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ENERGY BALANCE FOR RANDOM VIBRATIONS OF PIECEWISE-CONSERVATIVE SYSTEMS

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Vibrations of systems with instantaneous or stepwise energy losses, e.g., due to impacts with imperfect rebounds, dry friction forces(s) (in which case the losses may be treated as instantaneous ones by appropriate introduction of the response energy) and/or active feedback "bang–bang" control of the systems' response are considered. Response of such (non-linear) systems to a white-noise random excitation is considered for the case where there are no other response energy losses. Thus, a simple linear energy growth with time between "jumps" is observed. Explicit expressions for the expected response energy are derived by direct application of the stochastic differential equations calculus, which contains the expected time interval between two consecutive jumps. The latter may be predicted as a solution to the relevant first-passage problem. Perturbational analysis of the relevant PDE for this problem for a certain vibroimpact system demonstrated the possibility for using the solution to the corresponding free vibration problem as a zero order approximation. The method is applied to an s.d.o.f. system with a feedback inertia control, designed according to a certain previously introduced "generalized reversed swings law". Extensive Monte-Carlo simulation results are presented for this system as well as for several previously analyzed ones: system with impacts; system with dry friction; system with stiffness control; pendulum with controlled length. The results are compared with those due to the asymptotic stochastic averaging approach. Both methods are shown to provide adequate accuracy far beyond the expected applicability range of the asymptotic approach (which requires both excitation intensity and losses to be small), with direct energy balance being generally superior.

1. INTRODUCTION AND OUTLINE OF THE APPROACH

The name "piecewise-conservative" is used in this paper for vibrating systems with stepwise finite energy losses, which appear at discrete time instants only. A typical example is a vibroimpact system with the dominant mechanism of energy loss being impacts with imperfect rebounds. Another example is a system with externally imposed instantaneous stepwise variations, or "jumps", of parameters, which can either bring in or carry away the system's energy (pendulum clocks, swings, etc.). It may be added that certain non-conservative systems may be treated as piecewise-conservative ones. An example is a single-degree-of-freedom (s.d.o.f) system with dry friction, or a resistance force of a constant magnitude with its direction being always opposite to that of the system's velocity. By including work of this force into the system's total energy, one can describe

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energy losses in vibration as being instantaneous, corresponding to reversals of velocity. The described kind of phenomena may also be observed in systems with active control of a "bang–bang" type, whereby the available control force, as developed by an actuator, is of a bounded magnitude; the optimal bounded control law is usually obtained as a sequence of "switches" between the given bounds.

These losses make the vibrating systems non-linear in general, as long as the instants of stepwise variations are not known in advance but rather are governed by the equations of motion. This non-linearity greatly complicates analyses of the systems' response to a random excitation. Even for an s.d.o.f. system such an analysis requires either the use of some moment closure scheme, or the use of the stochastic averaging approach, which is valid only for small energy losses and excitation intensity.

An alternative method for response prediction had been proposed in reference [1] for an s.d.o.f. vibroimpact system, subjected to a white-noise excitation. The method is based on a direct balance of the expected response energy. A stochastic differential equation (SDE) ofor the total response energy $H(t)$ is derived from the original equation of motion. A conditional averaging is first applied to this SDE, denoted by bar, the condition being the initial value of H at the start of a certain response cycle. This results in the deterministic ODE $\dot{\bar{H}} = D/2$, with D being the intensity of the white-noise excitation, thereby implying linear growth of the (conditional) expected response energy with time. After deducting a properly evaluated energy loss within the cycle, the conditional expectation of energy at the start of the next cycle can be evaluated. (The actual value of energy at this instant will be random.) The concept of a "cycle" is problem-dependent, of course, but it is unambiguously defined by finite relation(s), which control the instantaneous energy losses. Thus, in case of a vibroimpact system with a single rigid barrier the cycle corresponds to a time interval between two consecutive rebounds (or impacts).

