*Journal of Sound and Vibration* (2001) **248**(5), 789–807
doi:10.1006/jsvi.2001.3838, available online at http://www.idealibrary.com on IDEAL®

# A NON-LINEAR INVERSE VIBRATION PROBLEM OF ESTIMATING THE EXTERNAL FORCES FOR A SYSTEM WITH DISPLACEMENT-DEPENDENT PARAMETERS

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(*Received* 31 *July* 2000, *and in final form* 4 *January* 2001)

The conjugate gradient method, i.e., the iterative regularization method, is used in an inverse non-linear force vibration problem of estimating the unknown time-dependent external forces in a damped system with the displacement-dependent spring constant and damping coefficients. The accuracy of the inverse analysis is examined by using the simulated exact and inexact measurements. Since the system parameters are functions of displacement, the present study, from a purely mathematical viewpoint, is thus a genuine non-linear inverse vibration problem. The numerical simulations are performed to test the validity of the present algorithm by using different types of system parameters, external forces and measurement errors. Results show that an excellent estimation on the external forces can be obtained with any arbitrary initial guesses with a couple of second's CPU time at Pentium III-500 MHz PC.



## 1. INTRODUCTION

The solutions of the direct problem for a non-linear damped force vibration system are concerned with the determination of the system displacement, velocity and acceleration at time $t$ when the initial conditions, external forces, displacement-dependent spring constant and damping coefficient are specified. In contrast, the non-linear inverse vibration problems for a damped system considered here involves the determination of the time-dependent external forces from the knowledge of the simulated measured displacements at different time $t$.

Inverse problems can be found in various fields of research in both science and social science problems. Many difficult but practical inverse heat transfer problems in thermal sciences have been solved by using a very powerful algorithm, i.e., the conjugate gradient method (CGM).

The CGM is also called an iterative regularization method, which means the regularization procedure is performed during the iterative processes. The CGM derives basis from the perturbational principles [1] and transforms the inverse problem to the solution of three problems, namely, the direct problem, the sensitivity problem and the adjoint problem.

Many papers regarding the use of CGM in inverse problems can be found in the open literature, for instance, Huang and Wang [2] used the CGM to estimate the unknown boundary heat fluxes in a three-dimensional inverse heat conduction problem. Huang and Chen [3] used the CGM to estimate the wall heat fluxes in a three-dimensional inverse forced heat convection problem. Huang and Chin [4] used the CGM in estimating thermal

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conductivity for a non-homogeneous medium. Moreover, it has also been used in engineering fracture mechanics. For example, Huang and Shih [5] used CGM to estimate the interfacial cracks in bimaterials, etc.

For the inverse vibration problems, the textbook by Gladwell [6] contains a general presentation of the inverse problem for undamped vibrating system. Stevens [7] has shown an overview in identifying the forces for the case of linear vibratory system. Desanghere and Snoeys [8] used a condition number in force identification problems and observed that it is a reliable indicator for ill-conditioned matrix. Bateman *et al.* [9] presented two force reconstruction techniques, i.e., the sum of weighted acceleration and the deconvolution techniques to evaluate the impact test. Michaels and Pao [10] presented an iterative method of deconvolution, which determined the inverse source problem for an oblique force on an elastic plate. More recently, Ma *et al.* [11] used the Kalman filter with a recursive estimator to determine the impulsive loads in a single-degree-of-freedom (s.d.o.f.) as well as for a multiple-degree-of-freedom (m.d.o.f.) lumped-mass systems.

In all the above references, the system parameters are all assumed as constants. Recently, Huang [12] used the CGM to solve the inverse force vibration problems in estimating the external forces. In reference [12], the system parameters are assumed as functions of time, for this reason it can be classified as a non-linear problem [13]. However, if we take a look at the problem from a purely mathematical viewpoint, it is not a genuine non-linear inverse vibration problem unless the system parameters are functions of the dependent variable, i.e., the displacement.

The purpose of the present study is to extend the technique used in reference [12] to estimate the unknown external forces for a non-linear vibration system with displacement-dependent system parameters. It is obvious that the inverse analysis of the present study will be more difficult than what had been done in reference [12].

In most considerations of vibration systems, certain idealizing assumptions such as linearity between force and displacement are taken into consideration. There is almost no real physical system that is linear under all conditions. Non-linear behavior can be the result of (1) material non-linearities (2) geometric non-linearities and (3) kinematic non-linearities. In the present study, we assumed that the system has material non-linearities, i.e., the spring constant and damping coefficient are functions of displacement.

Finally, the inverse solutions for a damped vibration problem with different types of external forces will be illustrated to show the validity of using the CGM in the present non-linear inverse vibration problem.

## 2. THE DIRECT PROBLEM

To illustrate the methodology for developing expressions for use in determining unknown time-dependent external forces in a genuine non-linear damped vibration system with displacement-dependent system parameters, (i.e., the damping coefficient $C(x)$ and spring constant $K(x)$ are both functions of displacement), we consider the following damped force vibration problem.

The initial displacement and velocity conditions of the system are $x(0) = x_0$ and $\mathrm{d}x(0)/\mathrm{d}t = y(0) = y_0$ respectively. When $t > 0$, the system parameters $K(x)$ and $C(x)$ are given, moreover, the time-dependent external forces $f(t)$ are also assumed to be known.

The system under consideration here is shown in Figure 1 and the mathematical formulation of this damped forced vibration problem is given by

$$M\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = -\,C(x)\frac{\mathrm{d}x(t)}{\mathrm{d}t} - K(x)x(t) + f(t), \quad t > 0 \tag{1a}$$

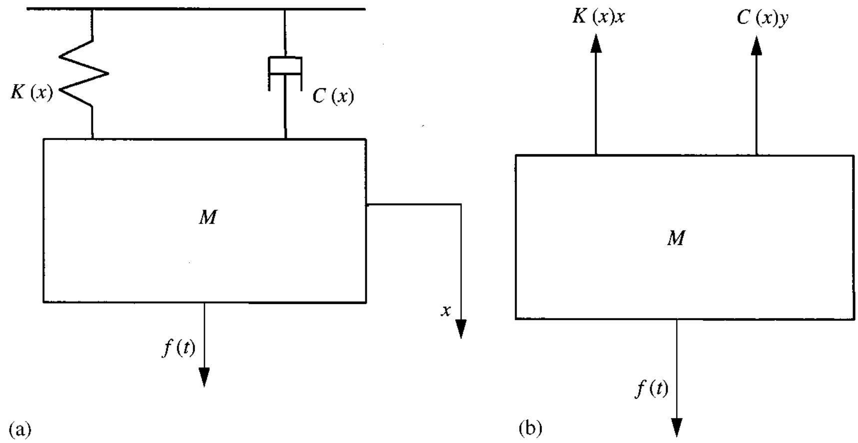


Figure 1. A non-linear force vibration system considered in the present study.

with initial conditions

$$x(0) = x_0 \quad \text{and} \quad \frac{\mathrm{d}x(0)}{\mathrm{d}t} = y(0) = y_0. \tag{1b}$$

Here, $M$ represents the mass of the system. There exists no exact solution for equation (1) for any arbitrary function of $K(x)$ and $C(x)$. For this reason, the numerical solution with the technique of the fourth order Runge–Kutta method will be applied to solve equation (1) by reducing it into two coupled first order differential equation as shown below:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = y(t), \quad t > 0, \quad x(0) = x_0, \tag{2a}$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = -\frac{C(x)}{M}y - \frac{K(x)}{M}x(t) + \frac{f(t)}{M}, \quad t > 0, \quad y(0) = y_0. \tag{2b}$$

The above direct problem considered here is concerned with the determination of the system displacement $x(t)$ and velocity $y(t)$ when the initial conditions, the displacement-dependent system parameters $K(x)$ and $C(x)$ and the time-dependent external forces $f(t)$ are all given.

