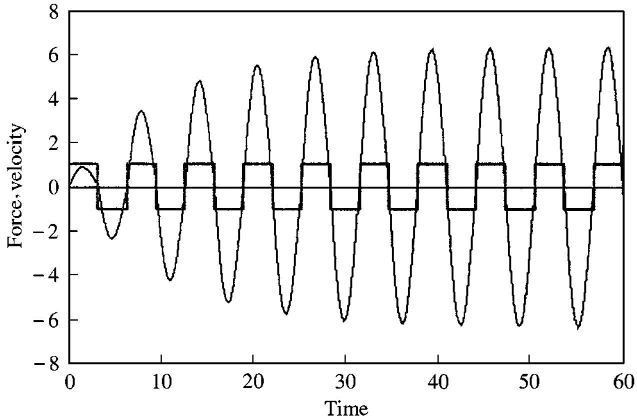

Figure 5. The elementary model behaviour.

method of control of regime and parameters of self-sustained vibration will be called *phase control*. This means of excitation and vibration control is devoid of drawbacks present in the traditional *frequency control* of forced vibration. The progress of modern electronics has made this control possible even for powerful machines.

The problem of forced vibration of linear and non-linear systems under external periodic excitation with unlimited power and variable frequency and amplitude is one of the most examined and common areas within vibration theory. The same problem in the case of limited power of excitation is less well known, but has been widely studied [1, 3]. The shape of the amplitude–frequency (resonance) curves determines the properties of traditional vibratory machines with forced excitation and frequency control. The shape of these curves has been thoroughly investigated for a wide range of vibrating systems. Ambiguity of resonant curves of non-linear systems determines the instability of some regimes of vibration and the jump phenomenon (Figure 2(a)).

Historically, a clock was one of the first self-sustaining oscillating systems. Airy [6] investigated the influence of the driving instant impulse phase on the period of a pendulum oscillation. This work was probably the first related to the phase control of self-sustained oscillation. His results are known in chronometry as the Airy theorem. Other authors (see reference [7] for example) emphasized the importance that phase shift plays in self-sustained oscillating systems, but mostly from the point of view of oscillation suppression [8]. The problem of phase control has assumed new significance with the expansion of power semiconductors giving new possibilities for synthesis of electro-mechanical self-sustaining vibratory machines.

If harmonic oscillation of a single-degree-of-freedom system with phase control and harmonic excitation is considered, it is obvious that the resonant regime takes place when the force is in phase with vibratory velocity (or lags $3\pi/2$ in phase from vibratory displacement). This system is designated as an *autoresonant* one [2, 5]. It maintains the resonant regime of oscillation when the natural frequency of a mechanical subsystem changes. As the phase shift changes, the regime of oscillation (amplitude, frequency) also changes. Amplitude–*phase* curves determine the system's behaviour when phase shift changes purposely or accidentally. These curves play the same role in phase-controlled systems as traditional resonance curves play in frequency-controlled systems. However, amplitude–*phase* curves have not been studied as thoroughly as conventional amplitude–frequency resonance characteristics. These curves for some representative single-degree-of-freedom systems and the simplest two-degree-of-freedom systems are considered below. This consideration reveals some features that are essential for the design of autoresonant vibrating machines.

2. SINGLE-DEGREE-OF-FREEDOM SYSTEM

2.1. GENERAL EXPRESSIONS

The stationary vibration of an oscillator with a single degree of freedom, linear damping and unlimited power of harmonic excitation will first be considered in order to understand the main properties of amplitude–phase curves and the behaviour of a phase-controlled system under variable phase shift. The dimensionless equation of motion of this oscillator can be written in the form

$$x'' + 2Dx' + x + f(x) = F(\eta)\cos\eta\tau, \tag{1}$$

where x is the dimensionless displacement, D is the dimensionless damping factor, $x' = \mathrm{d}x/\mathrm{d}\tau$, $\tau = \omega_0 t$ is the dimensionless time, ω_0 is the natural frequency of oscillation without non-linearity, $\eta = \omega/\omega_0$ is the dimensionless frequency, $f(x)$ is the small non-linear part of the dimensionless restoring force, and F is the amplitude of the dimensionless exciting force. In this expression, we take into account the fact that amplitude of force depends generally on frequency: $F = F(\eta)$. At present, it makes no difference whether the exciting force is external or produced by the feedback circuit.

Assuming that the resulting movement is close to the harmonic one $x = X\cos(\eta\tau - \psi)$, where X is the amplitude and ψ is the phase shift, we can easily obtain the general expression for amplitude–phase curves from the equation of energy balance for non-conservative forces during a period of vibration [4]: $2\pi D\eta X^2 = \pi F X \sin\psi$. This yields the following expression:

$$X = \frac{1}{2D}\frac{F}{\eta}\sin\psi = \frac{1}{2D}\frac{F[\eta(\psi)]}{\eta(\psi)}\sin\psi, \tag{2}$$

where $\eta(\psi)$ is the frequency–phase function, an inverse function of the traditional phase (phase–frequency) one. Equation (2) is the exact one for linear system ($f(x) \equiv 0$) and remains valid in the framework of the approximate method of harmonic balance for systems with non-linear restoring force ($f(x) \neq 0$). Figure 6 illustrates the shape and correlation of the amplitude–frequency $X(\eta)$, phase–frequency $\psi(\eta)$ and amplitude–phase $X(\psi)$ curves for the simplest example of a linear system with $F = const$. These curves are drawn as projections of the 3-D curve, the points of which correspond to the different possible regimes of oscillation with parameters X, η, ψ.

Our prime interest is to study the amplitude–phase characteristics from the standpoint of its single-valuedness and flatness near the resonance. The single-valuedness is important because the ambiguity causes instability of some regimes of vibration and hence difficulties in control. Changing vibration system parameters can cause a small deviation in initial resonant phase shift tuning, when the feedback circuit is non-ideal or the drive power is limited. The flatness determines how difficult it is to tune the system to the resonant regime and to maintain this regime in circumstances when the tuning varies due to parameter deviation.

An important general conclusion can be drawn from expression (2). It is sufficient, for function $X(\psi)$ to be single-valued, that the functions $\eta(\psi)$ and $F(\eta)$ are single-valued. In the case of phase-controlled vibration with unlimited power of excitation, the vibration is stable

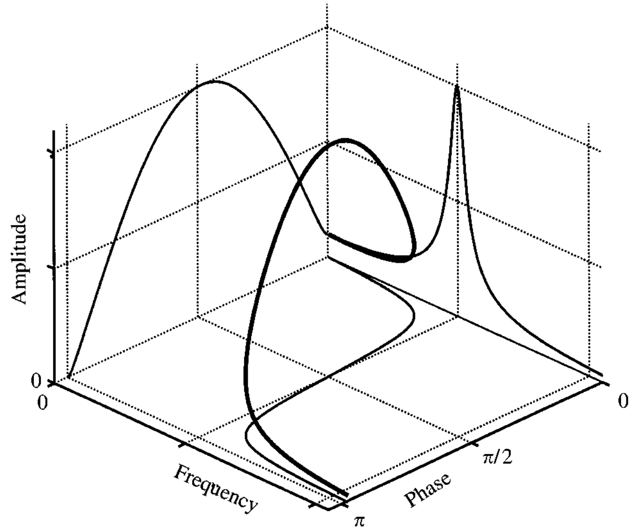

Figure 6. 3-D amplitude–phase–frequency curve and its projections.

at all points on the amplitude–phase curve if the curve is single-valued. As this takes place, vibration is stable at all points of the corresponding amplitude–frequency curve even if it is ambiguous and some regimes are unstable under traditional forced excitation with frequency control.

The usual function $F(\eta) = \eta^n$, $n = 0, 1, 2$ [4] are further considered:

$$F(\eta) = 1, \tag{3}$$

— excitation by force with constant amplitude or by reciprocal motion through a linear spring;

$$F(\eta) = \eta, \tag{4}$$

— excitation by reciprocal motion through a linear viscous damper;

$$F(\eta) = \eta^2, \tag{5}$$

— excitation by inertial force or rotating unbalance.