The above procedure results in a random sequence of values of H at the starting instants of various cycles. The unconditional averaging is applied then to this sequence, i.e., averaging over all response cycles, as denoted by angular brackets. As long as a stationary sequence has a constant mean value, the mean net energy increment per cycle should be zero. This results in a simple energy-balance relation:

$$\langle \Delta H \rangle = DT/2, \tag{1}$$

where the LHS is a total mean energy loss per cycle. It is related to the system's energy and/or other state variables by the specific equation for energy loss for a given problem. The RHS is the mean energy input per cycle, with T being the expected duration of the cycle. It can be identified as the solution to the relevant first-passage problem for the response—namely, as an expected time to arrive at the starting point of the next response cycle after the start of the present cycle with energy H. This (conditionally) expected time satisfies the relevant generalized Pontryagin equation [2], which had been identified and analyzed in reference [1] for the corresponding vibroimpact system. Solution $T(H)$ to this PDE, which is to be used in the exact (by itself) relation (1), with H replaced by its unconditional mean value, is the challenging part of the approach. It may be added that, in general, T may also be present in the LHS of relation (1); this will be the case where the magnitude of the energy drop depends not only, say, on the initial energy of the cycle, but on the instantaneous energy as well.

A perturbational analysis of the second order PDE for $T(H)$ has been made in reference [1], with the excitation intensity D regarded as a small parameter. In a zero order approximation, $D = 0$, the PDE is of reduced (first) order, and its exact solution is just the system's natural half-period, or $T = \pi/\Omega$, where Ω is the system's natural frequency. As long

as the solution satisfies both boundary conditions for the original PDE, this is the case of regular rather than singular perturbations. Thus, the solution for the deterministic cycle duration T of the system without excitation may naturally be used in relation (1). Of course, this would imply that the predictions are approximate only for not-very-small D's. It may be speculated, however, that their accuracy should be higher than those of the asymptotic stochastic averaging—as the latter requires not only small variations of the response period, but also small variations of the response energy per cycle (and thus, small losses). This general expectation had been confirmed in reference [1] for the vibroimpact system by direct Monte-Carlo simulation.

Thus, the energy balance approach may provide better accuracy than the asymptotic one whenever the system's losses and excitation level are not very small, and a single expected value of a certain response characteristic is adequate for a given application—for example, to evaluate the efficiency of a "bang–bang" control. The type of response characteristic to be obtained from relation (1) is problem-dependent. For example, the expected response energy is predicted in four of the five specific problems considered in this paper, whereas the expected response amplitude, or peak value of the displacement, is predicted in the fifth one. The superior accuracy of the energy balance approach is demonstrated, by an extensive Monte-Carlo simulation, for all these five problems (previously, it has been done only for a vibroimpact system with zero offset of the barrier). Relation (1) is shown to provide reasonable results far beyond the applicability range of the stochastic averaging. Actually, establishing the range of applicability of the energy balance method is the main purpose of this paper.

The possibility for extending this approach to systems with non-linear restoring forces should also be mentioned here. Introducing a relevant potential energy function, one can obtain the same linear growth law for the corresponding total energy, leading eventually to the same energy-balance equation (1) for the steady state response. The difference for the non-linear case is in the RHS of this equation, where T should now depend on H, that is, on the instantaneous starting energy value of the response cycle, even if it is predicted approximately as a natural cycle duration for a system without random excitation. For a slightly non-linear system with smooth non-linearity, with $T(H)$ being linear in H, the linear part may be included into the RHS of equation (1), together, with the constant one [3]. As long as the energy loss in the LHS depends on the same H, the mean response energy can be predicted indeed. (In general, however, the functions of H in two sides of equation (1) may appear to be different, thereby precluding the desired estimate without independent information on the relation between these functions.)

Thus, this paper provides the following:

(1) Results of extensive Monte-Carlo simulations, evaluating accuracy of the method; in general, it was found to be reasonably good beyond the supposed limits for applicability of the asymptotic stochastic averaging approach for not-too-small values of the supposedly small parameter(s).
(2) Analytical solution to certain new random vibration problems by the method of direct energy balance—namely, for a system with "bang–bang" control of its moment of inertia and for a vibroimpact system with non-zero offset of the barrier.

