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THE INFLUENCE OF A LIQUID FLOW ON SOUND FIELDS CONFINED BY CONICAL WALLS

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Sound propagation in conical waveguides is analyzed for the case where a background flow is maintained through the cone and points along the radial co-ordinate r only. A general expression for the acoustic pressure is derived and the perturbations of the acoustic field accommodated by the presence of a background flow are found by use of the Green function method. In the second part of this paper, flow measurement properties are discussed. The designation of propagation constants such as the wave number and phase speed lose much of their intuitive meaning in a conical waveguide. Instead, the so-called pseudo-guide wave number and pseudo-phase speed are introduced as they are well defined in the cone case and simplifies to the wave number and phase speed in the cylinder case respectively. It is shown that changes due to a background flow in pseudo-guide wave number, pseudo-phase speed, and zero-point crossing times all exhibit an oscillatory behavior as a function of the r co-ordinate. This is in contrast to the case of a cylinder where such changes become a linear function of the distance from the transmitter. The oscillatory behavior in the changes in zero-point crossing times as a function of r does not hamper flow measurement in the cone case since the only requirement to be fulfilled is, in principle, that a one-to-one correspondence between measured output (changes in zero-point crossing times) and actual flow (determined by $\tilde{v}_0$) at a specific receiver location exists. It is shown that this requirement is fulfilled as changes in zero-point crossing times depend linearly on the flow coefficient $\tilde{v}_0$ at a given r co-ordinate. There are, however, certain discrete r co-ordinate values where this is not the case, namely those where changes in zero-point crossing times become zero for any value of $\tilde{v}_0$. In other words, if the receiver is positioned near r co-ordinates where zero-point crossing times are at a maximum or a minimum for a given value of $\tilde{v}_0$, the truncated cone flow meter sensor is able to measure flow unambiguiosly by detection of changes in zero-point crossing times induced by a background flow.

1. INTRODUCTION

Sound propagation in conical horns or waveguides containing a moving fluid is of great interest in connection with ultrasonic flow measurement. Most liquid flow meter designs consist of several inter-connected tubes being either cylindrical or conical in shape. Theory for sound propagation in a moving fluid confined by cylindrical walls has been examined to some extent [1–10]. Similarly, several papers exist on sound propagation in a moving fluid for the case of a duct of varying cross-section [11–21]. Particular attention is paid to conical ducts in the interesting works by Easwaran and Munjal [15], Davies and Doak [17, 18], and Lung and Doige [19] for the case where a mean flow is present in the medium, and Munjal [20] and Benade [21] for the case of a stationary medium. However, a full analytical three-dimensional treatment of sound propagation in conical ducts carrying

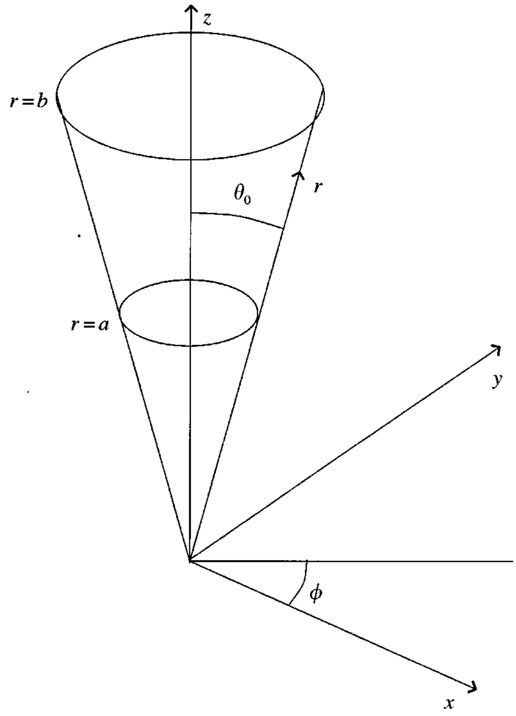

Figure 1. Figure of a cone and associated spherical co-ordinates used in the present work.

a moving medium does not exist as of today to the present author's knowledge. There is considerable and excellent work on electromagnetic field theory of conical horns [22–24] which can be easily transformed to the problem of sound propagation in a quiet medium confined by conical walls.

In the first part of the present work, the azimuthal symmetric three-dimensional problem (involving two space co-ordinates) of sound propagation in a conical duct confining a moving medium is described. Two coupled partial differential equations are derived for the case where flow points along the radial co-ordinate r (using spherical co-ordinates to represent the cone, refer to Figure 1). An analytical treatment is carried out using the Green function method for the analysis of the perturbation of the acoustic pressure due to the presence of a background flow if the medium flow is sufficiently small.

The remaining part of this work is concerned with ultrasonic liquid flow measurement in conical flow meter sensors. The designation of propagation constants such as wave number and phase speed lose much of their intuitive meaning in a conical waveguide. Instead, the so-called pseudo-guide wave number and pseudo-phase speed are introduced as they are well defined in the cone case. It is of particular relevance to consider changes in zero-point time crossings at a given r location accommodated by the presence of a background flow as such changes usually represent the measured output from the flow meter electronics. The measured change in zero-point crossing times with flow is then correlated to the mean flow in a one-to-one correspondence. It is found that changes in pseudo-guide wave number and pseudo-guide phase speed as well as changes in zero-point crossing times vary in an oscillatory manner as a function of the radial co-ordinate r but this does not necessarily hamper the possibility of measuring actual flow by detecting changes in zero-point crossing times. In principle, the only requirement to be fulfilled is that a one-to-one correspondence between measured changes in zero-point crossing times and actual flow is guaranteed. It will be shown that this requirement is fulfilled if the receiver is located near one of the r co-ordinate values where changes in zero-point crossing times are at a minimum or a maximum.

2. THEORY

Consider a truncated conical waveguide of radii a and b ($a < b$) and opening angle θ_0 (refer to Figure 1). Assume that a liquid flow $\mathbf{v}_0$ is maintained through the cone where $\mathbf{v}_0$ points in the r direction and $\mathbf{v}_0$ is a function of the r co-ordinate only, i.e., $\mathbf{v}_0(\mathbf{r}) = (v_{0r}, v_{0\theta}, v_{0\phi}) = (v_0(r), 0, 0)$. Spherical co-ordinates are chosen to represent the cone in the sense that the product of intervals [22–24]:

$$I_r \times I_\theta \times I_\phi = [a; b] \times [0; \theta_0] \times [0; 2\pi], \tag{1}$$

spans the whole truncated cone.