All these functions together with their linear combinations are single-valued. As this takes place, the amplitude–phase curve $X(\psi)$ is single-valued if, and only if, the frequency–phase curve $\eta(\psi)$ is also single-valued. It is known that the frequency–phase curve $\eta(\psi)$ is single-valued for a very wide range of vibrating systems. Among them are linear systems, non-linear ones with a hardening restoring force characteristic, including vibro-impact ones with a preliminary gap, most of the systems with a softening characteristic and many others.

Figure 7 presents the Mathlab-Simulink computer model of the oscillating system with hardening restoring force (see section 2.3 below) and autoresonant excitation. Unlike the model in Figure 4(a), this has a special block for shifting the phase of the square-wave signal, and the displacement signal is used in the feedback circuit. The amplitude–phase and amplitude–frequency curves of this model are shown in Figure 8. These curves are the result of simulation of the real oscillation under slow changes of the phase shift (its value in the figure defines the lag between the displacement and force). As expected, *all* points of the

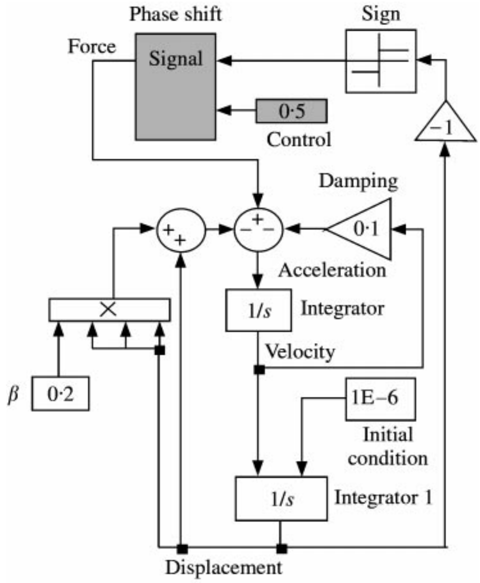

Figure 7. Computer model of non-linear system under phase control.

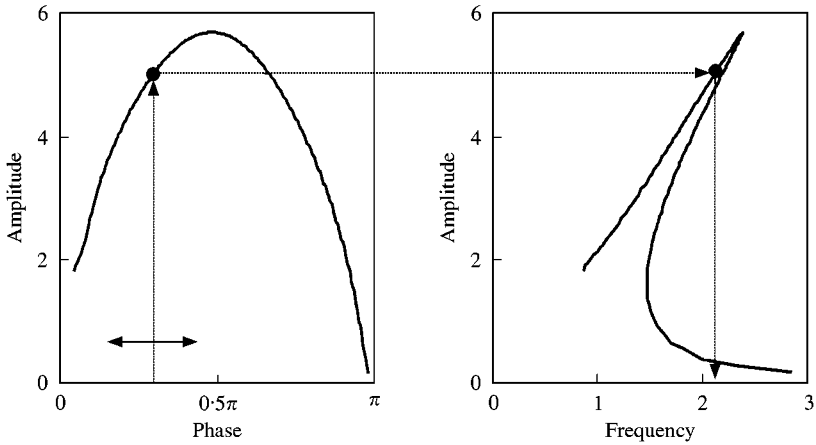

Figure 8. Result of simulation for non-linear system under phase control.

above curves correspond to the *stable* regimes of vibration. Similar results were obtained for the full-scale electro-mechanical model with synchronous motor [9, 10].

A case of excitation (4) is of special interest as it is the easiest way to investigate the amplitude–phase curve shape. Equation (2) takes the form: $X = (2D)^{-1} \sin \psi$ and the curve has the shape of sine (positive half-period). It is important to note that this shape does not depend on *the restoring force characteristic* (Figure 9).

The values of frequency and phase that correspond to the maximum amplitude generally depend on dimensionless damping. To compare the shape of different curves in the vicinity

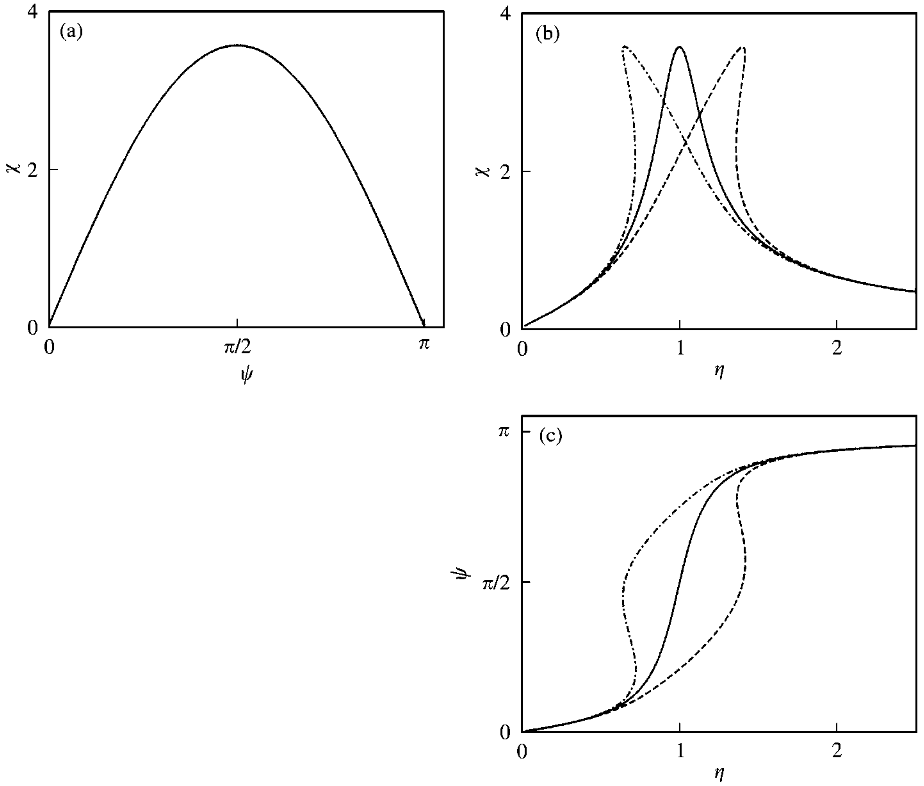

Figure 9. System with linear and non-linear spring, $D = 0{\cdot}14$, $F = \eta$.

of the maximum, new variables are introduced: X_{max} as the maximum of amplitude; $\psi^* = arg\,[X_{max}\,(\psi)]$, the value of phase shift that gives maximum amplitude. Hereinafter we shall consider, if possible, the normalized functions $\chi = \chi(\varphi)$, where $\chi = X/X_{max}$ and $\varphi = \psi/\psi^*$. This change of variables enables the maximum points to be superimposed and it makes the comparison of shapes more representative. In the case being considered, the normalized amplitude–phase function takes the form

$$\chi = \sin\frac{\pi}{2}\,\varphi. \tag{6}$$

Unlike the previous expression, this does not depend on either the dimensionless damping of the oscillating system or the exciter parameters. Relation (6) is very flat near the resonance irrespective of the Q-factor.

The amplitude–phase curves will be considered in detail for the particular vibrating systems: linear and non-linear systems with cubic non-linearity of restoring force.

2.2. LINEAR SYSTEM

In this case $f(x) \equiv 0$. It is convenient to obtain the parametric representation of amplitude–phase curve $X\,(\psi)$ using the expression for phase–frequency characteristic:

$$\psi = \arctan \frac{2D\eta}{1-\eta^2} + \frac{1}{2}(\operatorname{sgn}(\eta - 1) + 1)\,\pi,$$

$$X = \frac{1}{2D}\frac{F(\eta)}{\eta}\sin\psi. \tag{7}$$

Here η is the parameter. Note that from these expressions only the dimensionless damping factor D and the type of $F(\eta)$ function determine the shape of $X(\psi)$ curve. The first of expressions (7) gives the function $\psi(\eta)$ asymptotic behaviour near the resonance when $D \to 0$: $\psi_a(\eta) = 1 = const$. The corresponding asymptotic behaviour of function $X(\psi)$ is $X_a = (2D)^{-1} F(1) \sin\psi$ or $\chi_a = \sin\psi$. This means that the asymptotic behaviour of the linear system amplitude–phase curves does not depend on the Q-factor and the type of function $F(\eta)$.