Here, the fourth order Runge–Kutta method is used to solve the system of equations (2a) and (2b). Moreover, we should note that some special treatment should be considered for this genuine non-linear problem and this treatment will be stated as follows.

When solving equations (2a) and (2b) with Runge–Kutta method from time $t$ to $t + \Delta t$ for displacement $x(t + \Delta t)$ and velocity $y(t + \Delta t)$, the values of $K[x(t + \Delta t)]$ and $C[x(t + \Delta t)]$ should be given *a priori*. However, $K[x(t + \Delta t)]$ and $C[x(t + \Delta t)]$ are still unknown values at this stage since $x(t + \Delta t)$ and $y(t + \Delta t)$ are both unknown. For this reason, we should first use $K[(x(t)]$ and $C[x(t)]$ as the initial guesses and solve equations (2a) and (2b) for $x(t + \Delta t)$ and $y(t + \Delta t)$. Those solved values of $x(t + \Delta t)$ and $y(t + \Delta t)$ are still not the correct answer since the new calculated $K[x(t + \Delta t)]$ and $C[x(t + \Delta t)]$ are definitely not equal to the initial guesses of $K[x(t)]$ and $C[x(t)]$, therefore equation (2b) cannot be satisfied. We should then use the new calculated $K[x(t + \Delta t)]$ and $C[x(t + \Delta t)]$ as second guesses and resolve equations (2a) and (2b) again. One should repeat the above procedure until the difference between the values of new $K$, $C$, and old $K$, $C$, are converged to a specified value.

## 3. THE INVERSE PROBLEM

For the inverse problem, the time-dependent external forces $f(t)$ are regarded as being unknown, but everything else in equations (2a) and (2b) are known. In addition, system displacement and velocity measured at some appropriate time are considered available.

Let the measured system displacement be denoted by $X(t)$, here $t = 0$ to $t_f$, where $t_f$ represents the final time of the measurements. Then the inverse problem can be stated as follows: by utilizing the above-mentioned measured system displacement data $X(t)$ to estimate the unknown time-dependent external forces $f(t)$.

In the present study, we have not used real measured system displacement and velocity, rather, we used the exact external forces to generate the simulated values of measured displacement $X(t)$, then tried to retrieve the time-dependent external forces by using $X(t)$ and the initial guesses of external forces.

The solution of the present inverse vibration problem is to be obtained in such a way that the following functional is minimized:

$$J[f(t)] = \int_{t=0}^{t_f} [x(t) - X(t)]^2 \,\mathrm{d}t, \tag{3}$$

here, $x(t)$ is the estimated or computed displacement at time $t$. These quantities are determined from the solution of the direct problem given previously by using an estimated $\hat{f}(t)$ for the exact $f(t)$. Here, the hat " $\wedge$ " denotes the estimated quantities.

## 4. CONJUGATE GRADIENT METHOD FOR MINIMIZATION

The following iterative process based on the conjugate gradient method [1] is now used for the estimation of time-dependent external forces $f(t)$ by minimizing the functional $J[f(t)]$

$$\hat{f}^{n+1}(t) = \hat{f}^{n}(t) - \beta^n P^n(t) \quad \text{for } n = 0, 1, 2, \ldots, \tag{4}$$

where $\beta^n$ is the search step size in going from iteration $n$ to iteration $n+1$, and $P^n(t)$ are the directions of descent (i.e., search direction) given by

$$P^n(t) = J'^n(t) + \gamma^n P^{n-1}(t), \tag{5}$$

which are a conjugation of the gradient direction $J'^n(t)$ at iteration $n$ and the direction of descent $P^{n-1}(t)$ at iteration $n-1$. The conjugate coefficient is determined from

$$\gamma^n = \frac{\int_{t=0}^{t_f} (J'^n)^2 \,\mathrm{d}t}{\int_{t=0}^{t_f} (J'^{n-1})^2 \,\mathrm{d}t} \quad \text{with } \gamma^0 = 0. \tag{6}$$

We note that when $\gamma^n = 0$ for any $n$, in equation (6), the direction of descent $P^n(t)$ becomes the gradient direction, i.e., the "steepest descent" method is obtained. The convergence of the above iterative procedure in minimizing the functional $J$ is guaranteed in reference [14].

To perform the iterations according to equation (4), we need to compute the step size $\beta^n$ and the gradient of the functional $J'^n(t)$. In order to develop expressions for the determination of these two quantities, a "sensitivity problem" and an "adjoint problem" are constructed as described below.

### 4.1. SENSITIVITY PROBLEM AND SEARCH STEP SIZE

The sensitivity problem is obtained from the original direct problem defined by equations (2a) and (2b) in the following manner: it is assumed that when $f(t)$ undergoes a variation $\Delta f(t)$, $x(t)$, $y(t)$, $C(x)$ and $K(x)$ are perturbed by $\Delta x$, $\Delta y$, $\Delta C$ and $\Delta K$ respectively. Then replacing in the direct problem $f$ by $f+\Delta f$, $x$ by $x+\Delta x$, $y$ by $y+\Delta y$, $C$ by $C+\Delta C$ and $K$ by $K+\Delta K$, subtracting from the resulting expressions the direct problem, using the fact that $\mathbf{\Delta} C = (\mathrm{d}C/\mathrm{d}x)\,\mathbf{\Delta} x$ and $\mathbf{\Delta} K = (\mathrm{d}K/\mathrm{d}x)\mathbf{\Delta} x$ and neglecting the second order terms, the following sensitivity problems for the sensitivity functions $\Delta x$ and $\Delta y$ are obtained.

$$\frac{\mathrm{d}\Delta x(t)}{\mathrm{d}t} = \Delta y(t), \quad t>0, \quad \Delta x(0)=0, \tag{7a}$$

$$\frac{\mathrm{d}\Delta y(t)}{\mathrm{d}t} = -\frac{C(x)}{M}\Delta y(t) - \left[\frac{K(x)}{M} + \frac{(\mathrm{d}K/\mathrm{d}x)x}{M} + \frac{(\mathrm{d}C/\mathrm{d}x)\,y}{M}\right] \times \Delta x(t) + \frac{\Delta f(t)}{M}, \quad t>0, \quad \Delta y(0)=0. \tag{7b}$$

We should note that, from a purely mathematical viewpoint, equations (7a) and (7b) are not non-linear since $x$ is no longer the dependent variable, thus the standard technique of fourth order Runge–Kutta method can be used to solve these sensitivity problems, i.e., special iterative treatment is unnecessary here.

The functional $J(\hat{f}^{n+1})$ for iteration $n+1$ is obtained by rewriting equation (3) as

$$J[\hat{f}(t)] = \int_{t=0}^{t_f} [x(\hat{f}^n - \beta^n P^n) - X(t)]^2 \,\mathrm{d}t, \tag{8}$$

where we replaced $\hat{f}^{n+1}(t)$ by the expression given by equation (4). If the estimated displacement $x(\hat{f}^n - \beta^n P^n)$ is linearized by a Taylor expansion, equation (8) takes the form

$$J[\hat{f}^{n+1}(t)] = \int_{t=0}^{t_f} [x(\hat{f}^n) - \beta^n \Delta x(P^n) - X(t)]^2 \,\mathrm{d}t, \tag{9}$$

where $x(\hat{f}^n)$ is the solution of the direct problem by using estimate $\hat{f}^n(t)$ for exact $f(t)$ at time $t$. The sensitivity function $\Delta x(P^n)$ is taken as the solution of problem (7a) and (7b) at time $t$ by letting $\Delta f(t) = P^n(t)$ in equation (7b) [1].