Assume next that the flow $\mathbf{v}_0(\mathbf{r})$ is incompressible, i.e.,

$$\boldsymbol{\nabla} \cdot \mathbf{v}_0 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_0(r)) = 0 \tag{2}$$

and so

$$v_0(r) = \frac{\tilde{v}_0}{r^2}, \tag{3}$$

where $\tilde{v}_0$ is a constant.

In this paper, a thorough analysis of the influence of a background flow $v_0(r)$, as given by equation (3), on sound propagation properties will be discussed for the truncated cone case. The first task to accomplish is the derivation of a partial differential equation in the acoustic pressure including perturbative effects due to the presence of a small liquid background flow $v_0(r)$. This will be carried out in the following section.

2.1. DERIVATION OF A DIFFERENTIAL EQUATION DESCRIBING SOUND PROPAGATION IN A MOVING FLUID CONFINED BY CONICAL WALLS

In this section, a single equation describing sound propagation in a moving fluid confined by conical walls will be derived (equation (13)). This partial differential equation is subsequently separated in variables r and θ to obtain two (weakly) coupled ordinary differential equations. The coupling is weak since the θ equation is independent of the radial equation, however, the eigenvalues found by solving the θ equation must be used in solving the radial equation.

From the equation of continuity and the Euler equation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0, \qquad \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -\frac{\boldsymbol{\nabla} p}{\rho}, \tag{4, 5}$$

the following equations, exact to first order in the small quantity $\mathbf{v}_0$, can be derived upon assuming monofrequency operation (replacing all time derivatives $\partial/\partial t$ by $-\mathrm{i}\omega$):

$$-\mathrm{i}\omega p' + \rho_0 c^2 \nabla\cdot\mathbf{v} + (\mathbf{v}_0\cdot\nabla)p' = 0, \tag{6}$$

$$-\mathrm{i}\omega\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}_0 + (\mathbf{v}_0\cdot\nabla)\mathbf{v} = -\frac{\mathbf{1}}{\rho_0}\nabla p', \tag{7}$$

where

$$\mathbf{v} = \mathbf{v}_0(\mathbf{r}) + \mathbf{v}(\mathbf{r}), \qquad p = p_0 + p'(\mathbf{r}), \qquad \rho = \rho_0 + \rho'(\mathbf{r}). \tag{8–10}$$

In equations (8)–(10), $\mathbf{v}_0(\mathbf{r})$, p_0, and ρ_0 represent the medium flow, pressure, and mass density in the absence of pressure waves respectively. The parameters $\mathbf{v}(\mathbf{r})$, $p'(\mathbf{r})$, and $\rho'(\mathbf{r})$ are all assumed small (linear acoustics) and represent the velocity, pressure, and mass density variations, respectively, accommodated by the presence of pressure waves in the flowing medium. In deriving equations (6, 7) use has also been made of the isentropic condition

$$p'(\mathbf{r}) = c^2\rho'(\mathbf{r}), \tag{11}$$

where c is the sound speed at constant entropy in the quite medium. Finally, only linear terms in the primed quantities are kept in deriving equations (6, 7) since higher-order terms are negligibly small (again assuming operation in the linear acoustic regime).

Taking the divergence of equation (7) yields

$$-\mathrm{i}\omega\nabla\cdot\mathbf{v} + \nabla\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}_0 + (\mathbf{v}_0\cdot\nabla)\,\mathbf{v}] = -\frac{1}{\rho_0}\nabla^2 p'. \tag{12}$$

Employing equation (6) for $\nabla\cdot\mathbf{v}$ and approximating $\mathbf{v}$ by $(1/\mathrm{i}\rho_0\omega)\nabla p'$ in terms of equation (12) were $\mathbf{v}_0$ appears leads to a single ordinary differential equation in the acoustic pressure p':

$$\nabla^2 p' + \frac{\omega^2}{c^2}p' + \frac{\mathrm{i}\omega}{c^2}(\mathbf{v}_0\cdot\nabla)p' + \frac{1}{\mathrm{i}\omega}\nabla\cdot[(\nabla p'\cdot\nabla)\mathbf{v}_0 + (\mathbf{v}_0\cdot\nabla)\nabla p'] = 0. \tag{13}$$

Note that equation (13) is exact to first order in $\mathbf{v}_0$. In the case of ultrasonic liquid flow meter applications, considered in the following, the flow velocity range is in the order of 0·1–1·0 m/s and the liquid sound speed is above 1000 m/s implying: $|\mathbf{v}_0|/c \ll 1$. In addition, ultrasonic frequencies are assumed to be higher than 1 MHz so that $a\omega/c \gg 1$, where a is the smallest truncated cone radius (refer to Figure 1). These conditions guarantee that the second and third terms on the left-hand side of equation (7) are considerably smaller as compared to the first term on the left-hand side of equation (7). This justifies neglecting second order terms in $\mathbf{v}_0$ in equation (13).

By separating the acoustic pressure p' in functions depending on r and θ only (considering azimuthal symmetrical pressure wave excitations in the following):

$$p'(r, \theta) = f(r)g(\theta), \tag{14}$$

so that equation (13) becomes

$$g(\theta)H_1(r) + H_2(r)\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial g}{\partial\theta}\right) = 0, \tag{15}$$

where

$$H_1(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{\omega^2}{c^2}f(r) + \frac{\mathrm{i}\omega}{c^2}v_0(r)\frac{\partial f}{\partial r}$$

$$+\frac{1}{\mathrm{i}\omega}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\frac{\partial v_0}{\partial r}\right) + \frac{1}{\mathrm{i}\omega}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_0(r)\frac{\partial^2 f}{\partial r^2}\right), \tag{16}$$

$$H_2(r) = \frac{1}{\mathrm{i}\omega}\frac{v_0(r)}{r^2}\frac{\partial f}{\partial r} - \frac{1}{\mathrm{i}\omega}\frac{v_0(r)}{r^3}f(r) + \frac{f}{r^2}. \tag{17}$$

Note that azimuthal symmetry implies that p' cannot be a function of the azimuthal co-ordinate ϕ. It follows from equation (15) that

$$\frac{g(\theta)}{(1/\sin\theta)\,\partial/\partial\theta\,(\sin\theta\,\partial g/\partial\theta)} = -\frac{H_2(r)}{H_1(r)} = -\frac{1}{\kappa}, \tag{18}$$

where κ is a constant independent of r and θ. The possible values of κ are obtained by solving