Even if the resonant peak of the amplitude–frequency curve is extremely sharp, the amplitude–phase curve remains flat near the resonance. Figures 9–12 show the amplitude–phase curves $\chi(\varphi)$ for cases (3)–(5) of the excitation, respectively, and for different values of D. The traditional amplitude–frequency and phase–frequency characteristics are also presented in these figures. These curves are constructed using the new variable $\zeta = \eta/\eta^*$ instead of η, where $\eta^* = arg\,[X_{max}(\eta)]$ is the value of frequency η that gives maximum to amplitude X.

In the case of excitation (4), $\eta^* = 1$ and $\zeta = \eta$. It is evident that amplitude–phase curves are always flat near the resonance. The advantage of phase control over frequency control

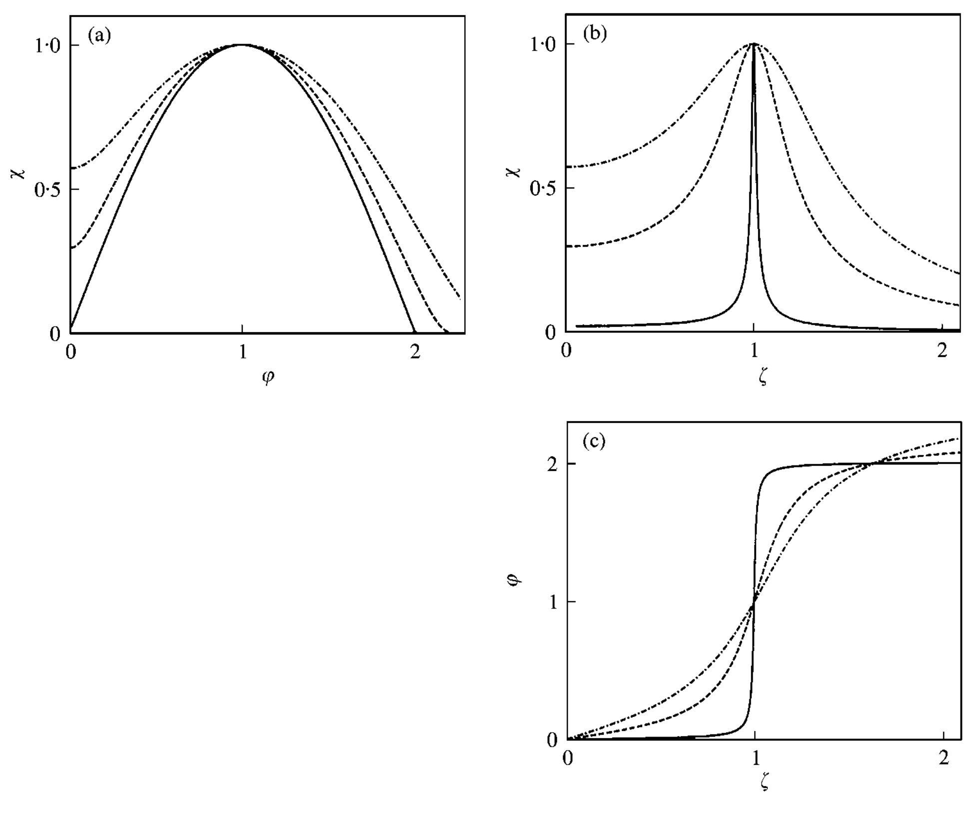

Figure 10. Linear system, $F = const$: ———, $D = 0{\cdot}01$; ---------, $D = 0{\cdot}15$; -·-·-·-·-, $D = 0{\cdot}3$.

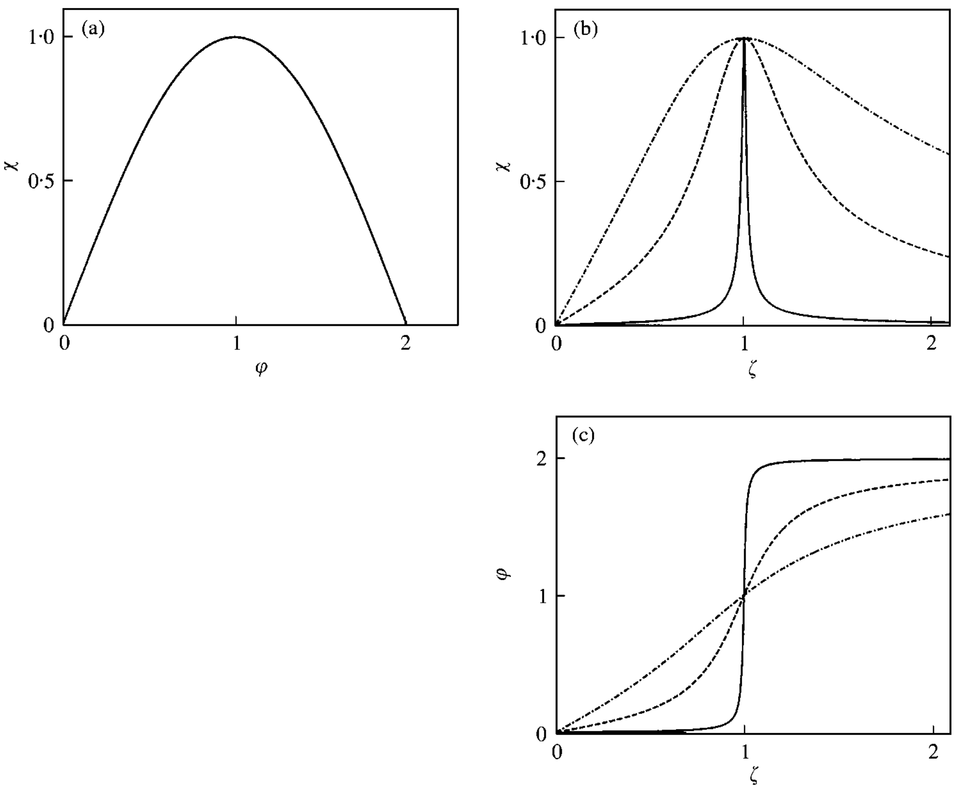

Figure 11. Linear system, $F = \eta$: ———, $D = 0{\cdot}01$; ---------, $D = 0{\cdot}2$; -·-·-·-·-, $D = 0{\cdot}6$.

when exciting resonant vibration in a linear system is obvious from these figures. As the amplitude–phase curves are always flat in the vicinity of maximum, the precise tuning of phase shift is not indispensable for maintaining a resonant regime. This advantage is most effective when the Q-factor is large. It is in this case that the resonant regime is most effective, but difficult to maintain under frequency control.

2.3. NON-LINEAR SYSTEM WITH HARDENING RESTORING FORCE

Now consider the system with non-linearity $f(x) = \beta x^3, \beta > 0$. To investigate the general properties of non-linear systems under phase control, approximate equations similar to equation (7) will be used, that can be obtained by the harmonic balance method. They have the following form [4]:

$$\psi = \arctan\frac{2D\eta}{\eta_e^2 - \eta^2} + \frac{1}{2}(\mathrm{sgn}(\eta - \eta_e) + 1)\pi,$$

