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LETTERS TO THE EDITOR

CLAMPED PLATES CONSIDERED AS THREE-DIMENSIONAL SOLIDS: A METHOD TO FIND ARBITRARY PRECISION FREQUENCIES

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1. INTRODUCTION

The problem of free vibrations of a plate clamped at four boundaries does not have, to the authors' knowledge, a classical solution within the theory of elasticity. A generalized solution previously developed by the authors for one, two and three dimensions [1–3] is employed in the analysis. Its application to the clamped plate considered as a three-dimensional solid (3-D) is a continuation of reference [3] in which the free vibration of arbitrary thickness plates with lateral shear diaphragms was solved by the same means.

The methodology is capable of solving a wide scope of differential problems—governed by either ordinary differential equations (ODEs) [4] or partial differential equations (PDEs) [5]. Additionally, non-linear problems have been handled successfully [6]. A linear combination of functions that belong to a complete set in L_2 is systematically stated as the *extremizing sequence*. Only the essential or geometric boundary conditions (BC) are to be satisfied, but this requirement is fulfilled by the sequence and not, in general, by each co-ordinate function. Eventual non-satisfied essential BC are taken into account using Lagrange multipliers. The extreme condition of a suitable functional stated in the sequences leads to arbitrary precision frequencies and mode shapes uniformly convergent to the classical solution. These assertions are based on theorems and corollaries not included here (see references [3, 7]). The authors name this methodology as whole element method (WEM) since the domain is considered as a unique element even when discontinuities such as intermediate supports, springs, masses, etc., are present [8]. An important conclusion has been drawn. Its use in any boundary-value problem reduces to the application of a *pseudo-theorem of Virtual Work* using the above-mentioned extremizing sequences. The present application provides a useful tool to verify technical theories such as the one developed by Mindlin for moderately thick plates and the adjustment of the factors involved therein.

The variational methods are well known [9–11] for linear problems governed by positive and symmetric (energy) functionals. In fact, the functional herein dealt with is of that type. However, it should be mentioned that WEM is able to handle other types of functionals. Additionally, in general, it should not be regarded as a traditional Ritz method. In effect, WEM makes use of certain extended series. The functions to be linearly combined

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a priori belong to a complete set in L_2 and each of them does not satisfy, in general, the essential BC.

Many authors have carried out relevant research on thick plate vibration problems taking into account diverse geometries and BC. Liew and co-workers [12, 13] have reported thorough papers on these subjects which include very complete bibliographic reviews. In particular, Liew *et al.* [13] report a Ritz formulation to tackle the vibration problem of thick plates including the present case.

In this paper, the plate is assumed as a regular prism of arbitrary aspect ratio and thickness with four consecutive faces clamped. By increasing the thickness from zero, one may model the thin, moderately thick and thick plate successively. By means of WEM, the frequency parameters are obtained for the transverse modes of vibration with arbitrary precision. Other modes (such as the "breathing" ones) might be easily considered. Also, a finite-element model using "brick" elements [14] was solved for comparison.

2. ENERGY FUNCTIONAL

The vibrational problem of deformable isotropic bodies in three dimensions is governed by the energy functional F given by the Theory of Elasticity as a function of the components u,v and w of the displacement vector:

$$F \equiv F[u, v, w] = \lambda/2\, \|\varepsilon_x + \varepsilon_y + \varepsilon_z\|^2 + \mu/4\,(\|\varepsilon_x\|^2 + \|\varepsilon_y\|^2 + \|\varepsilon_z\|^2)$$

$$+ \mu(\|\gamma_{xy}\|^2 + \|\gamma_{yz}\|^2 + \|\gamma_{zx}\|^2) - \rho\omega^2/2\,(\|u\|^2 + \|v\|^2 + \|w\|^2). \qquad (1)$$

where normal modes of frequency ω have been admitted, ρ is the body uniform density, $\mu = E/(1+v)$ and $\lambda = \mu v/(1-2v)$ are the Lamé's constants, with v being the Poisson coefficient, E the modulus of elasticity, ε_x, ε_y, ε_z and γ_{xy}, γ_{yz}, γ_{zx} are the components of the linear strain tensor.