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial g}{\partial\theta}\right) + \kappa g(\theta) = 0, \tag{19}$$

subject to the boundary conditions: (1) $g(0)$ must be finite; (2) $\partial g/\partial\theta|_{\theta=\theta_0} = 0$ (corresponding to rigid conical walls). The latter condition can be proved as follows. Note that the assumption of rigid conical walls implies

$$v'_\theta|_{\theta=\theta_0} = 0. \tag{20}$$

Employing one of equations (7), one has

$$-\mathrm{i}\omega v'_\theta + v_0(r)\frac{\partial v'_\theta}{\partial r} = -\frac{1}{\rho_0}\frac{1}{r}\frac{\partial p'}{\partial\theta}, \tag{21}$$

and writing v'_θ on separable form one has

$$v'_\theta = h(r)l(\theta), \tag{22}$$

which leads to

$$n\rho_0 l(\theta) + \frac{\partial g}{\partial\theta} = 0, \tag{23}$$

for some constant n. The expression in equation (20) implies $l(\theta_0) = 0$, and the sought result follows:

$$\left.\frac{\partial g}{\partial\theta}\right|_{\theta=\theta_0} = 0. \tag{24}$$

2.2. SOLVING FOR POSSIBLE κ VALUES

Firstly, note that equation (19) for $g(\theta)$ is independent of $\tilde{v}_0$ (or independent of $v_0(r)$). Therefore, the presence of a background flow $v_0(r)$ leaves the θ dependence of the acoustic pressure $p'(r, \theta)$ unchanged, however, the r-dependent part $f(r)$ will be modified.

Consider the possible κ values. Equation (19) is customarily expressed in terms of the variable $x = \cos\theta$,

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial g_v}{\partial x}\right] + v(v+1)g_v(x) = 0, \tag{25}$$

where the κ parameter has been replaced by $v(v+1)$ for reasons that will soon become clear. Similarly, g has been replaced by g_v to indicate that the function g depends on the parameter v. Equation (25) must be solved subject to the boundary conditions:

(1) $g_v|_{x=1}$ must be finite; (2) $\partial g_v/\partial x|_{x=\cos\theta_0} = 0$ (corresponding to rigid conical walls).

These boundary conditions are equivalent to those stated earlier. The solution to the differential equation given by equation (25), known as the Legendre equation, is the so-called Legendre functions [25]:

$$g_v(x) = {}_2F_1\left(-v;\ v+1;\ 1,\ \frac{1-x}{2}\right) \tag{26}$$

or

$$g_v(\cos\theta) = {}_2F_1\left(-v;\ v+1;\ 1,\ \frac{1-\cos\theta}{2}\right), \tag{27}$$

where ${}_2F_1$ is the hypergeometric function

$${}_2F_1(a;\ b;\ c;\ z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots . \tag{28}$$

The Legendre function defined by equation (26) fulfills

$$g_v(1) = 1, \tag{29}$$

ensuring the finiteness of g_v at $x = 1$ as required by the boundary conditions. The possible values for v are found by solving:

$$\left.\frac{\partial g_v(x)}{\partial x}\right|_{x=\cos\theta_0} = 0. \tag{30}$$

numerically by using equations (26)–(28).

In Table 1, the first five v values are given for the set of opening angles: $\theta_0 = 5, 10, 15, 20,$ 25, and 30° (where $360^\circ = 2\pi$ rad).

TABLE 1

Calculated v values for a cone with opening angles: $\theta_0 = 5$, 10, 15, 20, 25, *and* 30°. *The lowest five values for v are given for each opening angle* θ_0.

θ_0/mode index	1	2	3	4	5
5	0	43·4	79·9	116·1	152·2
10	0	21·5	39·7	57·8	75·8
15	0	14·1	26·3	38·4	50·4
20	0	10·5	19·6	28·6	37·7
25	0	8·3	15·6	22·8	30·0
30	0	6·8	12·9	18·9	25·0

2.3. DETERMINATION OF THE RADIAL FUNCTION f CORRESPONDING TO A VANISHING BACKGROUND FLOW

It is instructive first to determine the possible solutions $f_\nu^{(0)}$ corresponding to a vanishing background flow: $\tilde{v}_0 = 0$. Observe that the second expression in equation (18),

$$H_1(r) - \nu(\nu + 1)H_2(r) = 0, \tag{31}$$

reduces to

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial f_\nu^{(0)}}{\partial r}\right) - \left(\nu(\nu + 1) - \frac{\omega^2}{c^2} r^2\right) f_\nu^{(0)}(r) = 0 \tag{32}$$

for the case where $\tilde{v}_0 = 0$. The general solution to equation (32) is

$$f_\nu^{(0)}(r) = A\mathrm{h}_\nu^{(1)}\left(\frac{\omega}{c} r\right) + B\mathrm{h}_\nu^{(2)}\left(\frac{\omega}{c} r\right), \tag{33}$$

were $\mathrm{h}_\nu^{(1)}$ and $\mathrm{h}_\nu^{(2)}$ are the spherical Hankel functions:

$$\mathrm{h}_\nu^{(1)}(r) = \mathrm{j}_\nu(r) + \mathrm{i}\eta_\nu(r), \qquad \mathrm{h}_\nu^{(2)}(r) = \mathrm{j}_\nu(r) - \mathrm{i}\eta_\nu(r), \tag{34, 35}$$

defined in terms of the spherical Bessel and Neumann functions j_ν and η_ν. The spherical Hankel function choice of representing solutions to equation (32) is convenient for radiation problems because for large values of the argument z, they behave as [26]

$$\mathrm{h}_\nu^{(1)}(z) = (-\mathrm{i})^{\nu+1} \frac{\exp(\mathrm{i}z)}{z}, \qquad \mathrm{h}_\nu^{(2)}(z) = \mathrm{i}^{\nu+1} \frac{\exp(-\mathrm{i}z)}{z}, \tag{36, 37}$$

corresponding to out-going and in-going waves respectively.