$$X = \frac{1}{2D}\frac{F(\eta)}{\eta}\sin\psi, \qquad (8)$$

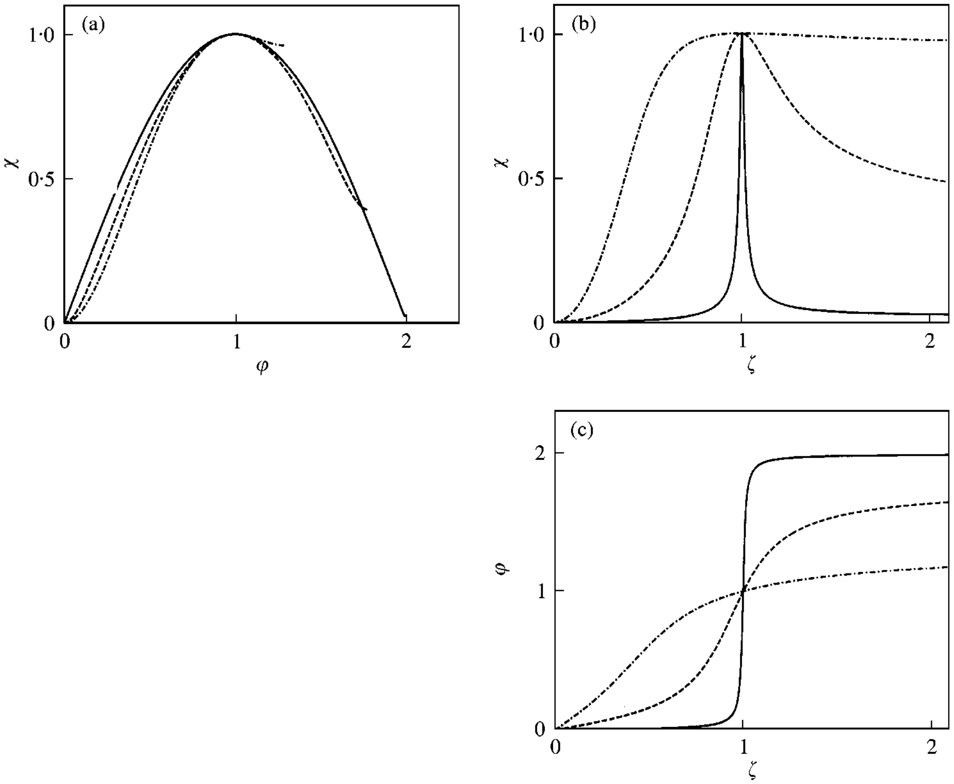

Figure 12. Linear system, $F = \eta^2$: ———, $D = 0{\cdot}01$; ---------, $D = 0{\cdot}2$; -·-·-·-·-, $D = 0{\cdot}6$.

where $\eta_e^2 = \eta_e^2(X) = 1 + \gamma^2 X^2$, $\gamma^2 = 3/4\beta$. Since η_e depends on X, equations (8) do not give, unlike equation (7), the parametric description of function $X(\psi)$. The expression that gives the implicit definition of an amplitude–frequency characteristic follows from (8):

$$X = \frac{F(\eta)}{\sqrt{(\eta_e^2 - \eta^2)^2 - 4D^2\eta^2}}. \tag{9}$$

The general regularity of the asymptotic behaviour of the considered curves near resonance when $D/\beta \to 0$ can be obtained from equations (8) and (9). The asymptotic behaviour of the amplitude–frequency curve is $\eta_a = \gamma X$. Consequently, the following equations determine the asymptotic behaviour of amplitude–phase and phase–frequency curves when $D/\beta \to 0$:

$$\frac{X^2}{F(\gamma X)} = \frac{1}{2D\gamma} \sin\psi \tag{10}$$

and

$$\frac{\eta^2}{F(\eta)} = \frac{\gamma}{2D} \sin\psi. \tag{11}$$

As discussed above, for the case of excitation in equation (4), the shape of the amplitude–phase curve does not depend on the type of restoring force characteristic and equation (10) yields equation (6) (Figure 13).

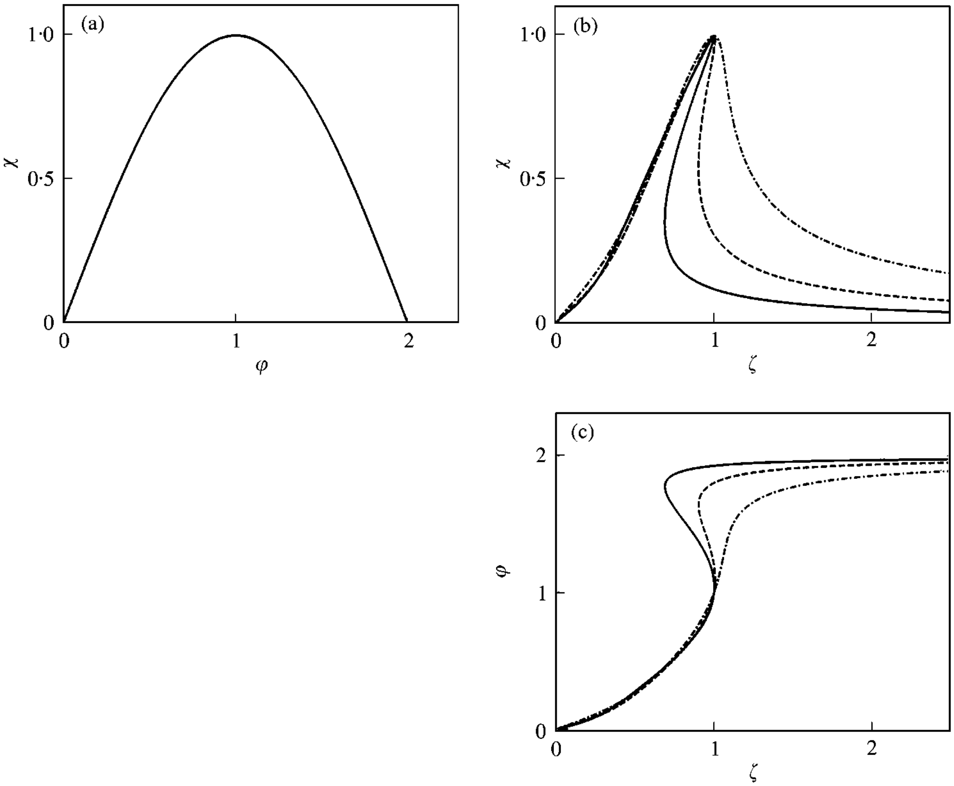

Figure 13. System with hard non-linearity, $F = \eta$: ————, $\beta = 0{\cdot}2$, $D = 0{\cdot}1$; ---------, $\beta = 0{\cdot}2$, $D = 0{\cdot}15$; -·-·-·-·-, $\beta = 0{\cdot}2$, $D = 0{\cdot}25$.

In the case of equation (3), the curves $\chi(\varphi)$ remain flat (Figure 14). Furthermore, these curves are flatter than the ones already considered: the asymptotic behaviour near resonance when $D/\beta \to 0$ is described by the expression $X_a = (1/\sqrt{2D\gamma}) \sin^{1/2} \psi$ or $\chi_a = \sin^{1/2} \psi$. The asymptotic behaviour of the frequency–phase curve is the same: $\varsigma_a = \sin^{1/2} \psi$.

The system behaviour in the case of excitation (5) depends on parameters D and β more strongly. The feature of a system with such excitation *under the model of damping considered* is that its resonance curve does not have a maximum if the ratio D/γ is sufficiently small (Figure 15). This fact results from energy consideration. Without going into detail, when the parameter $\xi = \gamma^2 - 4D^2 = \frac{3}{4}\beta - 4D^2$ is positive, the upper and lower branches of the amplitude–frequency curve tend to infinity asymptotically in parallel, and the distance between branches is proportional to $\xi^{1/2}$. If this takes place, the phase shift ψ cannot take all the values from the $(0, \pi)$ range. Both amplitude–phase and frequency–phase characteristics are not defined on the phase shift interval including the $\pi/2$ point. The width of this interval is 2 arctan $\xi^{1/2}/2D$. Since the maximum amplitude is not defined for all values of D and β, the curves in Figure 15 are not normalized.

From the point of phase control the last case is the least favourable, especially if ξ is close to zero or positive. The advantage of phase control related to flatness of amplitude–phase curves is not definite, but the main advantage related to single-valuedness remains in force.