2. VIBROIMPACT SYSTEM

As a first example, consider an s.d.o.f. mass–spring system, with a rigid barrier installed with an offset h from the system's static equilibrium position. The equations of motion

between impacts may then be written as

$$\dot{y} = v, \quad \dot{v} = -\Omega^2 y + \varsigma(t) \text{ for } y > -h. \tag{1a}$$

Thus, positive and negative values of h may imply, say, pretension and slack, respectively, in the mooring line of a floating moored body as excited by ocean waves. The excitation $\varsigma(t)$ is assumed here, as well as in all other examples in this paper, to be a zero-mean stationary Gaussian random white noise. Its intensity is denoted here by D. The impact/rebound condition, which should be satisfied at time instants t_*, when $y = -h$, may be written, by introducing a restitution factor r, as

$$v_+ = -rv_-, \quad v_\pm = v(t_* \pm 0), \quad y(t_*) = -h, \quad 0 < r \leqslant 1. \tag{2}$$

Introducing the total response energy $H(t)$ as

$$H = \dot{y}^2/2 + U(y), \qquad U(y) = \Omega^2 y^2/2, \qquad \dot{H} = \dot{y}(\ddot{y} + \Omega^2 y) = v\varsigma(t) \tag{3}$$

and applying conditional averaging for a given $H(0)$, yields, according to the basic SDE calculus [2, 4],

$$\dot{\bar{H}} = D/2, \qquad \bar{H}(t) = H(0) + Dt/2. \tag{4}$$

The energy evolution equation (4) may be applied to predict response energy at impact as well as conditional mean square impact velocity.

$$\bar{H}(t_* - 0) = H(0) + Dt_*/2, \qquad \bar{v}_-^2 = 2H(0) + Dt_* - 2H_*, \qquad H_* = U(-h) = \Omega^2 h^2/2. \tag{5}$$

The impact/rebound condition (2) is applied now to obtain mean square rebound velocity and response energy after rebound—that is, at the start of the next cycle:

$$\bar{v}_+^2 = r^2 \bar{v}_-^2, \qquad \bar{H}(t_* + 0) = r^2 [H(0) + Dt_*/2 - H_*] + H_*. \tag{6}$$

The unconditional averaging as denoted by angular brackets is applied now to relation (6). Imposing then the stationary condition for the expected energy at the start of a cycle yields the following reduced energy-balance relation:

$$\langle H(0) \rangle = \langle \bar{H}(T + 0) \rangle, \qquad T = \langle t_* \rangle, \qquad \langle H(0) \rangle = H_* + \frac{r^2 (DT/2)}{1 - r^2}. \tag{7}$$

The unconditional mean square impact velocity can be found now, using equations (5) and (7), as

$$\langle v_-^2 \rangle = 2(H(0) - H_*) = DT/(1 - r^2). \tag{8}$$

This result is the same as that obtained in reference [1] for the case $h = 0$. Of course, in general, the offset of the barrier h cannot but influence the response through the value of T. The cycle duration T in this work is approximated by the system's natural period. The latter can be easily obtained from equation (1a) with $\varsigma(t) \equiv 0$ as [4]

$$T(H) = \pi/\Omega + (2/\Omega)\sin^{-1}\sqrt{H_*/H} = \pi/\Omega + (2/\Omega)\sin^{-1}(\Omega h/\sqrt{2H}). \tag{9}$$

Thus, solution (8) is meaningful for sufficiently small h only, which leads to negligibly small variations of T due to the second term in expression (9). Thus, the system (1a) and (2), should be quasi-isochronous, although it should still be regarded s a strongly non-linear one.

It is interesting to compare the "exact" mean square velocity (8) (quotation marks are applied since the exact value of T is not available at present) with its limiting value for the case of small impact losses, i.e.,

$$\langle v_-^2 \rangle_{AS} = \lim_{r \to 1} \langle v_-^2 \rangle = DT/[2(1 - r)]. \tag{8'}$$

The latter expression can also be obtained by applying the asymptotic stochastic averaging method to the SDE (1a) with impact condition (2), as described in reference [1, 4]. Therefore, it should be valid only for values of $1 - r$, proportional to a small parameter. Actually, both "exact" and approximate solutions, (8) and (8′), respectively, rely on the approximation of the cycle duration by the system's natural period; therefore, they are based on the assumption of small D, and thus (implicitly) on that of small impact losses. However, Monte-Carlo simulations for the case $h = 0$ demonstrated a good accuracy of the energy-balance approach down to values $r = 0{\cdot}7$ [1]. (Actually, the expected response energy was predicted with a good accuracy by this approach with $T = \pi/\Omega$ even for $r = 0{\cdot}6$, the corresponding expected cycle duration was found to be rather lower than the natural period at such a high level of impact loss. Also they were certainly found to be superior to the asymptotic results for not-too-small values of $1 - r$. In other words, the superior convergence rate of the energy-balance approach has been confirmed indeed, for this example, compared with the asymptotic approach, which requires small energy variations per cycle.