In conclusion, the complete solution for the acoustic pressure in the case of a vanishing background flow becomes

$$p\left(r, \theta; \frac{\omega}{c}\right) = \sum_\nu \left[A_\nu \mathrm{h}_\nu^{(1)}\left(\frac{\omega}{c} r\right) + B_\nu \mathrm{h}_\nu^{(2)}\left(\frac{\omega}{c} r\right)\right] {}_2\mathrm{F}_1\left(-\nu; \nu + 1; 1, \frac{1 - \cos\theta}{2}\right), \tag{38}$$

where A_ν and B_ν are constant coefficients to be determined by the boundary conditions at the cross-sections $r = a$ and b of the truncated cone. In equation (38), $p(r, \theta; \omega/c)$ denotes the acoustic pressure corresponding to monofrequency conditions with angular frequency ω.

2.4. THE INFLUENCE OF A BACKGROUND FLOW $v_0(r)$ ON THE RADIAL FUNCTION $f_\nu(r)$

Consider next the case where a background flow is maintained through the cone: $\tilde{v}_0 \neq 0$. Let us rewrite the general equation for the radial function (f_ν) as given by the second equality in equation (18):

$$H_1(r) - \nu(\nu + 1) H_2(r) = 0. \tag{39}$$

Assume that the condition

$$(\omega/c)\, a \gg 1, \tag{40}$$

is fulfilled being a good assumption in ultrasonic flow meter applications [refer also to the discussion following equation (13)]. The condition given by equation (40) ensures that the relations in equations (36, 37) can be used in the interval $a < r < b$. In addition, the assumption: $(\omega/c)a \gg 1$ makes it possible to approximate for derivatives in the spherical Hankel functions as follows:

$$\frac{\partial \mathrm{h}_\nu^{(1)}((\omega/c)r)}{\partial r} = \mathrm{i}\,\frac{\omega}{c}\,\mathrm{h}_\nu^{(1)}\left(\frac{\omega}{c}\,r\right), \tag{41}$$

$$\frac{\partial \mathrm{h}_\nu^{(2)}((\omega/c)r)}{\partial r} = -\,\mathrm{i}\,\frac{\omega}{c}\,\mathrm{h}_\nu^{(2)}\left(\frac{\omega}{c}\,r\right). \tag{42}$$

In the following, consider the perturbation of out-going acoustic pressure waves (e.g., ultrasonic waves) by the presence of a background flow. In this case,

$$f_\nu^{(0)}(r) = \mathrm{h}_\nu^{(1)}\left(\frac{\omega}{c}\,r\right), \qquad \frac{\partial f_\nu^{(0)}(r)}{\partial r} = \mathrm{i}\,\frac{\omega}{c} f_\nu^{(0)}(r), \quad a \leqslant r \leqslant b. \tag{43, 44}$$

Next, assume that the changes in f_ν accommodated by the presence of a non-vanishing background flow are small such that f_ν can be replaced by $f_\nu^{(0)}$ in all terms involving $v_0(r)$ or derivatives of $v_0(r)$. With these assumptions, equation (39) simplifies to

$$\frac{\partial}{\partial r}\left(r^2\,\frac{\partial f_\nu}{\partial r}\right) - \left(\nu(\nu + 1) - \frac{\omega^2}{c^2}\,r^2\right) f_\nu(r) = \frac{2\omega^2 \tilde{v}_0}{c^3} f_\nu^{(0)}(r). \tag{45}$$

This equation can be solved using the Green function method (refer to the following section).

2.5. SOLVING FOR THE ACOUSTIC PRESSURE $p(r, \theta; \omega/c)$ IN THE CASE WHERE A NON-VANISHING BACKGROUND FLOW $v_0(r)$ IS MAINTAINED THROUGH THE CONE

Let us determine a partial solution to the inhomogeneous equation given by equation (45). This is conveniently done by use of the Green function method.

A Green function for the differential equation given by equation (45) satisfies

$$\frac{\partial}{\partial r}\left(r^2\,\frac{\partial G_\nu(r,r')}{\partial r}\right)-\left(\nu(\nu+1)-\frac{\omega^2}{c^2}\,r^2\right)G_\nu(r,r')=\delta(r-r'). \tag{46}$$

Solutions are known to this equation for $r \neq r'$, since in that case, $G_\nu(r, r')$ satisfies the homogeneous equation (32). In other words, $G_\nu(r, r')$ must be a linear combination of $\mathrm{h}_\nu^{(1)}((\omega/c)r)$ and $\mathrm{h}_\nu^{(2)}((\omega/c)r)$ whenever $r \neq r'$: i.e.,

$$G_\nu(r,r')=A_\nu^-\,\frac{\exp(\mathrm{i}\,(\omega/c)r)}{\mathrm{i}(\omega/c)r}+B_\nu^-\,\frac{\exp(-\mathrm{i}(\omega/c)r)}{\mathrm{i}(\omega/c)r};\quad a\leqslant r\leqslant r'\leqslant b, \tag{47}$$

$$G_\nu(r,r')=A_\nu^+\,\frac{\exp(\mathrm{i}\,(\omega/c)r)}{\mathrm{i}(\omega/c)r}+B_\nu^+\,\frac{\exp(-\mathrm{i}(\omega/c)r)}{\mathrm{i}(\omega/c)r};\quad a\leqslant r'\leqslant r\leqslant b. \tag{48}$$

Two of the coefficients A_ν^-, B_ν^-, A_ν^+, and B_ν^+ are determined by the conditions that (1) $G_\nu(r, r')$ must be continuous at $r = r'$,

$$G_\nu(r,r')|_{r=r'-}=G_\nu(r,r')|_{r=r'+}, \tag{49}$$

and (2) integration of equation (45) from $r = r' - \varepsilon$ to $r = r' + \varepsilon$, where ε is a small positive number, gives the second condition:

$$r'^2\left[\left.\frac{\partial G_\nu(r,r')}{\partial r}\right|_{r=r'+}-\left.\frac{\partial G_\nu(r,r')}{\partial r}\right|_{r=r'-}\right]=1. \tag{50}$$

The specification of boundary conditions is unnecessary at this point, since the aim is to find a particular solution. Accordingly, the coefficients A_ν^- and B_ν^+ may be chosen to equal zero:

$$A_\nu^-=0,\qquad B_\nu^+=0, \tag{51, 52}$$

upon keeping in mind that the addition of the full solution to the homogeneous differential equation (equation (32)) to any particular solution constitutes the complete solution to the inhomogeneous differential equation (equation (45)).