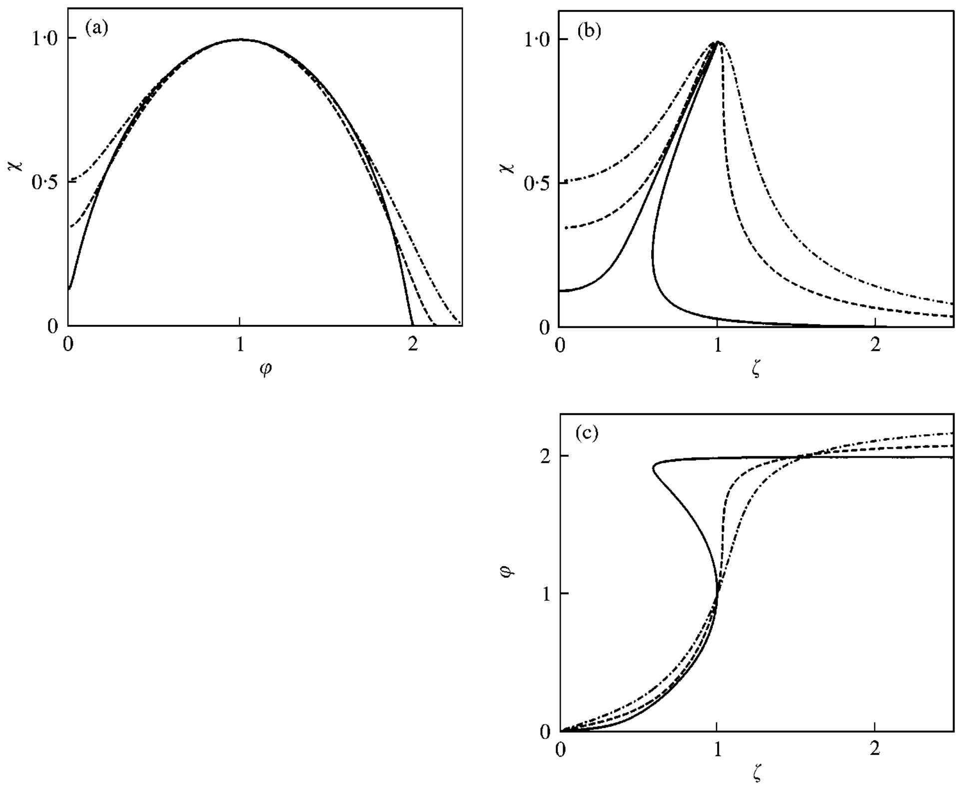

Figure 14. System with hard non-linearity, $F = const$: ————, $\beta = 0{\cdot}1$, $D = 0{\cdot}03$; ---------, $\beta = 0{\cdot}1$, $D = 0{\cdot}15$; -·-·-·-·-, $\beta = 0{\cdot}1$, $D = 0{\cdot}25$.

2.4. NON-LINEAR SYSTEM WITH SOFTENING RESTORING FORCE

The system with $f(x) = -\beta x^3$, $\beta > 0$ is now under consideration. Equations (8) and (9) remain unchanged provided $\eta_e^2 = \eta_e^2(X) = 1 - \gamma^2 X^2$, $\gamma^2 = \frac{3}{4}\beta$. It is important in this case to keep in mind that the restoring force becomes negative with a large displacement. Usually this fact does not receive proper attention, but vibration is impossible with amplitudes that result in a negative restoring force. The amplitude X^* that gives zero value to derivative $\mathrm{d}(P_{max}(X))/\mathrm{d}X$ will be called "*critical*", where P_{max} is the maximum value of potential energy during a period of vibration.† In the framework of the harmonic balance method $P_{max} = (1 - \gamma^2 X^2)\, X^2/2$ and (see footnote †) $X^* = \sqrt{2/3}/\sqrt{\beta}$. In this section, the curves for all values of amplitude will be plotted but the points corresponding to the critical one will be marked by dots. The parts of curves corresponding to stable regimes of vibration will be shown with a bold line; the parts corresponding to unfeasible regimes with amplitudes exceeding the critical one will be marked with a thin line. These regimes of vibration cannot take place as the non-linear force loses restoring ability. All the curves below are not normalized in this paragraph.

As shown above, the amplitude–phase curve has the sine shape for case (4) of excitation irrespective of D and β (Figure 16). A special feature of this case is that an amplitude–phase

† The value of "critical" amplitude deduced from the harmonic balance method is slightly below the exact value $X^* = 1/\sqrt{\beta}$.

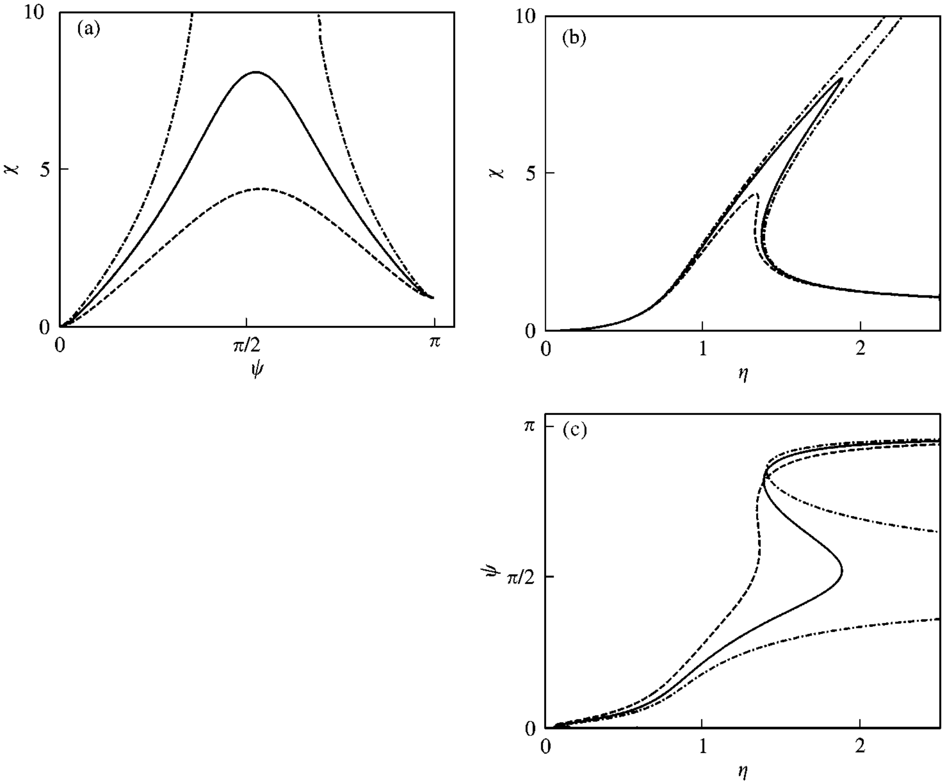

Figure 15. System with hard non-linearity, $F = \eta^2$: ·-·-·-·-·-, $\beta = 0{\cdot}05$, $D = 0{\cdot}095$; ———, $\beta = 0{\cdot}05$, $D = 0{\cdot}115$; ------, $\beta = 0{\cdot}05$, $D = 0{\cdot}15$.

curve is not defined on the phase interval including point $\pi/2$ when $2D/\gamma < 1$. However, this takes place only if $X > 2/\sqrt{3\beta}$: i.e., amplitude exceeds the critical value.

In case of excitation (5) (Figure 17), the soft non-linearity makes the amplitude–phase curve flatter in comparison with the corresponding linear system.

Excitation (3) is the most complicated. The amplitude–phase curves are two-valued (Figure 18). In addition, the zone of non-existent phase shifts appears when $D(1 - D^2) < \gamma$. When the amplitude–phase curve is defined for the whole interval $(0, \pi)$ of phase shifts, it remains flat. The curves are always two-valued, but the upper branches correspond to amplitudes exceeding the critical one.

All the bold branches of curves in Figures 16–18 correspond to stable regimes of vibration under phase control, even if these regimes are unstable under frequency control.

3. TWO-DEGREE-OF-FREEDOM SYSTEM

Any specific system with many degrees of freedom has a wide set of amplitude–frequency–phase curves, since different points of the system may be chosen to apply the exciting force and more than one force may be applied. Furthermore, different points may be chosen to define the phase shift between the vibration and force for any particular choice of the force application point. Moreover, even the phase shift of the relative vibration of any two points may be of interest. There is a multiple choice to build the phase-controlled excitation for the system with many degrees of freedom.