3. EXTREMIZING SEQUENCES

Since we are dealing with a symmetric and positive functional [10], the sequences are minimizing. As is known, series such as the Fourier ones ensure the convergence in the mean (L_2) of any square integrable function. It was shown in reference [3] that the following functions are of uniform convergence in a regular prismatic domain $\{D\colon 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1, 0 \leqslant z \leqslant 1\}$:

$$s_i s_j s_k,\quad s_i s_j c_k,\quad s_i c_j s_k,\quad s_i c_j c_k,\quad c_i s_j s_k,\quad c_i s_j c_k,\quad c_i c_j s_k,\quad c_i c_j c_k, \qquad (2)$$

where the following notation was introduced:

$$\alpha_i \equiv i\pi,\quad \beta_j \equiv j\pi,\quad \gamma_k \equiv k\pi,\quad s_i \equiv \sin\alpha_i x,\quad s_j \equiv \sin\beta_j y,\quad s_k \equiv \sin\gamma_k z,$$

$$c_i \equiv \cos\alpha_i x,\quad c_j \equiv \cos\beta_j y,\quad c_k \equiv \cos\gamma_k z \quad (i, j, k = 0, 1, 2, \ldots).$$

However, WEM requires the uniform convergence of the so-called *essential functions*, which in this problem are u, v and w. A systematic method of generation of such series is reported in references [3, 7]. In short, one may start with the following two possibilities (among infinite):

$$\varphi_M(x, y, z) = \sum_i^{M} B_i(y, z) \sin \alpha_i x + \underline{x B_0(y, z) + b_0(y, z)}, \tag{3a}$$

$$\varphi_M(x, y, z) = \sum_i^{M} A_i(y, z) \cos \alpha_i x + A_0(y, z). \tag{3b}$$

The underlined terms in equation (3a) represent a support function. In turn, each function of (y, z) must be expanded in a similar fashion in each variable. Obviously, an infinite number of combinations may be obtained from this procedure and all of them would be of uniform convergence in D.

In particular and without loss of generality, we will concentrate our study of the transversal (bending) vibrational mode of the clamped plate at $x = 0, 1$ and $y = 0, 1$:

$$w(x, y, z) = w(x, y, 1 - z), \quad u(w, y, z) = -u(w, y, 1 - z), \quad v(x, y, z) = -v(x, y, 1 - z). \tag{4}$$

Now, if the plates were "simply supported" (prism with shear diaphragms) the essential conditions would be [3]

$$\text{at } x = 0, 1: \quad w = 0, v = 0; \quad \text{at } y = 0, 1: \quad w = 0, u = 0. \tag{5}$$

In this case, the complete 3-D trigonometric series are systematically generated achieving uniform convergence of u, v and w:

$$u_{LMN}(x, y, z) = \sum_i^{L} \sum_j^{M} \sum_k^{N} A_{ijk} c_i s_j c_k + \sum_j^{M} \sum_k^{N} A_{0jk} s_j c_k, \tag{6a}$$

$$v_{LMN}(x, y, z) = \sum_i^{L} \sum_j^{M} \sum_k^{N} B_{ijk} s_i c_j c_k + \sum_i^{L} \sum_k^{N} B_{i0k} s_i c_k, \tag{6b}$$

$$w_{LMN}(x, y, z) = \sum_i^{L} \sum_j^{M} \sum_k^{N} C_{ijk} s_i s_j s_k + \sum_i^{L} \sum_j^{M} d_{ij} s_i s_j, \tag{6c}$$

where k is odd ($k = 1, 3, 5, \ldots$). The clamped plate case requires the additional conditions

$$\text{at} \quad x = 0, 1: \quad u = 0; \quad \text{at } y = 0, 1: \quad v = 0, \tag{7}$$

which when written in terms of sequences (6) give place to

$$\sum_i^{L} A_{ijk} + A_{ojk} = 0, \quad \sum_i^{L} A_{ijk}(-1)^i + A_{ojk} = 0,$$

$$\sum_j^{M} B_{ijk} + B_{iok} = 0, \quad \sum_j^{M} B_{ijk}(-1)^j + B_{iok} = 0, \tag{8}$$