The combination of equations (47)–(52) leads to the following result for the Green function:

$$G_\nu(r,r')=\frac{1}{2\mathrm{i}\omega/c}\,\frac{\exp(-\mathrm{i}\,(\omega/c)r)}{r}\,\frac{\exp(\mathrm{i}(\omega/c)r')}{r'},\quad a\leqslant r\leqslant r'\leqslant b, \tag{53}$$

$$G_\nu(r,r')=\frac{1}{2\mathrm{i}\omega/c}\,\frac{\exp(\mathrm{i}\,(\omega/c)r)}{r}\,\frac{\exp(-\mathrm{i}(\omega/c)r')}{r'},\quad a\leqslant r'\leqslant r\leqslant b \tag{54}$$

and a particular solution $y_p(r)$ to equation (45) becomes

$$\begin{aligned} y_p(r)&=\int_a^b G_\nu(r,r')\left(\frac{2\omega^2\tilde{v}_0}{c^3}f_\nu^{(0)}(r')\right)\mathrm{d}r'\\ &=-(-\mathrm{i})^{\nu+1}\,\frac{2\omega^2\tilde{v}_0}{c^3}\,\frac{\exp(-\mathrm{i}(\omega/c)r)}{(\omega/c)r}\left[\ln\left(2\mathrm{i}\,\frac{\omega}{c}r\right)+\sum_{n=1}^{\infty}\frac{(2\mathrm{i}(\omega/c)r)^n}{n\cdot n!}\right]\\ &\quad+C1(a)\mathrm{h}_\nu^{(1)}\left(\frac{\omega}{c}r\right)+C2(b)\mathrm{h}_\nu^{(2)}\left(\frac{\omega}{c}r\right), \end{aligned} \tag{55}$$

where $C1(a)$ and $C2(b)$ are constants determined uniquely by the radii a and b respectively. Recapitulating: in the absence of a background flow: $\tilde{v}_0 = 0$, the solution describing propagation of outgoing waves is $f_v^{(0)}(r) = \mathrm{h}_v^{(1)}((\omega/c)r)$. In the presence of a background flow, this function must be replaced by $f_v(r)$ defined by

$$f_v(r) = \mathrm{h}_v^{(1)}\left(\frac{\omega}{c} r\right) - (-\mathrm{i})^{v+1} \frac{2\omega^2 \tilde{v}_0}{c^3} \frac{\exp(-\mathrm{i}(\omega/c)r)}{(\omega/c)r} \left[\ln\left(2\mathrm{i}\frac{\omega}{c} r\right) + \sum_{n=1}^{\infty} \frac{(2\mathrm{i}(\omega/c)r)^n}{n \cdot n!}\right]$$
$$+ \tilde{v}_0 \left[C\,\mathrm{h}_v^{(1)}\left(\frac{\omega}{c} r\right) + C_2 \mathrm{h}_v^{(2)}\left(\frac{\omega}{c} r\right)\right], \tag{56}$$

where C_1 and C_2 are constants determined by the boundary conditions imposed. Note that $f_v(r)$ simplifies to $\mathrm{h}_v^{(1)}((\omega/\mathrm{c})r)$ when $\tilde{v}_0 = 0$ as it should.

In the next section, this solution for f_v will be used to examine the influence of a background flow on acoustic properties of truncated cones.

2.6. DEFINITION OF PSEUDO-GUIDE WAVE NUMBER AND PSEUDO-PHASE SPEED

By analogy with the cylindrical case, one can define the so-called pseudo-guide wave number:

$$k_v(r) = \mathrm{Im}\left(\frac{\partial f_v/\partial r}{f_v}\right), \tag{57}$$

where $\mathrm{Im}(\cdot)$ denotes the imaginary part of the argument. Observe that $k_v(r)$ becomes $k = \omega/c$ for the plane wave: $\exp(\mathrm{i}kr - \mathrm{i}\omega t)$, i.e., the pseudo-guide wave number equals the wave vector in the case of propagating waves in a cylinder (if, for the cylinder, the co-ordinate r corresponds to the axial co-ordinate). The designation of propagation constants (such as the wave number k) for the conical wave guide does not follow in the usual sense. In a conical wave guide, the acoustic fields are not periodic functions of the propagation co-ordinate r, and propagation constants lose much of their intuitive meaning. However, the pseudo-guide wave number is well defined in the conical wave guide for any r position and represents the relative change in the acoustic pressure as a function of r. Obviously, the pseudo-guide wave number generally depends on the position co-ordinate r where it is evaluated.

The pseudo-guide wave number becomes, to first order in the small quantity $\tilde{v}_0$,

$$k_v(r) = \mathrm{Im}\left(\frac{\partial f_v^{(0)}/\partial r}{f_v^{(0)}}\right) + \mathrm{Im}\left(\frac{\partial f_v^{(1)}/\partial r}{f_v^{(0)}}\right) - \mathrm{Im}\left(\frac{\partial f_v^{(0)}}{\partial r} \frac{f_v^{(1)}}{(f_v^{(0)})^2}\right)$$
$$= \frac{\omega}{c} + \mathrm{Im}\left(\frac{\partial f_v^{(1)}/\partial r}{f_v^{(0)}}\right) - \mathrm{Im}\left(\frac{\partial f_v^{(0)}}{\partial r} \frac{f_v^{(1)}}{(f_v^{(0)})^2}\right), \tag{58}$$

where

$$f_v^{(1)}(r) = f_v(r) - f_v^{(0)}(r) = -(-\mathrm{i})^{v+1} \frac{2\omega^2 \tilde{v}_0}{c^3} \frac{\exp(-\mathrm{i}(\omega/c)r)}{(\omega/c)r} \left[\ln\left(2\mathrm{i}\frac{\omega}{c} r\right) + \sum_{n=1}^{\infty} \frac{(2\mathrm{i}(\omega/c)r)^n}{n \cdot n!}\right]$$
$$+ \tilde{v}_0 \left[C_1 h_v^{(1)}\left(\frac{\omega}{c} r\right) + C_2 h_v^{(2)}\left(\frac{\omega}{c} r\right)\right], \tag{59}$$

$$\frac{\partial f_v^{(1)}}{\partial r} = -(-\mathrm{i})^{v+1}\frac{2\omega^2\tilde{v}_0}{c^3}\left[-\mathrm{i}\frac{\exp(-\mathrm{i}(\omega/c)r)}{r}\left(\ln\left(2\mathrm{i}\frac{\omega}{c}r\right)+\sum_{n=1}^{\infty}\frac{(2\mathrm{i}(\omega/c)r)^n}{n\cdot n!}\right)+\frac{\exp(\mathrm{i}(\omega/c)r)}{(\omega/c)r^2}\right]$$
$$+\tilde{v}_0\left[\mathrm{i}\frac{\omega}{c}C_1h_v^{(1)}\left(\frac{\omega}{c}r\right)-\mathrm{i}\frac{\omega}{c}C_2h_v^{(2)}\left(\frac{\omega}{c}r\right)\right]. \tag{60}$$

A pseudo-phase speed is conveniently defined as

$$v_{pv}(r) = \omega/k_v(r). \tag{61}$$

Note that this definition ensures that the relation between pseudo-guide wave number and pseudo-phase speed is the same as the corresponding relation between wave number and phase speed.