Equation (9) is differentiated with respect to $\beta^n$ and equating them equal to zero. Finally, the step size can be obtained as

$$\beta^n = \frac{\int_{t=0}^{t_f} \Delta x(P^n)\,[x(\hat{f}^n) - X]\,\mathrm{d}t}{\int_{t=0}^{t_f} [\Delta x^2(P^n)]\,\mathrm{d}t}. \tag{10}$$

### 4.2. ADJOINT PROBLEM AND GRADIENT EQUATION

To obtain the adjoint problems when perturbing $f(t)$, equations (2a) and (2b) are multiplied by the Lagrange multipliers (or adjoint functions) $\lambda_1(t)$ and $\lambda_2(t)$ respectively. The resulting expression is integrated over the correspondent time domain, then the result is added to the right-hand side of equation (3) to yield the following expression for the functional

$$J[f(t)] = \int_{t=0}^{t_f} [x(t) - X(t)]^2 \, \mathrm{d}t + \int_{t=0}^{t_f} \lambda_1(t) \left[\frac{\mathrm{d}x(t)}{\mathrm{d}t} - y(t)\right] \mathrm{d}t + \int_{t=0}^{t_f} \lambda_2(t) \left[\frac{\mathrm{d}y(t)}{\mathrm{d}t} + \frac{C(x)}{M} y(t) + \frac{K(x)}{M} x(t) - \frac{f(t)}{M}\right] \mathrm{d}t. \tag{11}$$

The variation $\Delta J$ is obtained by perturbing $f$ by $f + \Delta f$, $x$ by $x + \Delta x$, $y$ by $y + \Delta y$, $C$ by $C + \Delta C$ and $K$ by $K + \Delta K$ in equation (11), subtracting the original equation (11), from the resulting expression using the fact that $\Delta C = (\mathrm{d}C/\mathrm{d}x)\Delta x$ and $\Delta K = (\mathrm{d}K/\mathrm{d}x)\Delta x$ and neglecting the second order terms. We thus find

$$\Delta J[f(t)] = 2\int_{t=0}^{t_f} [x(t) - X(t)]\Delta x \, \mathrm{d}t + \int_{t=0}^{t_f} \lambda_1(t) \left[\frac{\mathrm{d}\Delta x(t)}{\mathrm{d}t} - \Delta y(t)\right] \mathrm{d}t + \int_{t=0}^{t_f} \lambda_2(t) \left\{\frac{\mathrm{d}\Delta y(t)}{\mathrm{d}t} + \frac{C(x)}{M}\Delta y(t) + \left[\frac{K(x)}{M} + \frac{(\mathrm{d}K/\mathrm{d}x)x}{M} + \frac{(\mathrm{d}C/\mathrm{d}x)y}{M}\right] \times \Delta x(t) - \frac{\Delta f(t)}{M}\right\} \mathrm{d}t. \tag{12}$$

In equation (12), the second and third integral terms are integrated by parts; the initial condition of the sensitivity problem are utilized. The vanishing of the integrands leads to the following adjoint problems for the determination of $\lambda_1(t)$ and $\lambda_2(t)$:

$$-\frac{\mathrm{d}\lambda_1(t)}{\mathrm{d}t} = -\left[\frac{K(x)}{M} + \frac{(\mathrm{d}K/\mathrm{d}x)x}{M} + \frac{(\mathrm{d}C/\mathrm{d}x)y}{M}\right]\lambda_2(t) - 2(x - X), \quad t > 0,\ \lambda_1(t_f) = 0, \tag{13a}$$

$$-\frac{\mathrm{d}\lambda_2(t)}{\mathrm{d}t} = -\frac{C(x)}{M}\lambda_2(t) + \lambda_1(t), \quad t > 0,\ \lambda_2(t_f) = 0. \tag{13b}$$

The adjoint problems are different from the standard initial value problems in that the final time conditions at time $t = t_f$ is specified instead of the customary initial condition. However, this problem can be transformed into an initial value problem by the transformation of the time variables as $\tau = t_f - t$.

Again, from a purely mathematical viewpoint, adjoint equations (13a) and (13b) are not non-linear, the standard techniques of fourth order Runge–Kutta method can be used to solve the above adjoint problems.

Finally, the following integral term is left:

$$\Delta J = \int_{t=0}^{t_f} -(\lambda_2 \Delta f/M) \, \mathrm{d}t. \tag{14}$$

From definition [1], the functional increment can be presented as

$$\Delta J = \int_{t=0}^{t_f} (J' \Delta f)\,\mathrm{d}t. \tag{15}$$

A comparison of equations (14) and (15) leads to the following expression for the gradient of functional $J'$:

$$J'[f(t)] = \frac{-\lambda_2(t)}{M}. \tag{16}$$

### 4.3. STOPPING CRITERION

If the problem contains no measurement errors, the traditional check condition is specified as

$$J[\hat{f}^{n+1}(t)] < \varepsilon, \tag{17}$$

where $\varepsilon$ is a small specified number. However, the measured displacement data may contain measurement errors. Therefore, we do not expect the functional equation (3) to be equal to zero at the final iteration step. Following the experiences of the authors [1–5], we use the discrepancy principle as the stopping criterion, i.e., we assume that the residuals for the displacement may be approximated by

$$x(t) - X(t) \approx \sigma, \tag{18}$$

where $\sigma$ is the standard deviation of the displacement measurements, which is assumed to be a constant. By substituting equation (18) into equation (3), the following expression is obtained for stopping criteria:

$$\varepsilon = \sigma^2 t_f. \tag{19}$$

Then, the stopping criterion is given by equation (17) with $\varepsilon$ determined from equation (19).

## 5. COMPUTATIONAL PROCEDURE

The computational procedure for the solution of this non-linear inverse problem using conjugate gradient method may be summarized as follows:

Suppose $\hat{f}^n(t)$ is available at iteration $n$.

Step 1. Solve the direct problems given by equations (2a) and (2b) for $x(t)$ and $y(t)$.

Step 2. Examine the stopping criterion given by equation (19). Continue the iteration if not satisfied.

Step 3. Solve the adjoint problems given by equations (13a) and (13b) for $\lambda_1(t)$ and $\lambda_2(t)$.

Step 4. Compute the gradient of the functional $J'(t)$ from equation (16).

Step 5. Compute the conjugate coefficients $\gamma^n$ and direction of descent $P^n$ from equations (6) and (5) respectively.

Step 6. Set $\Delta f = P^n$. Then solve the sensitivity problems given by equations (7a) and (7b) for $\Delta x$ and $\Delta y$.

Step 7. Compute the search step size $\beta^n$ from equation (10).
Step 8. Compute the new estimations for $\hat{f}^{n+1}(t)$ from equation (4) and return to step 1.

## 6. RESULTS AND DISCUSSION

The objective of this work is to show the validity of the CGM in estimating the external forces $f(t)$ in the inverse non-linear force vibration problems with no prior information on the functional form of the unknown quantities.

To illustrate the accuracy of the conjugate gradient method in predicting external forces $f(t)$ in a damped vibration problem from the knowledge of transient displacement recordings, three specific examples having different forms of system parameters and external forces are considered here.

In order to compare the results for situations involving random measurement errors, we assume normally distributed uncorrelated errors with zero mean and constant standard deviation. The simulated inexact measurement displacement **X** can be expressed as

$$\mathbf{X} = \mathbf{X}_{exact} + \omega\sigma, \tag{20}$$

where $\mathbf{X}_{exact}$ is the solution of the direct vibration problem with exact external forces $f(t)$; $\sigma$ is the standard deviation of the measured displacement and $\omega$ is a random variable that is generated by subroutine DRNNOR of the IMSL [15] and will be within $-2{\cdot}576$–$2{\cdot}576$ for a 99% confidence bound.

One of the advantages of using the conjugate gradient method to solve the inverse problems is that the initial guesses of the unknown quantities can be chosen arbitrarily. In all the test cases considered here, the initial guesses of $\hat{f}(t)$ are taken as $\hat{f}(t)_{initial} = 0{\cdot}0$.

### 6.1. NUMERICAL TEST CASE 1

We now present below the numerical experiments in determining $f(t)$ by the inverse analysis using the CGM.