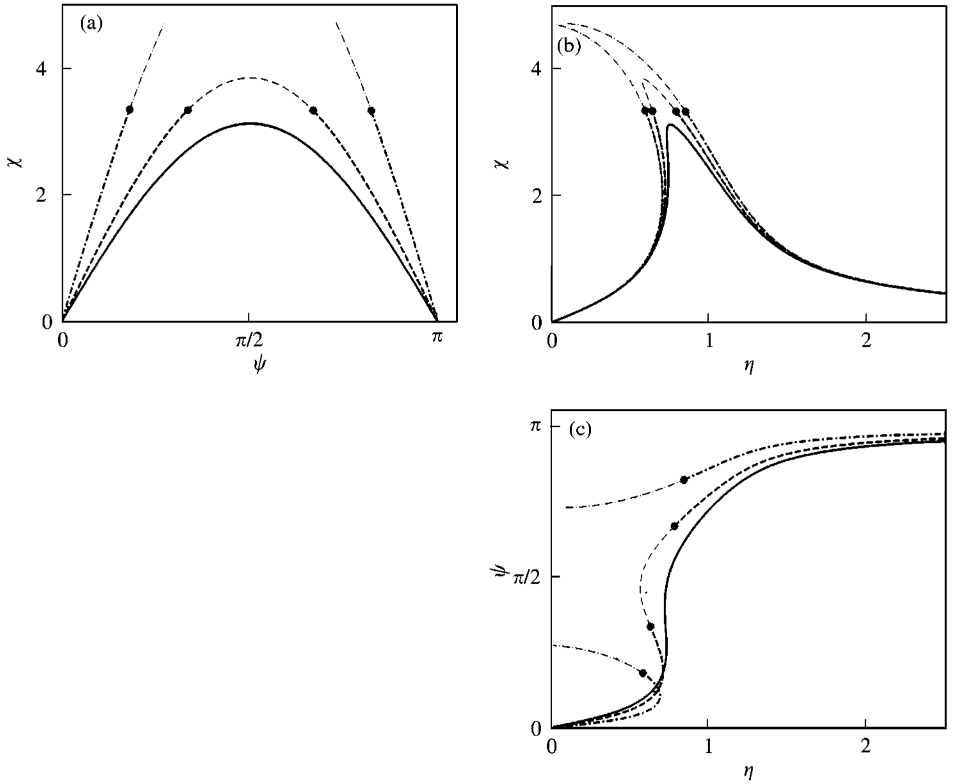

Figure 16. System with soft non-linearity, $F = \eta$: ———, $\beta = 0{\cdot}06$, $D = 0{\cdot}16$; ---------, $\beta = 0{\cdot}06$, $D = 0{\cdot}13$; -·-·-·-·-·-, $\beta = 0{\cdot}06$, $D = 0{\cdot}08$.

The point at which vibration is used to produce the signal for auto-resonant feedback will be called the *point of observation.*

To investigate the main properties of such systems under phase control, the simplest two-degree-of-freedom linear model will be considered (Figure 19).

Its dimensionless matrix differential equation of motion is

$$\mathbf{M}\mathbf{x}'' + 2\mathbf{D}\mathbf{x}' + \mathbf{C}\mathbf{x} = \mathbf{f}\cos\eta\tau. \tag{12}$$

Here

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}, \quad 2\mathbf{D} = \begin{pmatrix} \lambda\mu D_1 + D_2 & -D_2 \\ -D_2 & D_2 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 + \lambda^2\mu & -1 \\ -1 & 1 \end{pmatrix},$$

$$\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad \mathbf{x}' = \frac{\mathrm{d}\,\mathbf{x}}{\mathrm{d}\,\tau},$$

$$\mu = \frac{m_1}{m_2}, \quad \lambda = \frac{\omega_{01}}{\omega_{02}}, \quad \omega_{01}^2 = \frac{c_1}{m_1}, \quad \omega_{02}^2 = \frac{c_2}{m_2}, \quad D_1 = \frac{m_1}{2b_1\omega_{01}}, \quad D_2 = \frac{m_2}{2b_2\omega_{02}}, \quad \eta = \frac{\omega}{\omega_{02}},$$

$$\tau = \omega_{02}t.$$

The forced vibration of this system will be of the form

$$x_1 = X_1\cos(\eta\tau - \psi_1), \qquad x_2 = X_2\cos(\eta\tau - \psi_2), \qquad x_{21} = X_{21}\cos(\eta\tau - \psi_{21}). \tag{13}$$

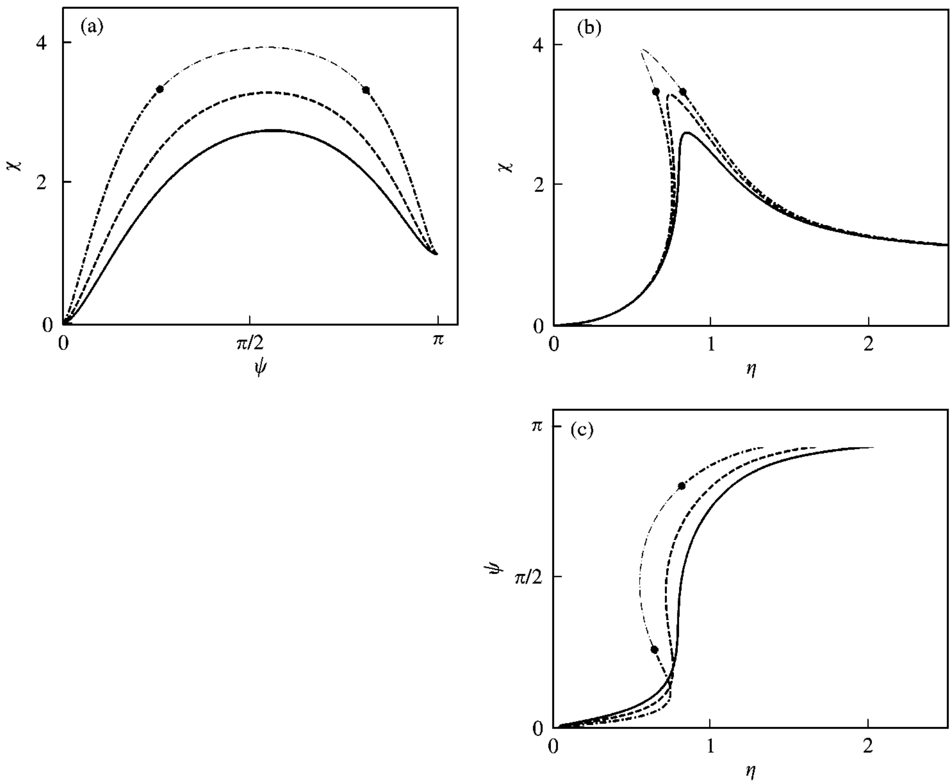

Figure 17. System with soft non-linearity, $F = \eta^2$: ———, $\beta = 0{\cdot}06$, $D = 0{\cdot}15$; ---------, $\beta = 0{\cdot}06$, $D = 0{\cdot}11$; -·-·-·-·-, $\beta = 0{\cdot}06$, $D = 0{\cdot}07$.

Here variables describing the relative oscillation of two masses have an index of "21". The example of amplitude–frequency–phase curves is shown in Figure 20(a–c) for the case $\mathbf{f} = \binom{1}{0}$, i.e., force is applied only to the first mass. The curves in these and subsequent figures are not normalized. The bold dotted and solid lines refer to masses 1 and 2, respectively; the thin dashed line refers to the relative vibration of two masses. The following qualities can be seen from this figure:

— the amplitude–phase curves of the system with two degrees of freedom remain bell-shaped and flat near the resonance;

— the amplitude–phase characteristic of mass 1 is ambiguous, unlike other amplitude–phase curves considered above.

The final result means that three regimes of vibration are possible in the system with the considered parameters under phase control for any phase shift $\psi_1 = \psi_0$ near the resonant $\pi/2$ value, when the same mass m_1 is used both for excitation and observation. The corresponding points of curves are marked with bold dots. Two regimes with high amplitude are close to the resonant normal vibration and the third regime with the lowest amplitude corresponds to anti-resonant vibration.