3. NUMERICAL RESULTS AND DISCUSSIONS

In the previous section, a general solution involving two space co-ordinates r and θ for the acoustic pressure in a truncated cone was derived for the case where flow is directed parallel to the **r** direction (equation (56)). Earlier papers on the same topic [15–21] involve one space co-ordinate only (r), and, therefore, fail to describe the complete picture of *ultrasound* propagation in truncated cones confining a flowing media. It is, however, possible to compare results of the present work with, e.g., reference [17], if we restrict our analysis to the first mode for which $v = 0$ (fundamental mode) allowed to propagate at any frequency. In actual fact, equation (6) in reference [17],

$$\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + 2\left[\frac{1}{r} - \mathrm{i}\bar{k}M_r\right]\frac{\mathrm{d}f}{\mathrm{d}r} + \bar{k}^2 f = 0, \tag{62}$$

becomes equation (45) of the present work by realizing that (to first order in $\mathbf{v}_0$),

$$-2\mathrm{i}\bar{k}M_r\frac{\mathrm{d}f}{\mathrm{d}r} \approx -2\mathrm{i}\bar{k}M_r\frac{\mathrm{d}f^{(0)}}{\mathrm{d}r} = 2\bar{k}^2M_rf^{(0)} = 2\frac{\omega^2 v_0(r)}{c^3}f^{(0)}(r), \tag{63}$$

where $f^{(0)}$ is the zeroth order approximation to f in the presence of a background flow $\mathbf{v}_0$. The last equality in equation (63) follows as M_r in reference [17] equals $\mathbf{v}_0(r)/c$ in the present work, and $\bar{k} = \omega/c$. The analytical results obtained here, therefore, agree with those of reference [17]. Note that the results displayed in Figures 2–5 which follow are solely based on equations (57)–(61) which again are analytically derivable from equation (45) as can be proved by direct insertion.

In order to assess the possibilities for measuring flow in conical waveguides by using ultrasonic time-of-flight measurements, properties such as the pseudo-guide wave number, pseudo-phase speed, phase changes and associated changes in zero-point crossing times at a given r co-ordinate induced by the presence of a background flow will be examined. However, before addressing such properties, some introductory remarks must be made.

In applications where the transducer excites a given pressure burst or continuous wave signal at a specified location, i.e., at a specified r co-ordinate (defined to be $r = a$ in the following), it is of interest to determine the acoustic pressure at a different r co-ordinate, say r', where $r' \ll b$. In particular, let us compare the two cases where (a) a pressure signal is

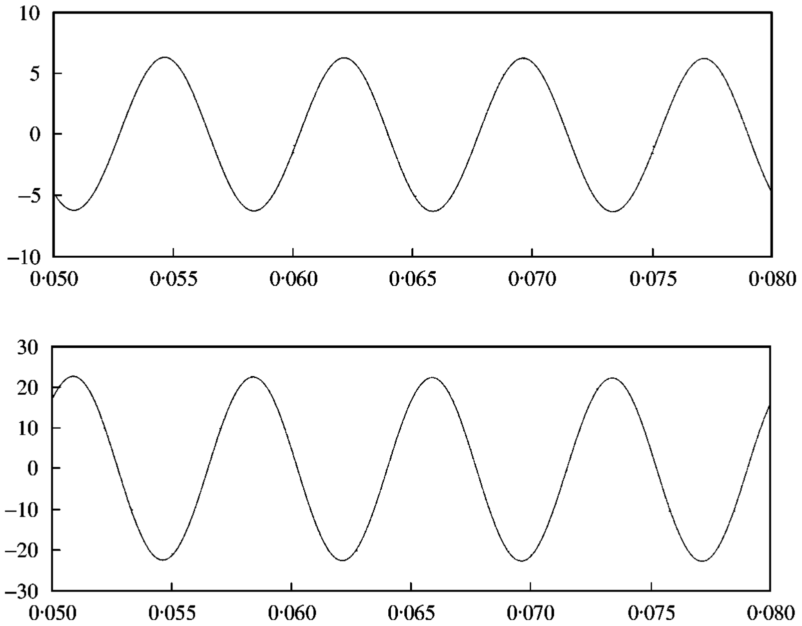

Figure 2. The upper [lower] figure shows the pseudo-guide wave number $k_v(r)$ [pseudo-phase speed $v_{pv}(r)$] as a function of the radial position co-ordinate r. Parameter values used in the calculations are: $f = \omega/2\pi = 100$ kHz, $a = 0{\cdot}05$ m, $c = 1500$ m/s, and $\tilde{v}_0 = 1 \times 10^{-5}$ m^3/s equivalent to a flow: $v_0(r) = 0{\cdot}10$ m/s at $r = a$.