These non-identically satisfied conditions are taken into account by means of Lagrange multipliers P_{lm}, Q_{lm}, R_{lm} and S_{lm} with an extended functional

$$F_{LMN_e} = F_{LMN} - \sum_j^M \sum_k^N \left[P_{jk}\left(\sum_i^L A_{ijk} + A_{ojk} \right) + Q_{jk}\left(\sum_i^L A_{ijk}(-1)^i + A_{ojk} \right) \right]$$
$$- \sum_i^L \sum_k^N \left[R_{ik}\left(\sum_j^M B_{ijk} + B_{iok} \right) + S_{ik}\left(\sum_j^M B_{ijk}(-1)^j + B_{iok} \right) \right]. \quad (9)$$

4. FOUNDATIONS

As was mentioned, the minimizing sequences yield the uniform convergence of u, v and w that are continuous functions. But some derivatives of the sequences only comply with convergence in the mean (in L_2). However, it is sufficient to demonstrate that the frequencies obtained with WEM are exact. For brevity, neither the theorems and corollaries statements nor the demonstrations are included herein. The interested reader may refer to references [2, 3, 7].

4.1. APPLICATION PROCEDURE

The extreme condition (corollary, reference [3]) should be imposed on the extended functional stated in the extremizing sequence

$$\delta F_{LMN_e} = 0, \quad (10)$$

where δ denotes the first variation with respect to the sequence constants. It was shown [15] that this equation is equivalent to a *pseudo-theorem of Virtual Work* stated in the extremizing sequences and adding the terms involving the Lagrange multipliers. This conclusion would allow working with the differential equation avoiding the statement of the functional. Equation (10) results in

$$\lambda(I_{LMN}, \delta I_{LMN}) + \mu[(\varepsilon_{x_{LMN}}, \delta\varepsilon_{x_{LMN}}) + (\varepsilon_{y_{LMN}}, \delta\varepsilon_{y_{LMN}}) + (\varepsilon_{z_{LMN}}, \delta\varepsilon_{z_{LMN}})]$$
$$+ \frac{\mu}{2}[(\gamma_{xy_{LMN}}, \delta\gamma_{xy_{LMN}}) + (\gamma_{yz_{LMN}}, \delta\gamma_{yz_{LMN}}) + (\gamma_{zx_{LMN}}, \delta\gamma_{zx_{LMN}})]$$
$$- \rho\omega^2[(u_{LMN}, \delta u_{LMN}) + (v_{LMN}, \delta v_{LMN}) + (w_{LMN}, \delta w_{LMN})]$$
$$- \sum_j \sum_k \delta P_{jk}\left(\sum_i A_{ijk} + A_{ojk} \right) - \sum_j \sum_k P_{jk}\left(\sum_i \delta A_{ijk} + \delta A_{ojk} \right)$$
$$- \sum_j \sum_k \delta Q_{jk}\left(\sum_i A_{ijk}(-1)^i + A_{ojk} \right) - \sum_j \sum_k Q_{jk}\left(\sum_i \delta A_{ijk}(-1)^i + \delta A_{ojk} \right)$$
$$- \sum_i \sum_k \delta R_{ik}\left(\sum_j B_{ijk} + B_{iok} \right) - \sum_i \sum_k R_{ik}\left(\sum_j \delta B_{ijk} + \delta B_{iok} \right)$$
$$- \sum_i \sum_k \delta S_{ik}\left(\sum_j B_{ijk}(-1)^j + B_{iok} \right) - \sum_i \sum_k S_{ik}\left(\sum_j \delta B_{ijk}(-1)^j + \delta B_{jok} \right) = 0. \quad (11)$$