The parameters that are used in the test case 1 are taken as

$$M = 1{\cdot}0\,\text{kg}, \qquad K(x) = 15{\cdot}0 + 0{\cdot}5 \times \sin(10x)\,\text{N/m} \quad \text{and} \quad C(x) = 3{\cdot}0 + 0{\cdot}5 \times \cos(10x)\,\text{N s/m}.$$

The initial conditions for displacement and velocity are both assumed to be zero, i.e., $x(0) = 0$ and $y(0) = 0$. Time interval is chosen as 120 s, i.e., $t_f = 120\,\text{s}$, and a time step $\Delta t = 1\,\text{s}$ is used. Therefore, a total of 120 unknown discretized external forces are to be determined in the present study. The number of measured displacement is also 120. The unknown transient external forces are assumed as

$$f(t) = 10{\cdot}0 + 1{\cdot}0 \times \sin\left(\frac{2\pi t}{120}\right)\text{N} \quad \text{for } 0 < t \leqslant 120\,\text{s}. \tag{21}$$

The non-linear inverse analysis is first performed by using the displacement measurements and assuming no measurement errors, i.e., $\sigma = 0{\cdot}0$. When the stopping criteria is set as $\varepsilon = 10^{-5}$, after only 10 iterations the inverse solutions are converged, $J$ is calculated as $7{\cdot}3 \times 10^{-7}$ and CPU time at Pentium III-500 MHz PC is about 2 s. The measured and estimated displacements, $X(t)$ and $x(t)$ are shown in Figure 2 while Figure 3 shows the exact and estimated external forces, $f(t)$ and $\hat{f}(t)$ respectively.

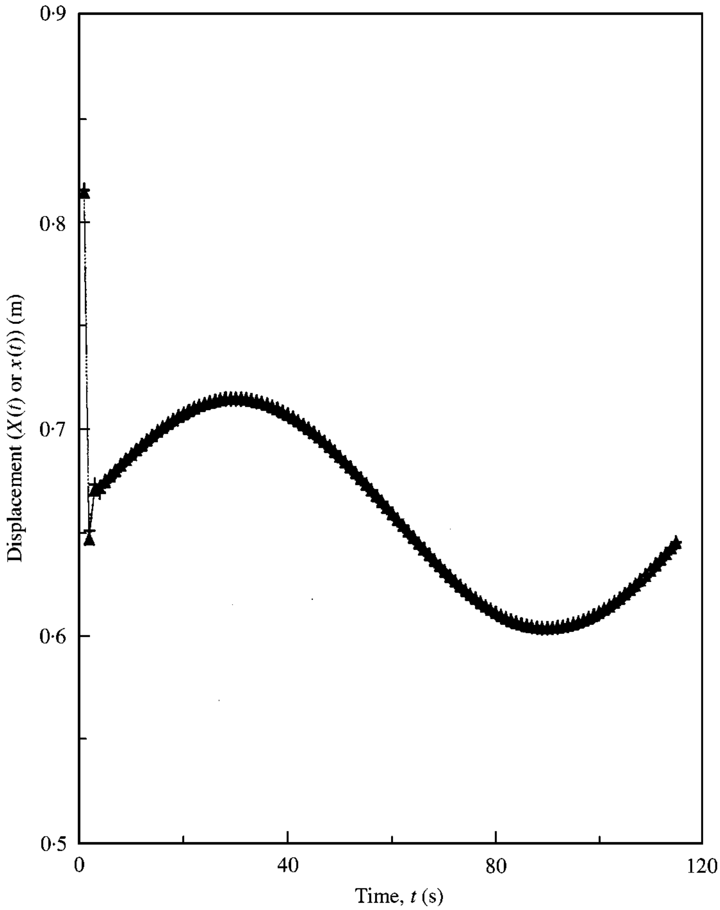


Figure 2. The measured $X(t)$, estimated $x(t)$ and estimated $x_1(t)$ with $\sigma = 0{\cdot}0$ in numerical test case 1: ——, measured displacement, $X(t)$; – –▲– –, estimated displacement, $x(t)$; – + –, estimated displacement, $x_1(t)$.

The average errors for the estimated external forces and displacements are ERR1 = 0·005% and ERR2 = 0·002%, respectively, where the definition for ERR1 and ERR2 is given as

$$\mathrm{ERR1}\% = \left[\sum_{n=1}^{N}\left|\frac{f(t_n)-\hat{f}(t_n)}{f(t_n)}\right|\right] \Big/ N \times 100\% \tag{22a}$$

and

$$\mathrm{ERR2}\% = \left[\sum_{n=1}^{N}\left|\frac{X(t_n)-x(t_n)}{X(t_n)}\right|\right] \Big/ N \times 100\%. \tag{22b}$$

From these figures, we concluded that the present algorithm has been applied successfully in the inverse vibration problem in estimating external forces since the estimated results are very accurate.

It may also be interesting to compare the differences between the estimated results for this non-linear analysis and the estimated results form linear analysis (i.e., assumed constant

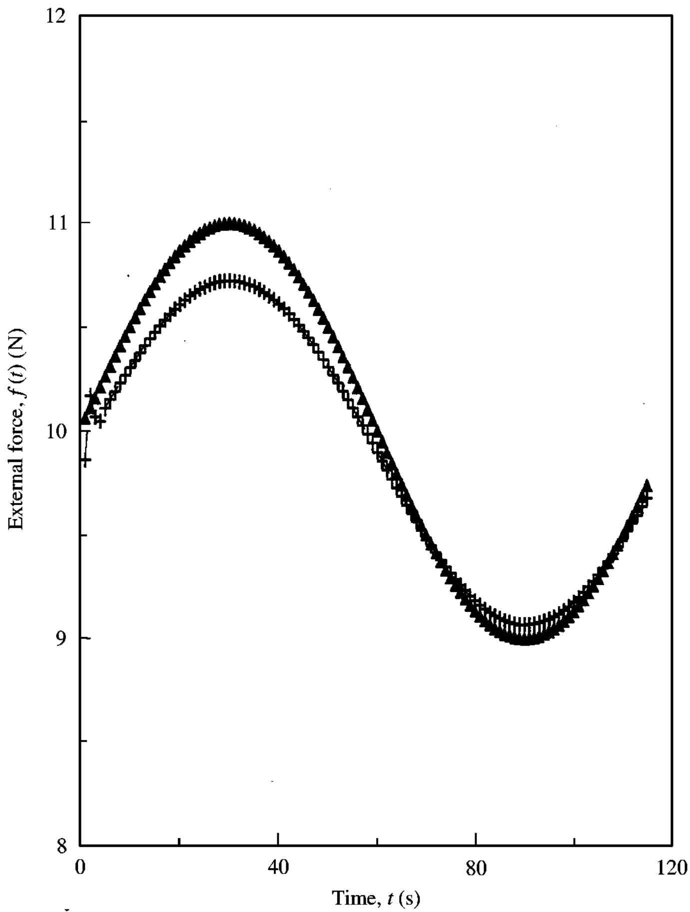


Figure 3. The exact ($f$), estimated $\hat{f}$ and estimated $\hat{f}_1$ using displacement measurements with $\sigma = 0{\cdot}0$ in numerical test case 1: ———, exact force, $f(t)$; –▲–, estimated force, $\hat{f}(t)$; – + –, estimated force, $\hat{f}_1(t)$.

system parameters, $K = 15$ and $C = 3$) based on same displacement measurements. The estimated displacement $x_1(t)$ and external forces $\hat{f}_1(t)$ are also shown in Figures 2 and 3 respectively. It is obvious that the requirement for displacements can be satisfied quite well for this linear analysis but the estimated external forces somehow deviate from the exact value of external force. Only the non-linear inverse analysis can estimate accurate unknown external forces for the non-linear vibration problem.