Thus, in this example, the energy-balance approach provides certain reasonably accurate predictions of the random response far beyond the applicability range of the asymptotic stochastic averaging method. The derived formula for mean response energy at time instant $T/2$ was used, in particular, to obtain an improved "equivalent" viscous damping ratio $\alpha_{eq} = (\Omega/\pi)(1 - r^2)/(1 + r^2)$ to account for impact losses [1].

3. SYSTEM WITH DRY FRICTION

Consider now an s.d.o.f. system with Coulomb, or dry-friction damping, as governed by the following equation of motion:

$$\ddot{x} + R \operatorname{sgn} \dot{x} + \Omega^2 x = \varsigma(t),$$

where

$$R > 0 \text{ and } \operatorname{sgn} x = +1 \text{ for } x > 0, \operatorname{sgn} x = -1 \text{ for } x < 0. \tag{10}$$

This equation may also appear for a system with active response control, whenever magnitude of the control force is bounded [5], as long as the dry-friction control law is found to be the optimal one to reduce the steady state expected response energy $\langle H \rangle$. Namely, replacing the second term in the LHS of equation (10) by any other control law $u(x, \dot{x}, t)$ with $|u| \leqslant R$ may only increase $\langle H \rangle$ [5].

Introducing the response energy H, equation of motion (10) may be rewritten in a space-state form as follows:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\Omega^2 x_1 - R \operatorname{sgn} x_2 + \varsigma(t),$$
$$H = (1/2)(\Omega^2 x_1^2 + x_2^2), \qquad \dot{H} = -R|x_2| + x_2 \varsigma(t). \tag{11}$$

A conditional averaging is applied to this set of "physical" or Stratonovich SDEs denoted by bar, with the condition being the given values of state variables at a certain selected time instant. Using the Wong-Zakai correction for $H(t)$ yields

$$\dot{\bar{H}} = -R|x_2| + D/2. \tag{12}$$

The ODE (12) may be integrated directly within any time interval that does not contain reversals of velocity. The resulting variation of the conditional mean energy will be

TABLE 1

Non-dimensional expected response amplitudes $\langle A\rangle\Omega^2/R$ *for various values of the non-dimensional "dry-friction" force* $\mu = R/\sqrt{D\Omega}$

	$\mu = 1{\cdot}414$	$\mu = 1{\cdot}0$	$\mu = 0{\cdot}8$	$\mu = 0{\cdot}5$	$\mu = 0{\cdot}2$
Analytical	0·3927	0·7854	1·2272	3·1416	19·635
Numerical	0·3353	0·7118	1·141	3·0788	19·58

$-Rs + Dt/2$, where s is the traversed distance. Let this distance be just the instantaneous range x_{peak}–x_{trough}, denoted as $2A$, or the doubled-response amplitude, where the initial and final instant of time correspond to the pair of consecutive trough and peak of $x(t)$. Then the unconditional averaging for steady state response results in

$$\langle \Delta\bar{H}\rangle = -2R\langle A\rangle + DT/2 \cong -2R\langle A\rangle + D\pi/2\Omega = 0 \tag{13}$$

so that $\langle A\rangle \cong D\pi/4\Omega R$.

Here, T is the expected value of the time interval between the consecutive trough and peak, which once again is approximated here by the system's natural half-period. The resulting expression for the expected response amplitude is found to be the same as that obtained by stochastic averaging [5]. However, its range of applicability should not be restricted by the condition for small D and R, as long as the energy-balance approach does not require the variations of energy to be small within any response cycle. It should be added, that the stochastic averaging method has its advantage in that it permits to predict probability density of the response and also estimates the system's reliability with respect to the first-passage failure—see reference [5] for such analyses for system (10).