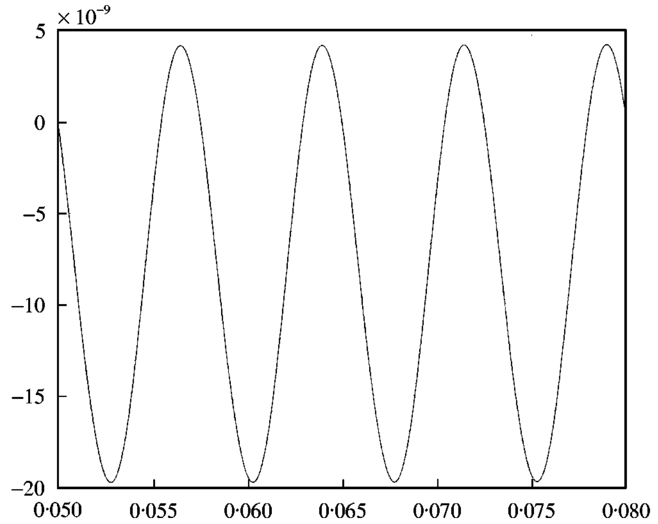

Figure 3. Changes in zero-point crossing times Δt as a function of the radial position co-ordinate r. Parameter values are the same as for Figure 2.

generated by the transmitter at $r = a$ and the acoustic pressure at the receiver position $r = r'$ is measured corresponding to zero-flow conditions: $\tilde{v}_0 = 0$, and (b) the same pressure signal is generated by the transmitter at $r = a$ and the acoustic pressure at the receiver location r' is measured corresponding to the case where a background flow: $v_0(r) = \tilde{v}_0/r^2 \neq 0$ is

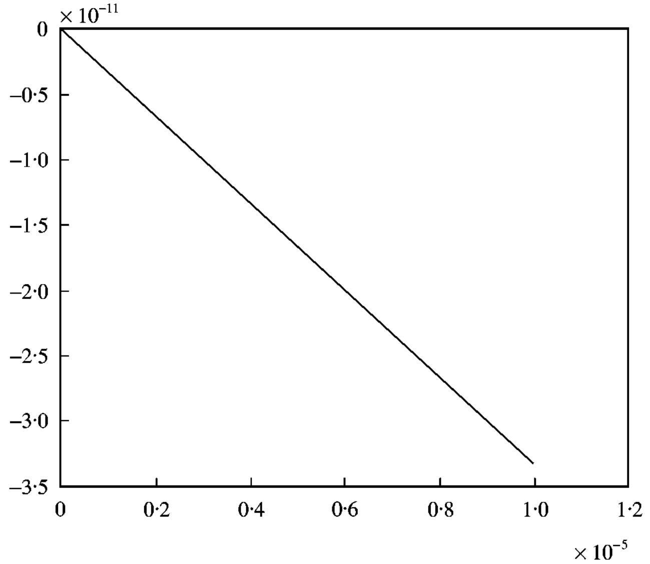

Figure 4. Changes in zero-point crossing times at a given r position co-ordinate ($r = 0{\cdot}08$ m) as a function of the flow coefficient $\tilde{v}_0$. Parameter values used in the calculations are: $f = \omega/2\pi = 100$ kHz, $a = 0{\cdot}05$ m, $c = 1500$ m/s.

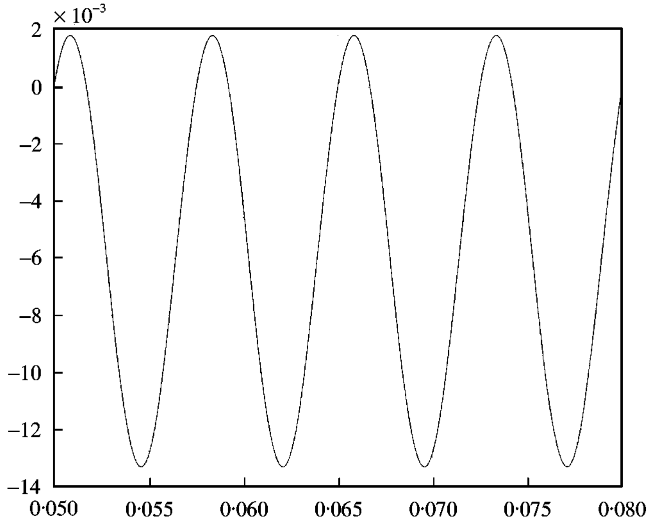

Figure 5. Relative change in acoustic pressure $\Delta f_v(r)$ (defined by equation (66)) plotted as a function of the radial co-ordinate r using the same parameter values as in Figure 2.

maintained through the cone. In both cases (a) and (b), the pseudo-guide wave guide wave number and pseudo-phase speed are calculated at $r = r'$. The difference in these parameters accommodated by the presence of the background flow allows us to examine flow measurement properties of flow meter sensor designs consisting of one or several pieces of truncated cones. In the case where several pieces of truncated cones are connected, the cross-sectional area will still be assumed to vary in a continuous manner as a function of distance along the duct centerline. In order to assess sound propagation characteristics of such sensor duct geometries (one or several pieces of truncated cone(s)), it is necessary to understand sound propagation in a single truncated cone in the first place. This will be discussed next.

Consider a single truncated cone with radii a and b ($a < b$) as discussed in the previous section. Let us analyze the particular solution to equation (56) corresponding to the parameter choices $C_2 = 0$ and $f_v(r = a) = f_v^{(0)}(r = a)$. The choice $C_2 = 0$ implies that in-going acoustic waves (propagating from b to a) are not excited by the presence of the background flow. Such (in-going) waves are certainly not excited in the absence of a medium flow as medium mass density and medium sound speed in that case are homogeneous parameters and transducer excitation takes place at $r = a$. In this case, acoustic waves must propagate from $r = a$ to b. The requirement, $f_v(r = a) = f_v^{(0)}(r = a)$, simply states that the acoustic pressure generated by the transmitter is independent of the background flow (reasonable since the vibration pattern of the transducer aperture is left unaffected by the presence of a small background flow).

In Figure 2, changes in the pseudo-guide wave number $k_v(r)$ and the associated pseudo-phase speed $v_{pv}(r)$ are shown as a function of the r co-ordinate. It follows from the definition of $k_v(r)$ and $v_{pv}(r)$ that both must be independent of the v value (f_v and $\partial f_v/\partial r$ are proportional to $(-1)^{v+1}$). It is evident that the changes in pseudo-guide wave number and pseudo-phase speed accommodated by the presence of the background flow oscillates around 0 with amplitudes of approximately $6{\cdot}5\ \mathrm{m}^{-1}$ and 23 m/s respectively, as the radial position co-ordinate r increases from $a = 0{\cdot}05$ m to $r = 0{\cdot}08$ m (keep in mind that the transmitter is located at $r = a$). In the calculations, the ultrasound frequency is assumed to be $f = \omega/2\pi = 100$ kHz and the sound speed in the quite medium is $c = 1500$ m/s corresponding to sound propagation in water at 25°C. With these parameter values, the ratio $(\omega/c)a \approx 21 \gg 1$ as required for the theory in the previous section to apply. Note that the frequency of the oscillations shown in Figure 2 is approximately $-2f$, i.e., twice the ultrasonic frequency f in absolute terms. This comes about as the term in $f_v^{(1)}(r)$ proportional to