Next, let us discuss the influence of the measurement errors on the inverse solutions. When the standard deviation of measurement error for the displacements measured by sensors is taken as $\sigma = 0{\cdot}006$ (about 1% of the average measured displacement), then the stopping criteria $\varepsilon$ can be calculated from equation (19). After four iterations (CPU time is about 1 s), the inverse solutions can be obtained and plotted in Figure 4 for the measured and estimated displacements and in Figure 5 for the exact and estimated external forces. The average errors for the estimated external forces and displacements are ERR1 $= 0{\cdot}69\%$ and ERR2 $= 0{\cdot}065\%$ respectively.

Then, the measurement error is increased to $\sigma = 0{\cdot}018$ (about 3% of the average measured displacement). After three iterations (CPU time is about 1 s), the inverse solutions can be obtained. In Figure 6, the measured and estimated displacements are shown and in

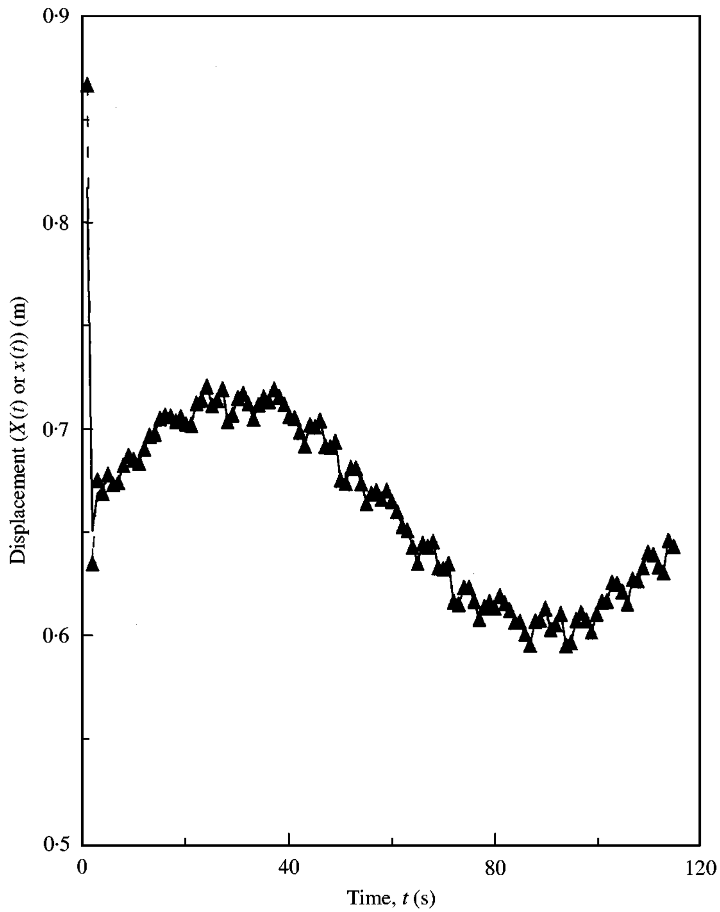


Figure 4. The measured ($X$) and estimated ($x$) displacement with $\sigma = 0{\cdot}006$ in numerical test case 1: ———, measured, $X(t)$; –▲–, estimated, $x(t)$.

Figure 7 the exact and estimated external forces are plotted. The average errors for the estimated external forces and displacements are ERR1 = 2·22% and ERR2 = 0·39% respectively.

From the above figures and data, we learned that the accuracy of the solution would get worse as the measurement error increased.

### 6.2. NUMERICAL TEST CASE 2

The system parameters that were used in test case 2 are now taken as

$$M = 1{\cdot}0\ \text{kg}, \quad K(x) = 15{\cdot}0 + 0{\cdot}5x + 0{\cdot}02x^2\ \text{N/m} \quad \text{and} \quad C(x) = 3{\cdot}0 + 0{\cdot}5x - 0{\cdot}01x^2\ \text{N s/m}.$$

The initial conditions are taken as $x(0) = 0$ and $y(0) = 0$. Time interval is also chosen as 120 s and a time step $\Delta t = 1$ s is used. Therefore, a total of 120 unknown discretized external forces are to be determined in the present study. The number of measured displacements is 120.

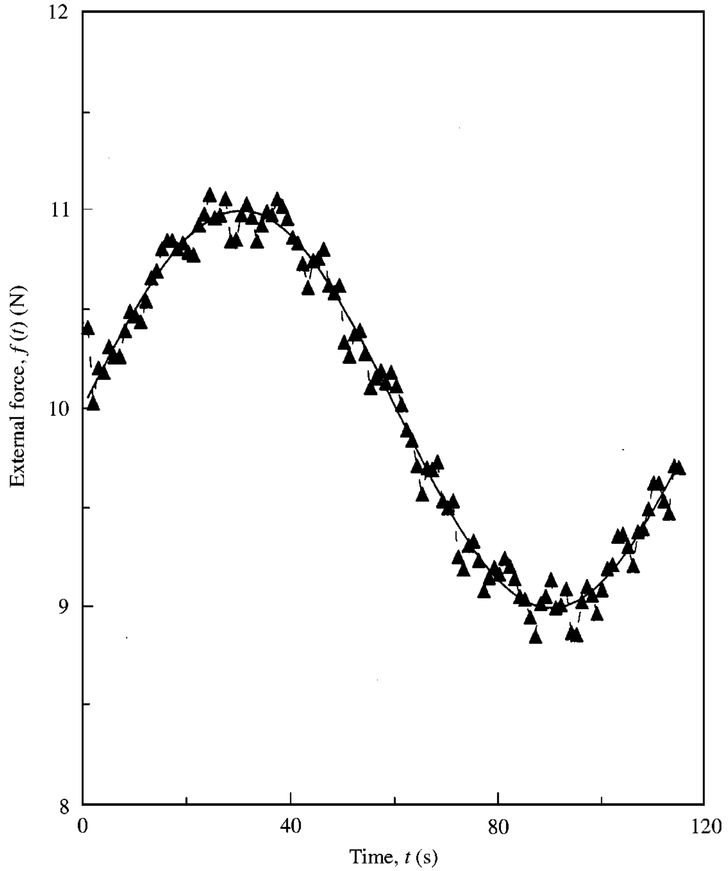


Figure 5. The exact ($f$) and estimated ($\hat{f}$) external forces using displacement measurements with $\sigma = 0{\cdot}006$ in numerical test case 1: ———, exact; – ▲ –, estimated.

The unknown transient external forces $f(t)$ are assumed as the following functions:

$$f(t) = \begin{cases} 1{\cdot}0 + 10 \times \sin\left(\dfrac{2\pi t}{50}\right) \mathrm{N} & 0 < t \leqslant 40\,\mathrm{s}, \\ 1{\cdot}0 + 0{\cdot}3t\,\mathrm{N} & 41 < t \leqslant 50\,\mathrm{s}, \\ 20{\cdot}0\,\mathrm{N} & 51 < t \leqslant 70\,\mathrm{s}, \\ 15{\cdot}0 - 0{\cdot}1t\,\mathrm{N} & 71 < t \leqslant 80\,\mathrm{s}, \\ 10 \times \cos\left(\dfrac{2\pi t}{50}\right) \mathrm{N} & 81 < t \leqslant 120\,\mathrm{s}. \end{cases} \tag{23}$$

The inverse analysis is first performed by using exact displacement measurements, i.e., assuming no measurement errors $\sigma = 0{\cdot}0$. When the stopping criteria is set as $\varepsilon = 10^{-3}$, after six iterations the inverse solutions are converged, $J$ is calculated as $1{\cdot}4 \times 10^{-4}$ and CPU time is about 1·5 s. The average errors for the estimated external forces and