Table 1 presents numerical (Monte-Carlo) simulation data for the expected response amplitude, normalized with respect to the "dead zone" $\Delta = R/\Omega^2$. These data are compared with calculations according to the energy-balance formula (13), which yields $\langle A\rangle/\Delta = \pi/4\mu^2$, where $\mu = R/\sqrt{D\Omega}$ is a non-dimensional parameter of the "dry-friction" force. The agreement is observed to be very good for values of μ, which are small compared with unity. It is also reasonably good for values of the order of unity—that is, far beyond the expected range of applicability of the asymptotic methods. (In actual numerical simulations, values $D = 1$ and $\Omega = 1$ were assigned, whereas R was varied.)

4. SYSTEMS WITH THE GENERALIZED REVERSED SWINGS CONTROL

In this section, two externally excited s.d.o.f. systems are considered that are controlled through their parameter variation: a basic mass–spring system

$$(\mathrm{d}/\mathrm{d}t)(J\dot{\theta}) + k\theta = \varsigma(t) \tag{14}$$

and a pendulum

$$(\mathrm{d}/\mathrm{d}t)(L^2\dot{\theta}) + gL\theta = -L\varsigma(t). \tag{15}$$

The cases of feedback-controlled temporal variations of inertia J and of stiffness k will be considered for the first of these systems, whereas pendulum (15) will be controlled through imposed variations of its length L. The following law of the feedback-controlled variations

will be considered for all three cases:

$$q(t) = q_0\ [1 + R\,\mathrm{sgn}(\theta\dot{\theta})] \quad 0 < R < 1; \quad R = (q_+ - q_-)/(q_+ + q_-), \tag{16}$$

where plus and minus subscripts are applied to the maximal and minimal values, respectively, of the positive unifying parameter q. The latter may be the pendulum's length $L(t)$; or stiffness $k(t)$ or moment of inertia $J(t)$ of the s.d.o.f system (15) (dependence of the feedback-controlled parameters on the state variables is not shown explicitly just for brevity).

Equation (15) with zero RHS and $q = L$ is reduced to that of a person on swings if the term with R in equation (16) is taken with the opposite sign [6, 7]. Thus, the control law (16) was introduced in reference [6] as a "generalized reversed swings law", and this name has been used for s.d.o.f. system (14) as well (which has nothing to do with swings), just to describe the timing for stepwise variations of parameters. This law has been shown in reference [3] to be the optimal one for the so-called "long-term" response control for system (14) in case $q = k$, namely, it provides minimal expected steady state response energy among all laws with the given bound on magnitude R. Whilst the proof of optimality was obtained by a solution to the relevant Hamilton–Jacobi–Bellman PDE [3], it can also be obtained for other cases by using asymptotic theory for small R, as has been done for the case of swings in reference [6].

The energy-balance method will now be used to evaluate efficiency of the generalized reversed swings control.

4.1. INERTIA-CONTROLLED SYSTEM (14, 16) WITH $q = J$

The equations are rewritten, by introducing a new state variable p, as two first order SDEs, which are then supplemented with that for the response energy (per unit J_0):

$$\dot{\theta} = \frac{p}{[1 + R\,\mathrm{sgn}(\theta p)]}, \quad \dot{p} = -\Omega^2\theta + \bar{\varsigma}(t), \text{ where } \Omega^2 = k/J_0,$$

$$H = \frac{p^2}{2[1 + R\,\mathrm{sgn}(\theta p)]} + \frac{\Omega^2\theta^2}{2}, \qquad \dot{H} = \frac{p\bar{\varsigma}(t)}{[1 + R\,\mathrm{sgn}(\theta p)]}, \qquad \bar{\varsigma}(t) = J_0^{-1}\,\varsigma(t). \tag{17}$$

The last Stratonovich SDE is transformed to the Ito one, by applying the Wong-Zakai correction, and the conditional averaging is applied then, with the condition being the initial values of the state variables at $t = 0$. This results in the deterministic equation for the conditional expected energy, which describes linear growth of the response energy between stepwise parameter variations (the notation D is used here for the intensity of the scaled white noise $\bar{\varsigma}(t)$, so that the original white-noise excitation in the RHS of equation (14) has intensity DJ_0^2):

$$\dot{\bar{H}} = \frac{D}{2[1 + R\,\mathrm{sgn}(\theta p)]}, \qquad \bar{H} = \bar{H}(0) + \frac{Dt}{2[1 + R\,\mathrm{sgn}(\theta p)]}. \tag{18}$$

(Whilst these equations are similar to their counterparts (4) for the dry-friction case, the rate of (linear) energy growth is seen to be different for the two values of J.)