5. RESULTS

After the application of equation (11) and taking factors (according to equations (6) and (9)), $\sum_i\sum_j\sum_k \delta A_{ijk}$, $\sum_i\sum_j\sum_k \delta B_{ijk}$, $\sum_i\sum_j\sum_k \delta C_{ijk}$, $\sum_i\sum_j \delta d_{ij}$, $\sum_j\sum_k \delta P_{jk}$, $\sum_j\sum_k \delta Q_{jk}$, $\sum_i\sum_k \delta R_{ik}$, $\sum_i\sum_k \delta S_{ik}$, the following equations are obtained for the transverse mode of vibration (index O denotes odd); the last four lead to two equations if symmetric and antisymmetric conditions are considered:

$$(\lambda + 2G)\left(\frac{\alpha_i^2}{a^2} A_{ijk}\right) + \lambda\left(\frac{\alpha_i\beta_j}{ab} B_{ijk} - \frac{\alpha_i\gamma_k}{ah} C_{ijk}\right) + G\left[-\frac{\gamma_k}{h}\left(\frac{\alpha_i}{a} C_{ijk} - \frac{\gamma_k}{h} A_{ijk} + \frac{2\alpha_i}{a} d_{ij} I_k\right) + \frac{\beta_j}{b}\left(\frac{\beta_j A_{ijk}}{b} + \frac{\alpha_i B_{ijk}}{a}\right)\right] + q_{jk} - \Omega^{*^2} A_{ijk} = 0, \quad (12)$$

$$(\lambda + 2G)\left(\frac{\beta_j^2}{b^2} B_{ijk}\right) + \lambda\left(\frac{\alpha_i\beta_j}{ab} A_{ijk} - \frac{\beta_j\gamma_k}{bh} C_{ijk}\right) + G\left[-\frac{\gamma_k}{h}\left(\frac{\beta_j}{b} C_{ijk} - \frac{\gamma_k}{h} B_{ijk} + \frac{2\beta_j}{b} d_{ij} I_k\right) + \frac{\alpha_i}{a}\left(\frac{\beta_j A_{ijk}}{b} + \frac{\alpha_i B_{ijk}}{a}\right)\right] + s_{ik} - \Omega^{*^2} B_{ijk} = 0, \quad (13)$$

$$(\lambda + 2G)\left(\frac{\gamma_k^2}{h^2} C_{ijk}\right) - \lambda\left(\frac{\alpha_i\gamma_k}{ah} A_{ijk} + \frac{\beta_j\gamma_k}{bh} B_{ijk}\right) + G\left[\frac{\beta_j}{b}\left(\frac{\beta_j}{b} C_{ijk} - \frac{\gamma_k}{h} B_{ijk} + \frac{2\beta_j}{b} d_{ij} I_k\right) + \frac{\alpha_i}{a}\left(\frac{\alpha_i C_{ijk}}{a} - \frac{\gamma_k A_{ijk}}{h} + \frac{2\alpha_i}{a} d_{ij} I_k\right)\right] - \Omega^{*^2}[C_{ijk} + 2d_{ij} I_k] = 0, \quad (14)$$

$$G\left[\frac{\beta_j^2}{b^2} + \frac{\beta_j^2}{b^2}\sum_l C_{ijl} I_l - \frac{\beta_j}{bh}\sum_l \gamma_l B_{ijl} I_l + \frac{\alpha_i^2}{a^2} d_{ij} + \frac{\alpha_i^2}{a^2}\sum_l C_{ijl} I_l - \frac{\alpha_i}{ah}\sum_l \gamma_l A_{ijl} I_l\right] - \Omega^{*^2}\left[d_{ij} + \sum_l C_{ijl} I_l\right] = 0, \quad (15)$$

$$\sum_{i=O} A_{ijk} = 0, \qquad \sum_{j=O} B_{ijk} = 0, \quad (16,\ 17)$$

with $q_{jk} \equiv P_{jk} - Q_{jk}$, $s_{ik} \equiv R_{ik} - S_{ik}$, $\Omega^{*^2} \equiv \rho\omega^2$, $I_m = (1, \sin mz)$, $G = v/2$.