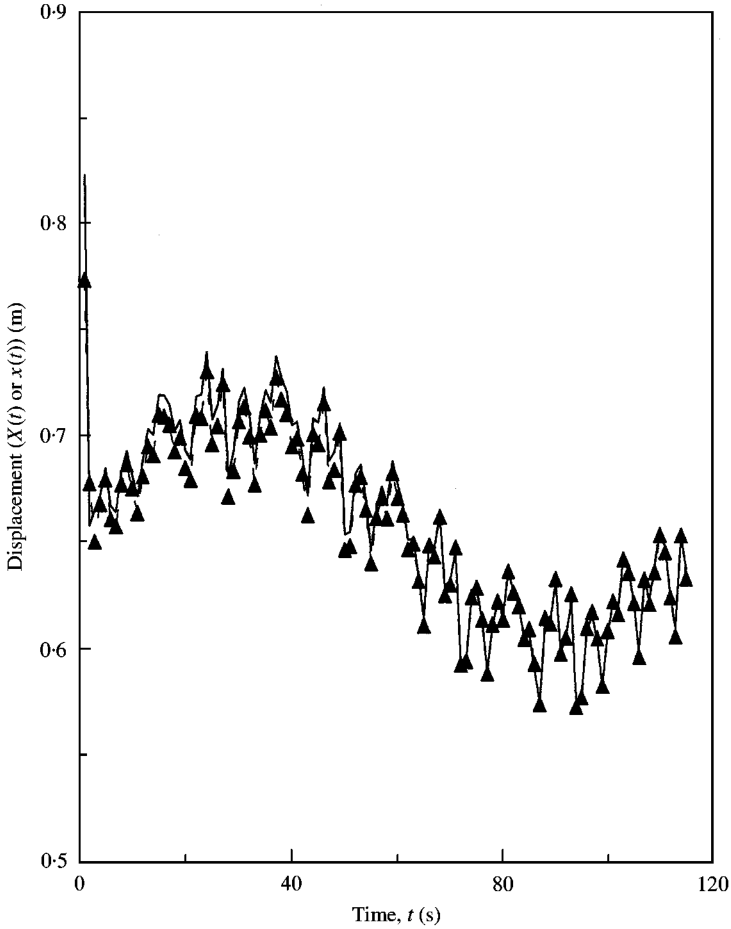


Figure 6. The measured ($X$) and estimated ($x$) displacement with $\sigma = 0{\cdot}018$ in numerical test case 1: ———, measured, $X(t)$; –▲–, estimated, $x(t)$.

displacements are ERR1 $= 0{\cdot}12\%$ and ERR2 $= 0{\cdot}091\%$ respectively. The exact and estimated external forces are shown in Figure 8. From this figure, we know that for this test the estimated external forces are still very accurate.

Next, when the standard deviation of measurement error for the displacements measured by sensors is taken as $\sigma = 0{\cdot}02$ (about 5% of the average measured displacement), then the stopping criteria $\varepsilon$ can be calculated from equation (19). The inverse solutions are converged after only four iterations and CPU time is about 1 s. The average errors for the estimated external forces and displacements are ERR1 $= 4{\cdot}26\%$ and ERR2 $= 0{\cdot}59\%$. The exact and estimated external forces are plotted in Figure 9. Again, from these figures, we learned that reliable inverse solutions can still be obtained when the large measurement errors are considered.

### 6.3. NUMERICAL TEST CASE 3

The system parameters that were used in test case 2 are now taken as

$$M = 2{\cdot}0\,\text{kg}, \qquad K(x) = 20{\cdot}0 + 0{\cdot}2x + 0{\cdot}02x^2\ \text{N/m} \quad \text{and} \quad C(x) = 1{\cdot}0 + 0{\cdot}3x - 0{\cdot}005x^2\ \text{N s/m}.$$

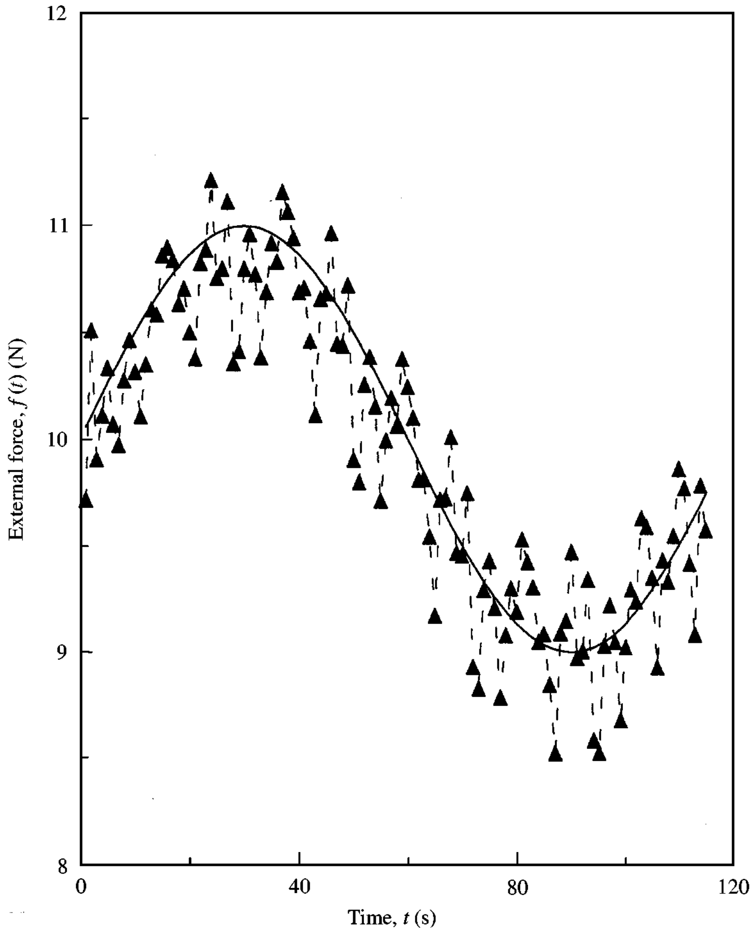


Figure 7. The exact ($f$) and estimated ($\hat{f}$) external forces using displacement measurements with $\sigma = 0{\cdot}018$ in numerical test case 1: ———, exact; –▲–, estimated.

The initial conditions are taken as $x(0) = 0$ and $y(0) = 0$. Time interval is also chosen as 120 s and a time step $\Delta t = 1$ s is used. Therefore, a total of 120 unknown discretized external forces are to be determined in the present study. The number of measured displacements is 120.

The unknown transient external forces $f(t)$ are assumed as the impulse function, which means large external forces will be applied to the system for only certain time and forces $f(t) = 10{\cdot}0$ for the rest of the time. The impulse function can be expressed as:

$$f(t) = \begin{cases} 40\delta(t-15)\,\mathrm{N} \\ -20\delta(t-30)\,\mathrm{N} \\ 30\delta(t-60)\,\mathrm{N} \quad \text{for } 0 < t \leqslant 120\,\mathrm{s}, \\ -30\delta(t-80)\,\mathrm{N} \\ 25\delta(t-100)\,\mathrm{N} \end{cases} \tag{24}$$