Consider now the variation of the response energy within a half-cycle, which starts slightly to the right of the system's equilibrium position (after the stepwise drop of the kinetic energy), so that both state variables are positive at $t = 0$. The random durations of the half-cycle and quarter-cycles are denoted by Θ with subcripts $\frac{1}{2}$ and $\frac{1}{4}$, respectively, and additional "plus" and "minus" subscripts for the quarter-cycles corresponding to the signs

in the basic generalized reversed swings law (16). The system's energy growth within each quarter-cycle can be obtained, by applying equation (18), as

$$\bar{H}(\Theta_{1/4}-0)=\bar{H}(0)+\frac{D\Theta_{1/4+}}{2[1+R]},\quad\text{and}\quad\bar{H}(\Theta_{1/2}-0)=\bar{H}(\Theta_{1/4}+0)+\frac{D\Theta_{1/4-}}{2[1-R]}. \tag{19}$$

The total energy does not experience any changes at the system's extreme positions, whereas the total energies before and after the stepwise parameter variation at the equilibrium position are related by the continuity condition for the angular momentum p as

$$\bar{H}(\Theta_{1/2}+0)=\bar{H}(\Theta_{1/2}-0)\left(\frac{1-R}{1+R}\right). \tag{20}$$

Combining equations (18–20), one can relate the response energy at the end of the half-cycle to that at the start of the half-cycle as

$$\bar{H}(\Theta_{1/2}+0)=\bar{H}(\Theta_{1/2}-0)\left(\frac{1-R}{1+R}\right)=\left\{\bar{H}(0)+\frac{D\Theta_{1/4+}}{2[1+R]}+\frac{D\Theta_{1/4-}}{2[1-R]}\right\}\left(\frac{1-R}{1+R}\right). \tag{21}$$

Whilst the response energy varies (randomly) from cycle to cycle, the basic response pattern repeats itself within all half-cycles, and the unconditional averaging (once again denoted by angular brackets) may be applied to equation (21). As long as the steady state response $H(t)$ is a stationary process, its expected value at the instants of zero-crossings by $\theta(t)$, i.e., at $t=0$, $t=T_{1/2}$, etc., should be a constant, so that

$$\langle\bar{H}(\Theta_{1/2}+0)\rangle=\langle\bar{H}(0)\rangle=\frac{(D/4\alpha_{eq})}{2}(1-R)\left[\frac{1}{\sqrt{1+R}}+\frac{1}{\sqrt{1-R}}\right], \tag{22}$$

as long as $\langle\Theta_{1/4\pm}\rangle=T_{1/4\pm}=(\pi/2\Omega)\sqrt{1+R}$, where $\alpha_{eq}=R\Omega/\pi$. (The expected time between stepwise parameter variations is once again approximated here by the corresponding natural quarter-periods of the free system (16), (18).)

The overall mean energy may be calculated now as the average-over-the-half-period of the piecewise-linear conditionally expected energy (19)

$$\langle H(t)\rangle=\frac{1}{T_{1/2}}\int_0^{T_{1/2}}\bar{H}(t)\,\mathrm{d}t=\sigma^2\phi(R),\quad\sigma^2=D/4\alpha_{eq},\quad\phi(R)=(1/2)(\sqrt{1+R}+\sqrt{1-R}). \tag{23}$$

The first co-factor in the final expression for the expected response energy is clearly seen to correspond to the limiting case $R\ll 1$, $\phi(R)\cong 1$. This case can be handled by the asymptotic stochastic averaging method. The latter also shows that the system behaves as one with a linear viscous damping, with the "equivalent" damping ratio $\Omega R/\pi$, and the angular response is asymptotically Gaussian, so that the response energy has an asymptotically exponential, stationary probability density. With increasing R, the expected response energy is seen to decrease from its limiting asymptotic value.

These analytical results are compared in Figure 1 with Monte-Carlo simulation data, as shown by the dotted line. The dashed line represents scaled expected response energy $\langle H\rangle/\sigma^2$, as calculated according to formula (23), whereas the horizontal solid line is its limiting (unity) value, as obtained by the asymptotic approach. The latter is seen to provide a reasonable accuracy (within 5%) up to $R=0{\cdot}5$, that is far beyond the expected applicability range of the theory for the supposedly small parameter R. However, the direct energy balance is seen to provide even better results for the not-too-small values of R, up to as high as $R=0{\cdot}9$.