$$\frac{\exp(-\mathrm{i}(\omega/c)r)}{(\omega/c)r}\left[\ln\left(2\mathrm{i}\frac{\omega}{c}r\right) + \sum_{n=1}^{\infty}\frac{(2\mathrm{i}(\omega/c)r)^n}{n\cdot n!}\right]$$

oscillates with a frequency equal to approximately $-f$ while $f_v^{(0)}(r)$ oscillates with a frequency $+f$. The double frequency oscillation in $k_v(r)$ found at small $\tilde{v}_0$ values now follows from the definition of k_v as the ratio between $\partial f_v(r)/\partial r$ and $f_v(r)$. This result is very different from the analog case of propagation of out-going waves in a cylinder carrying a moving fluid. In the cylinder case, the pseudo-guide wave number (being equal to the usual wave number) is a linear function of the axial distance [2–4]. It may seem at first sight that flow measurement in a truncated cone is hampered due to the oscillatory behavior of the pseudo-guide wave number and the pseudo-phase speed as a function of r. This is not necessarily the case as we will now address.

In Figure 3, the change in zero-point time crossings Δt for the first acoustic pressure mode $f_v(r, t)$ ($v = 0$) accommodated by the presence of a background flow is shown as a function of the r co-ordinate. Zero-point time crossings t_n are determined by imposing the condition.

$$\mathrm{Re}(f_v(r, t_n)) = \mathrm{Re}(f_v(r)\exp(-\mathrm{i}\omega t_n)) = 0, \tag{64}$$

where n is a zero-point crossing index number, and $\mathrm{Re}(\cdot)$ denotes the real part of the argument. Equation (64) follows as the physical signal is the real part of $f_v(r, t)$. The change in zero-point time crossings due to the presence of a background flow $v_0(r)$ becomes

$$\Delta t = t_n - t_{n0}, \tag{65}$$

where t_n and t_{n0} denote the nth zero-point crossing times in the presence and absence of a background flow $v_0(r)$ respectively.

It can be argued that changes in zero-point crossing times i.e., Δt as defined by equation (65) are more relevant in connection with practical ultrasonic flow measurement discussions than, e.g., pseudo-phase speed changes (being shown in Figure 2). The reason is that typically, in practice, variations in zero-point crossing times with flow at a certain specified receiver position co-ordinate are detected by the flow-meter electronics. The measured change in zero-point crossings with flow is then correlated to the mean flow in a one-to-one correspondence.

Nevertheless, Figures 2 and 3 reveal that $k_v(r)$, $v_{pv}(r)$ as well as Δt all are oscillating functions of the receiver position. As a result, it is beneficial (if not essential) to locate the receiver close to a position co-ordinate r where the absolute value of Δt is at a maximum to achieve the highest possible sensitivity for measuring flow. In other words, if the receiver is located at a r co-ordinate where $\Delta t = 0$ even when flow is maintained through the cone, flow measurement based on changes in zero-point time crossings is completely obstructed.

It is important to realize that ultrasonic flow measurement based on Δt variations with flow is possible if there is a one-to-one correspondence between $\tilde{v}_0$ and Δt. This is indeed the case if the receiver is located near one of the minima or maxima for Δt (refer to Figure 3). In Figure 4, Δt as a function of the flow coefficient $\tilde{v}_0$ is depicted for an r co-ordinate corresponding to a minimum in Δt ($r = 0{\cdot}08$ m). Evidently, Δt is a linear function of $\tilde{v}_0$, and so Δt measurements can be used to determine flow unambiguously. In other words, Figure 4 serves as a calibration curve between measured Δt values and $\tilde{v}_0$.

In Figure 5, the relative change in the acoustic pressure $\Delta f_v(r)$ accommodated by the presence of a background flow is shown as a function of the r position co-ordinate for a fixed value of the flow coefficient $\tilde{v}_0$. The relative change in the acoustic pressure is defined by

$$\Delta f_v(r) = \frac{f_v^{(1)}(r)}{f_v^{(0)}(r = a)}. \tag{66}$$

4. CONCLUSIONS

Sound propagation in conical waveguides is analyzed for the case where a background flow is maintained through the cone. The flow velocity $\mathbf{v}_0 = (v_{0r}, v_{0\theta}, v_{0\phi}) = (v_{0r}, 0, 0)$ points in the $\mathbf{r}$ direction everywhere and $v_0(r)$ decreases as r^{-2} so as to fulfill the equation of continuity. In the first part of the paper, the general theory for sound propagation in a moving fluid confined by conical walls is described and a general expression for the perturbations of the acoustic pressure due to the presence of a background flow is derived by use of the Green function method. In a conical waveguide, however, acoustic properties such as the wave number and phase speed lose much of their intuitive meaning. Instead, the so-called pseudo-guide wave number and pseudo-phase speed are often employed as they are well defined in the cone waveguide geometry. In addition, the pseudo-guide wave number and pseudo-guide phase speed simplify to the usual wave number and phase speed in the case of a cylinder, where acoustic waves become travelling waves: $\exp(ikr - i\omega t)$.

In the second part of this work, flow measurement properties of a conical waveguide are discussed. This problem is of practical interest as ultrasonic flow meters often consist of several inter-connected truncated conical waveguides. The cone geometry design is often taken so as to (a) tailor the flow profile, and (b) allow for physical space inside the flow meter sensor to the transmitting and receiving transducers. It is shown that changes in the pseudo-guide wave number and pseudo-phase speed as well as the better representative for ultrasonic flow measurement: changes in zero-point crossing times accommodated by the presence of a background flow all exhibit an oscillatory behavior as a function of the r co-ordinate. This is in contrast to the cylinder case where such properties depend linearly on distance from the transmitter. Fortunately, flow measurement by detection of, e.g., changes in zero-point crossing times is still possible if the receiver is located near the r co-ordinates where such changes are at a maximum or a minimum. It is necessary that the receiver is not positioned near r co-ordinates where flow-induced changes in zero-point crossing times vanish. In principle, flow measurement requires no more than a one-to-one correspondence between the measured output and the actual flow. It is shown that this requirement is fulfilled since a linear relationship between measured output (changes in zero-point crossing times) and the actual flow (determined uniquely by $\tilde{v}_0$) exists.

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