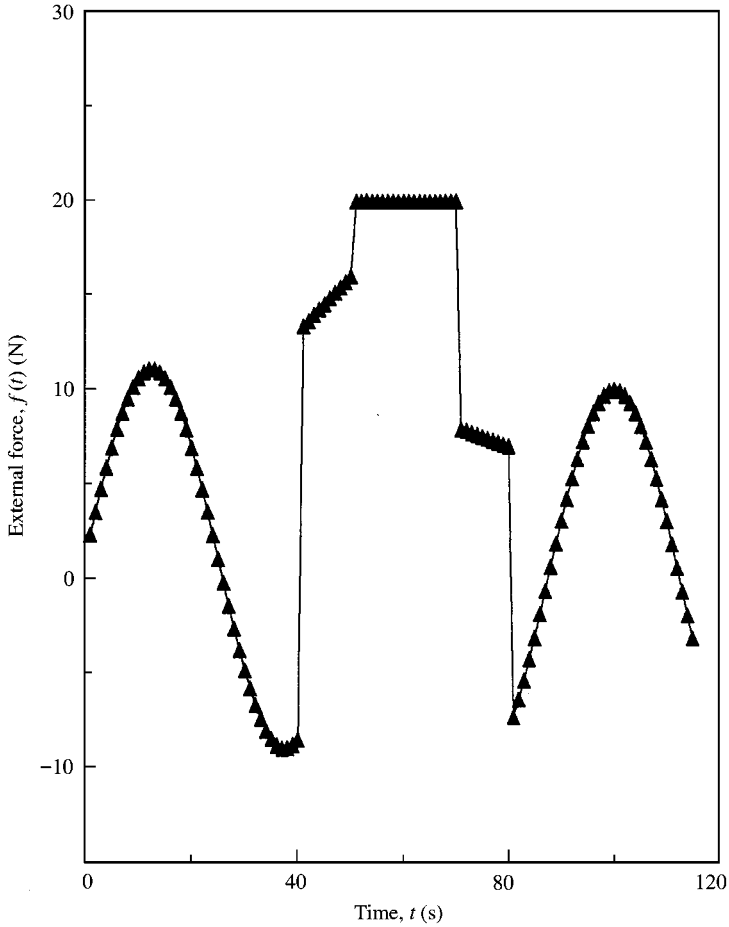


Figure 8. The exact ($f$) and estimated ($\hat{f}$) external forces using displacement measurements with $\sigma = 0{\cdot}0$ in numerical test case 2: ———, exact; –▲–, estimated.

where $\delta(\bullet)$ is the Dirac delta function. The inverse analysis is first performed by using exact displacement measurements, i.e., assuming no measurement errors $\sigma = 0{\cdot}0$. When the stopping criteria is set as $\varepsilon = 10^{-3}$, after seven iterations the inverse solutions are converged, $J$ is calculated as $6{\cdot}9 \times 10^{-4}$ and CPU time is about 2 s. The average errors for the estimated external forces and displacements are ERR1 = 0·32% and ERR2 = 0·21%. The exact and estimated external forces are shown in Figure 10. From this figure, we know that for this test the estimated external forces are still very accurate.

Next, when the standard deviation of measurement error for the displacements measured by sensors is taken as $\sigma = 0{\cdot}025$ (about 5% of the average measured displacement), then the stopping criteria $\varepsilon$ can be calculated from equation (19). The inverse solutions are converged after only five iterations and CPU time is about 1 s. The average errors for the estimated external forces and displacements are ERR1 = 3·7% and ERR2 = 1·0%. The exact and estimated external forces are plotted in Figure 11. Again, from these figures, we learned that reliable inverse solutions can still be obtained when the large measurement errors are considered.

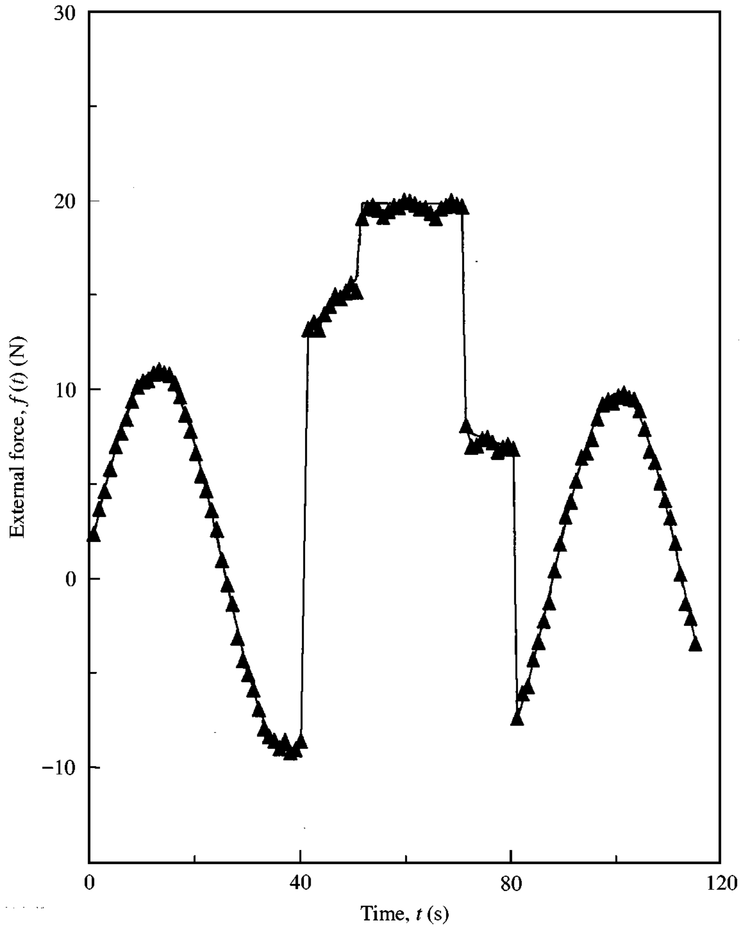


Figure 9. The exact ($f$) and estimated ($\hat{f}$) external forces using displacement measurements with $\sigma = 0{\cdot}02$ in numerical test case 2: ———, exact; –▲–, estimated.

To test the influence of the non-linear behavior of $K(x)$ and $C(x)$ on the present inverse problem, we increase the non-linearity of $K(x)$ and $C(x)$ as

$$K(x) = 20{\cdot}0 + 1{\cdot}0x + 0{\cdot}02x^2\,\mathrm{N/m} \quad \text{and} \quad C(x) = 1{\cdot}0 + 0{\cdot}5x - 0{\cdot}005x^2\,\mathrm{N\,s/m}.$$

By assuming no measurement errors $\sigma = 0{\cdot}0$ and setting the stopping criteria $\varepsilon = 10^{-3}$, after 67 iterations the inverse solutions are converged. $J$ is calculated as $9{\cdot}9 \times 10^{-4}$ and CPU time is about 19 s. The average errors for the estimated external forces and displacements are ERR1 = 0·42% and ERR2 = 0·28%. It is obvious that as the non-linear behavior of $K(x)$ and $C(x)$ increased, more iterations and computer time are needed, however, the solution of the inverse problem is still very accurate.

From the above three test cases, we learned that an inverse non-linear damped force vibration problem in estimating unknown external forces is now complete. Reliable estimations can be obtained when using either exact or error measurements.

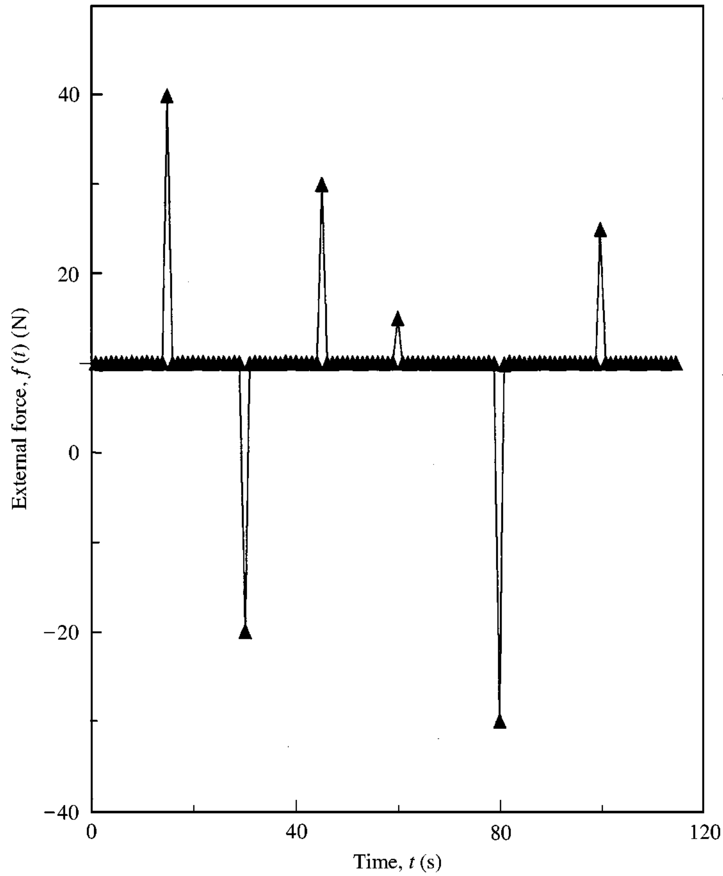


Figure 10. The exact ($f$) and estimated ($\hat{f}$) external forces using displacement measurements with $\sigma = 0{\cdot}0$ in numerical test case 3: ———, exact; –▲–, estimated.