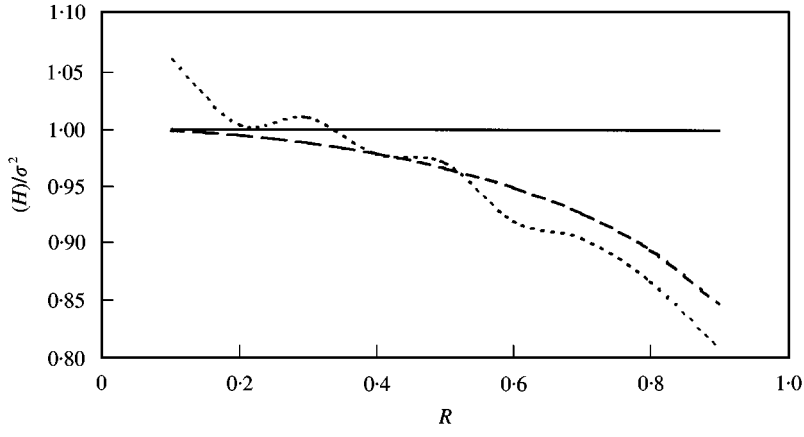

Figure 1. Efficiency of the inertia control: scaled expected response energy $\langle H\rangle/\sigma^2$ versus R according to the analytical solution by the direct energy balance approach (– – –) and Monte-Carlo simulation (·····). The horizontal line represents the asymptotic (stochastic averaging) value for small R.

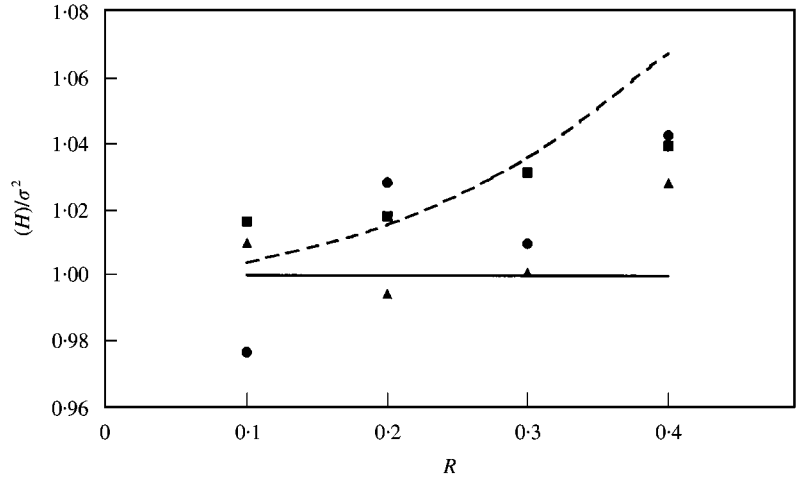

Figure 2. Efficiency of the stiffness control: scaled expected response energy $\langle H\rangle/\sigma^2$ versus R according to the analytical solution by the direct energy balance approach (– – –) and Monte-Carlo simulation for the following values of the excitation intensity: ●, $D = 10$; ▲, $D = 1$; ■, $D = 0{\cdot}1$. The horizontal line represents the asymptotic (stochastic averaging) value for small R.

4.2. STIFFNESS-CONTROLLED SYSTEM (14, 16) WITH $q = k$

Solution for the expected response energy has been obtained for this case in reference [3] as

$$\langle H\rangle = \sigma^2\psi(R), \quad \psi(R) = (1/2)[(1+R)^{-1/2} + (1-R)^{-1/2}] \tag{24}$$

with the same expressions for σ and α_{eq} as before. These analytical results for $\langle H\rangle/\sigma^2$ are represented in Figure 2 by the dashed line, whereas the horizontal solid line represents the limiting (unity) asymptotic value. Comparison with Monte-Carlo simulation data, shown by various symbols for three different values of D (and $\Omega = 1$) indicates a reasonable accuracy of both analytical approaches within the range $R < 0{\cdot}4$—once again, even for the not-very-small R's.