To estimate approximately the stability of these three possible regimes in the system with the ideal feedback circuit, the energy balance will be considered. The solid curve in Figure 20(d) represents the difference $E_f - E_d$ between the energy supplied by the excitation

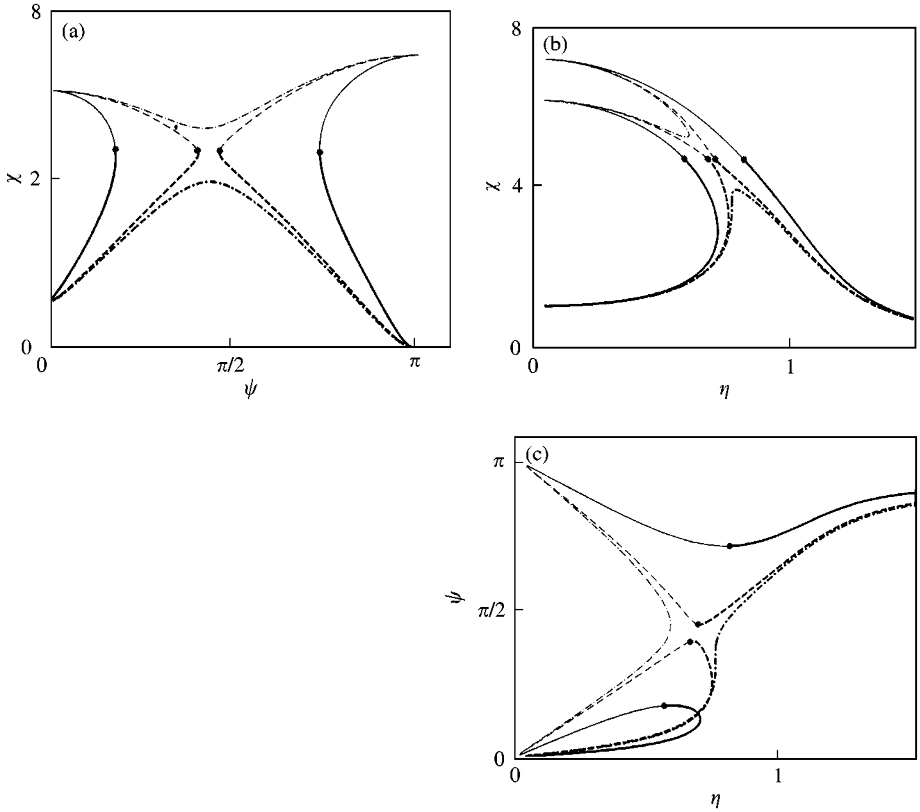

Figure 18. System with soft non-linearity, $F = const$: ————, $\beta = 0{\cdot}03$, $D = 0{\cdot}1$; ---------, $\beta = 0{\cdot}03$, $D = 0{\cdot}153$; - · - · - · - · -, $\beta = 0{\cdot}03$, $D = 0{\cdot}16$.

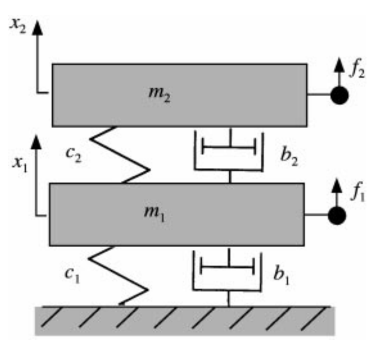

Figure 19. Linear model with two degrees of freedom.

over a period of vibration $E_f = \pi X_1 \sin \psi_0$, provided that $\psi_1 = \psi_0 = const$, and the energy dissipated by damping over the same period $E_d = 2\pi\eta(\lambda\mu D_1 X_1^2 + D_2 X_{21}^2)$. The dash–dotted line represents the sum of oscillating system potential and kinetic energy $E_p + E_k$ at any point in time. This point is chosen as $\eta\tau = 2k\pi$, $k = 0, 1, 2, \dots$; and thus $E_p = (\lambda^2 \mu X_1^2 \cos^2 \psi_1 + X_{21}^2 \cos^2 \psi_{21})/2$, $E_k = (\mu X_1^2 \sin^2 \psi_1 + X_2^2 \sin^2 \psi_2)/2$. Both curves are plotted against frequency. It is obvious from examination of the plots that the regime with

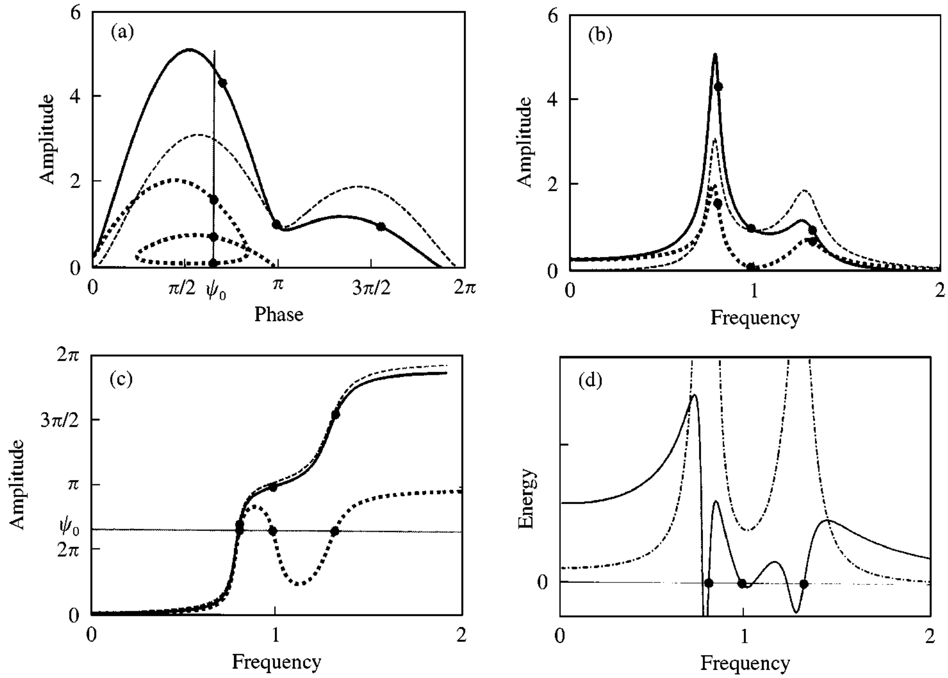

Figure 20. Two-degree-of-freedom system. Force is applied to mass 1: $D_1 = 0{\cdot}05$, $D_2 = 0{\cdot}05$, $\mu = 4$, $\lambda = 1$, ·······, mass 1; ———, mass 2.

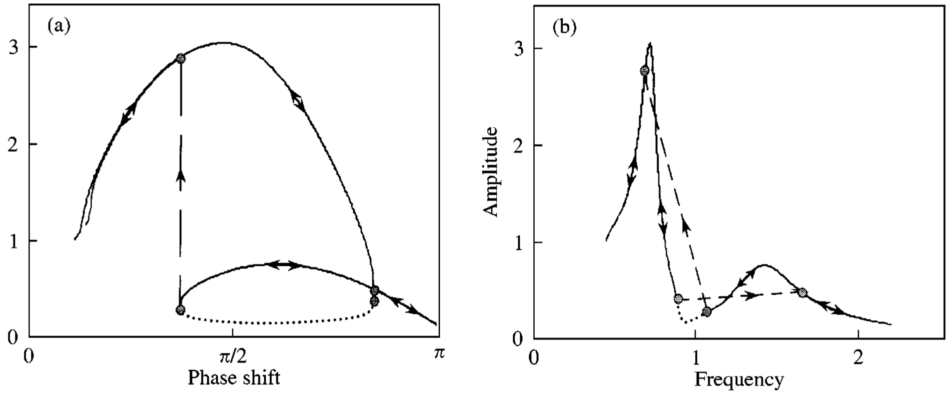

Figure 21. Two-degree-of-freedom system behaviour under phase control.