Moreover, even though the algorithm developed in this paper is for only a single-degree-of-freedom problem, it can readily be extended to a multiple-degree-of-freedom problem. This problem is more complicated and is now being undertaken.

## 7. CONCLUSIONS

An iterative regularization method, i.e., the conjugate gradient method (CGM), was used successfully for the solution of the genie inverse non-linear force vibration problem (i.e., the system parameters are functions of displacement) to determine the unknown transient external forces by utilizing simulated displacement readings obtained from sensors. Many test cases involving different system parameters, measurement errors and external forces were considered. The results show that the inverse solutions obtained by CGM are still reliable as the measurement errors are large. Moreover, the CPU time needed for the inverse calculations is very short and the initial guesses for external forces can be arbitrarily chosen as zero.

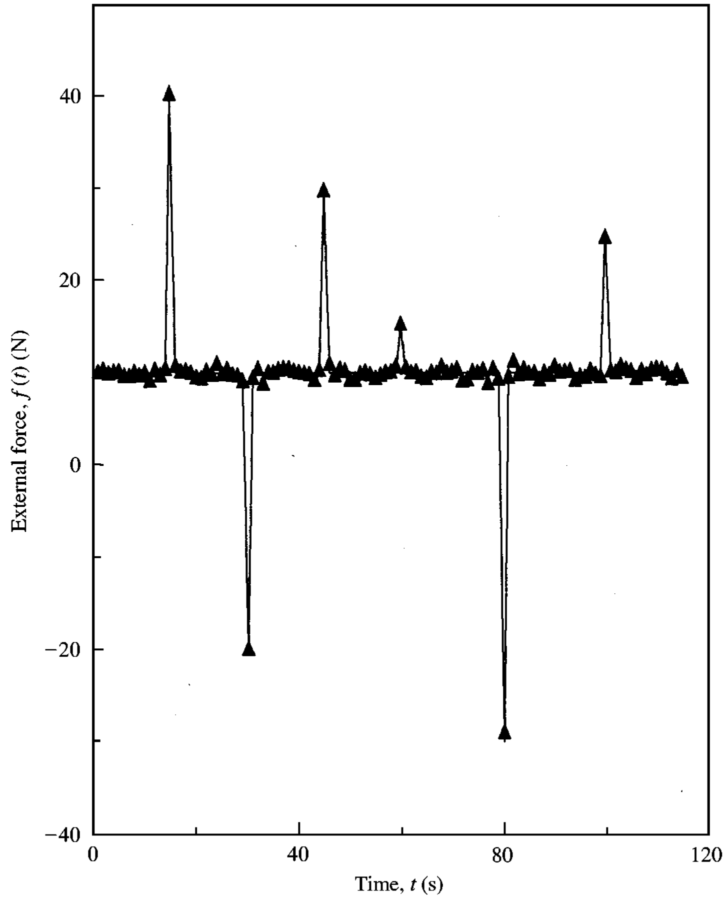


Figure 11. The exact ($f$) and estimated ($\hat{f}$) external forces using displacement measurements with $\sigma = 0{\cdot}025$ in numerical test case 3: ———, exact; – ▲ –, estimated.

## ACKNOWLEDGMENT

This work was supported in part through the National Science Council, Republic of China Grant No. NSC-90-2611-E-006-012.

## REFERENCES

1. O. M. Alifanov 1974 *Journal of Engineering Physics* **26**, 471–476. Solution of an inverse problem of heat conduction by iteration methods.
2. C. H. Huang and S. P. Wang 1999 *International Journal of Heat and Mass Transfer* **42**, 3387–3403. A three-dimensional inverse heat conduction problem in estimating surface heat flux by conjugate gradient method.
3. C. H. Huang and W. C. Chen 2000 *International Journal of Heat and Mass Transfer* **43**, 3171–3181. A three-dimensional inverse forced convection problem in estimating surface heat flux by conjugate gradient method.
4. C. H. Huang and S. C. Chin 2000 *International Journal of Heat and Mass Transfer* **43**, 4061–4071. A three-dimensional inverse forced convection problem in estimating surface heat flux by conjugate gradient method.

5. C. H. HUANG and W. Y. SHIH 1999 *International Journal of Numerical Methods in Engineering* **45**, 1547–1576. An inverse problem in estimating interfacial cracks in bimaterials by boundary element technique.
6. G. M. L. GLADWELL 1986 *Inverse Problem in Vibration*. The Netherlands: Kluwer Academic Publishers.
7. K. K. STEVENS 1987 *Proceedings of the SEM Spring Conference on Experimental Mechanics*, *Houstan*, *TX*, *June* 14–19, 838–844. Force identification problems—an overview.
8. G. DESANGHERE and R. SNOEYS 1985 *Proceedings of the* 1985 *International Modal Analysis Conference*, *Orlando*, *FL*, 685–690. Indirect identification of excitation force by modal coordinate transformation.
9. V. I. BATEMAN, T. G. CARNE, D. L. GREGORY, S. W. ATTAWAY and H. R. YOSHIMURA 1991 *Transactions of the American Society of Mechanical Engineers*, *Journal of Vibration and Acoustics* **113**, 192–200. Force reconstruction for impact tests.
10. J. E. MICHAELS and Y. H. PAO 1985 *Journal of the Acoustical Society of America* **77**, 2005–2011. The inverse source problem for an oblique force on an elastic plate.
11. C. K. MA, P. C. TUAN, D. C. LIN and C. S. LIU 1998 *International Journal of System Science* **29**, 663–672. A study of an inverse method for impulsive loads estimation.
12. C. H. HUANG 2001 *Journal of Sound and Vibration* **242,** 749–756. An inverse nonlinear force vibration problem of estimating the external forces in a damped system with time-dependent system parameters.
13. R. W. CLOUGH and J. PENZIEN 1993 *Dynamics of Structures*. New York: McGraw-Hill, Inc.
14. L. S. LASDON, S. K. MITTER and A. D. WARREN 1967 *IEEE Transactions on Automatic Control* **AC-12**, 132–138. The conjugate gradient method for optimal control problem.
15. IMSL Library Edition 10.0 1987 *User's Manual*: *Math Library Version* 1.0. Houston, TX: IMSL.


## APPENDIX A: NOMENCLATURE

| | |
|---|---|
| $C(x)$ | damping coefficient |
| $f(t)$ | external force |
| $J$ | functional defined by equation (3) |
| $J'$ | gradient of functional defined by equation (16) |
| $K(x)$ | spring constant |
| $M$ | system mass |
| $P$ | direction of descent defined by equation (5) |
| $t$ | time |
| $x$ | estimated displacement |
| $X$ | measured displacement |
| $y$ | estimated velocity |
| $Y$ | measured velocity |

*Greek letters*

| | |
|---|---|
| $\beta$ | search step size defined by equation (10) |
| $\gamma$ | conjugate coefficient defined by equation (6) |
| $\lambda_1$, $\lambda_2$ | adjoint functions defined by equation (13) |
| $\Delta x$, $\Delta y$ | sensitivity functions defined by equation (7) |
| $\varepsilon$ | convergence criteria |
| $\omega$ | random number |
| $\sigma$ | standard deviation of measurement errors |

*Superscript*

| | |
|---|---|
| $\wedge$ | estimated values |