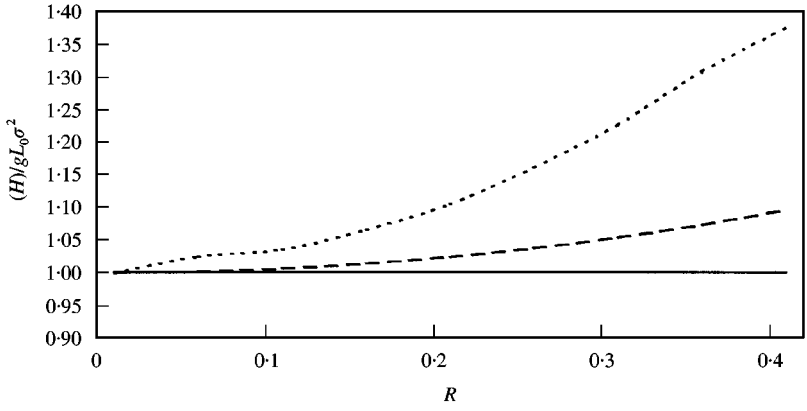

Figure 3. Efficiency of the length control for the pendulum (swings): scaled expected response energy $\langle H\rangle/gL_0\sigma^2$ versus R according to the analytical solution by the direct energy balance approach (– – –) and Monte-Carlo simulation (·····). The horizontal line represents the asymptotic (stochastic averaging) value for small R.

4.3. LENGTH-CONTROLLED PENDULUM, OR SWINGS (15, 16) WITH $q = L$

Solution for the expected response energy has been obtained for this case in reference [6] as

$$\langle H\rangle/gL_0 = \sigma^2\phi_L(R), \quad \phi_L(R) = \phi_0(R) - (3/2)R(\sqrt{1+R} + \sqrt{1-R})$$
$$- 4R[\phi_0(R) + 3R\sqrt{1-R}](1+R)^{-3/2}[\sqrt{1+R} + \sqrt{1-R})^{-1},$$
$$\sigma^2 = D_\xi\Omega^2/4\alpha_{eq}, \quad D_\xi = D/g^2, \quad \alpha_{eq} = 3\Omega R/\pi,$$
$$\phi_0(R) = (1/2)(1-R)[(1+R)^{5/2} + (1-R)^{5/2}](1+R^2/3)^{-1}, \tag{25}$$

where D_ξ is seen to be the intensity of the non-dimensional horizontal support acceleration in g's. Once again, in the asymptotic case of small R, the system behaves as one with the linear damping; the equivalent damping ratio, however, is found to be 3 times higher than that for system (14). This case is represented in Figure 3 by a solid horizontal line at the unit height, where the scaled expected response energy $\langle H\rangle/gL_0\sigma^2$ is given as a function of R. The dotted line represents the results of the Monte-Carlo simulations, whereas the dashed line represents the analytical solution (25). The latter is seen to provide some improvement of accuracy compared with the asymptotic stochastic averaging approach. This analysis may be used to evaluate efficiency of the reversed swings effect for controlling oscillations of load on shipboard cranes in rough seas.

5. CONCLUSIONS

The direct energy balance has proved itself to be an efficient and accurate approach for predicting the (non-linear) response of "piecewise-conservative" systems to white-noise random excitations for those cases. The method is certainly not universal. Firstly, it can be used only for those cases, where a simple estimate of the expected response level (expected energy) is sufficient for the given application. In particular, the method may be convenient for estimating the efficiency of the active feedback control systems, based on the use of "bang–bang" control laws. Such estimates may be used as important benchmarks, in spite of the fact that in real-life applications a low-pass filter may be included into the feedback

loop in order to avoid high-frequency chatter, whereas excitation may not be a white noise one, but rather just a broadband random process.

Secondly, the present version of the method will not be exact, as long as the expected time interval between two consecutive stepwise energy variations ("cycle duration") is approximated by a solution for the corresponding free vibration problem. However, extensive Monte-Carlo simulation studies for a variety of specific problems indicate a reasonable accuracy of the method far beyond the expected applicability range of the asymptotic approaches—for values of a supposedly small (compared with unity) non-dimensional parameter up to 0·4 and higher. Furthermore, the approach has a clear potential for improving its accuracy through the use of higher approximation for the expected cycle duration.

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