the lowest amplitude is unstable, unlike the resonant ones. Hence, two stable regimes with different amplitudes and frequencies can exist for the same phase in the considered model under phase control, when the first mass is chosen to observe an oscillation and to apply exciting force. This system under phase control demonstrates a phenomenon similar to the jump phenomenon in non-linear systems under frequency control. Figure 21 presents the results of a computer simulation for the model similar to the one shown in Figure 7, but now with two degrees of freedom. Two jumps take place under slow changing phase shift. They

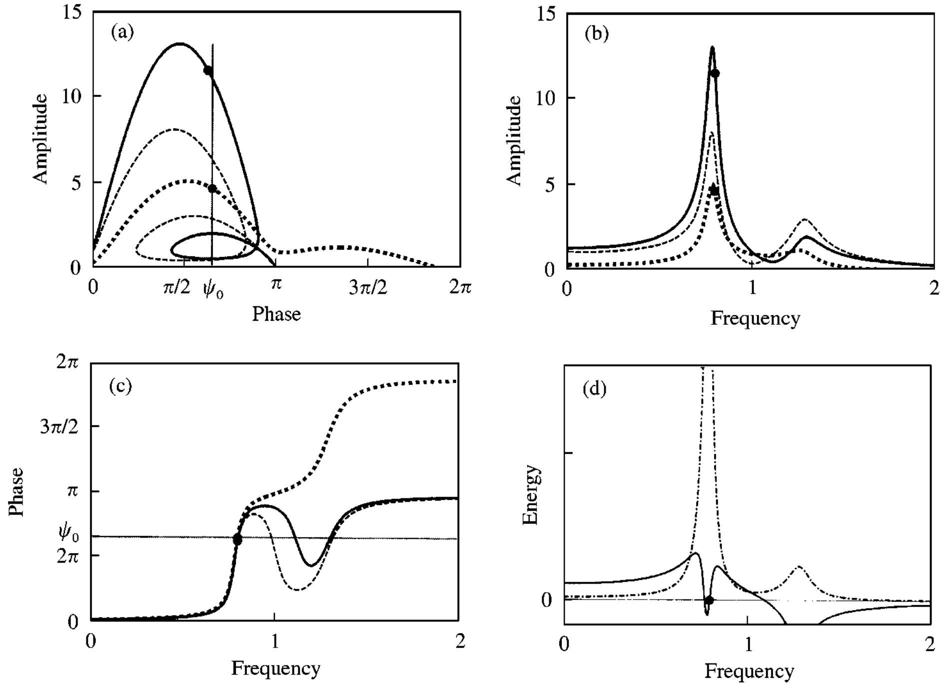

Figure 22. Two-degree-of-freedom system. Force is applied to mass 2: $D_1 = 0{\cdot}05$, $D_2 = 0{\cdot}05$, $\mu = 4$, $\lambda = 1$, ·······, mass 1; ———, mass 2.

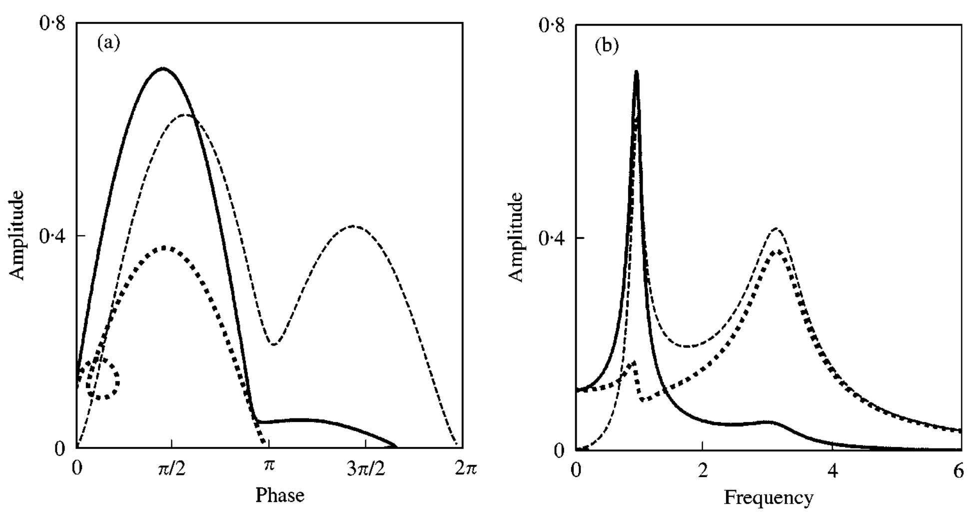

Figure 23. Two-degree-of-freedom system. Force is applied to mass 1: $D_1 = D_2 = 0{\cdot}1$, $\lambda = 3$, $\mu = 1$.

are shown with dashed lines. The dotted parts of the curves correspond to the regimes that cannot be obtained in this system under phase control.

The signal of the second body vibration or of the relative vibration of two bodies should be used in the feedback circuit to avoid this ambiguity and to get a one-to-one dependence between the phase shift and the regime in our case. The same situation takes place when $\mathbf{f} = \binom{0}{1}$, i.e., force is applied to body 2. In this case mass 1 vibration signal should be used in the feedback to get one-to-one dependence of the vibration regime on the phase shift. It can be seen from Figure 22 that in this case, only one regime of oscillation corresponds to any specific value of the phase shift ψ_2, and this regime is stable.

Under some values of two-degree-of-freedom system parameters, all the amplitude–phase curves can be single-valued within most of the phase shift interval, as is shown in Figure 23. However, in order to get precise one-to-one dependence of the vibration regime on the phase shift, an excitation point and a sensor of the vibration signal have to be positioned on the different bodies.

4. CONCLUSIONS

Phase control of the resonant vibration is robust and reliable due to the specific properties of amplitude–phase characteristics. These curves are usually bell-shaped, and this shape depends only slightly on the vibrating system and actuator parameters.

Unlike amplitude–frequency curves, amplitude–phase curves of single-degree-of-freedom systems are single-valued and flat near the resonance for many cases of practical importance, regardless of the Q-factor and non-linearity of the vibrating system.

The two-degree-of-freedom linear system has a set of amplitude–phase curves. All these curves are bell-shaped and flat near resonances. At least one of these characteristics is single-valued.

All these features determine the following advantages of the phase-controlled excitation over the traditional frequency-controlled forced excitation of the resonant vibration in various systems:

— resonant regimes are easy to maintain under changing system parameters;
— resonant regimes become stable even for non-linear systems.

In the case of the two-degree-of-freedom vibrating system, the feedback circuit design demands special attention in choosing the right points of observation and excitation.

Although only the linear viscous damping was considered in the paper, it is understood that other types of friction (pure Coulomb damping, for example) can influence the shape of amplitude–phase curves. This could be the subject of a further study.

REFERENCES

1. V. KONONENKO 1969 *Vibrating Systems with a Limited Power Supply*. London: Iliffe Books Ltd.
2. V. ASTASHEV, V. BABITSKY and M. KOLOVSKY 2000 *Dynamics and Control of Machines*. New York: Springer.
3. R. M. EVAN-IVANOVSKI 1976 *Resonance Oscillations in Mechanical Systems*. Amsterdam: Elsevier Scientific Publishing Company.
4. K. MAGNUS 1965 *Vibrations*. London: Blackie & Son Limited.
5. V. I. BABITSKY 1995 *Mechatronics* **5**, 483–495. Autoresonant mechatronic systems.
6. G. B. AIRY and W. AIRY 1896 *Autobiography of Sir George Biddell Airy*. Cambridge: University Press.
7. A. A. KHARKEVICH 1965 *Nonlinear and Parametric Phenomena in Radio Engineering*. London: Iliffe Books Ltd.
8. J. K. HWANG, C. H. CHOI and C. K. SONG 2000 *Journal of Vibration and Control* **6**, 257–272. Phase delay control of a cantilever beam.
9. I. SOKOLOV 1991 *Problems of Mechanical Engineering and Reliability of Machines* **2**, 93–99 (in Russian). On the problem of the phase control of self-sustained oscillation.
10. I. SOKOLOV 1993 *Ph.D. Thesis, Institute of Mechanical engineering of Russian Academy of Science, Moscow*: (in Russian). Autoresonant excitation of vibrating machines by synchronous rotary actuator.