TABLE 1

Fundamental frequency parameters for a clamped rectangular plate. The Poisson coefficient: $\nu = 0{\cdot}3$

Aspect ratio a/b	Height/side ratio h/b	$(a/b)^2\ \Omega$		
		Liew [13]	WEM ($M = 700$)	FEM [14] (4800 el.)
1	0·2	26·9055	26·8922	26·9854
	0·3	21·8690	21·8555	21·9111
	0·5	15·2939	15·2814	15·3104
1·5	0·2	47·6907	47·6685	47·9493
	0·3	39·5588	39·5372	39·6865
	0·5	28·3089	28·2907	28·3541
2	0·2	77·9736	77·9420	78·6235
	0·3	64·8236	64·7922	65·1606
	0·5	46·4660	46·4388	46·5773

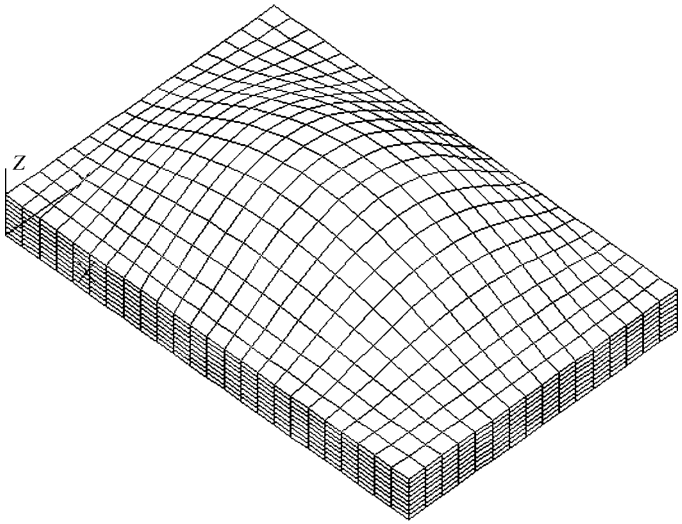

Figure 1. Fundamental transverse mode shape of a prismatic solid (rectangular plate clamped at four consecutive boundaries of arbitrary thickness).

Numerical results of the transverse frequency parameter $\Omega = \sqrt{\rho h/D}$. ($D = Eh^3/12(1 - \nu^2)$) were obtained for odd values of i, j and k using the WEM solution—equations (12)–(17). The results are compared with a 3-D Ritz solution [13] and with a finite-element model using 3-D (brick)-type elements solved with ALGOR [14]. Table 1 depicts the non-dimensional natural frequencies for a prism with aspect ratio $a/b = 1$, 1·5, 2 and height ratio $h/b = 0{\cdot}2$, 0·3, 0·5. In all cases, the reported values correspond to the first transverse (bending) mode and $\nu = 0{\cdot}3$. The WEM values were found with 700 terms in the sums. The procedure to achieve a certain number of exact digits consists in fixing the desired accuracy and increasing the number of terms until the goal is attained. For simplicity, in this example, all the plates were studied with the same number of terms. An example of convergence is reported in reference [3]. Figure 1 shows the fundamental mode of the plate considered as a 3-D solid and clamped at four consecutive boundaries.

6. CONCLUSIONS

The exact natural frequencies corresponding to the transverse mode of vibration of a rectangular prism clamped at four consecutive boundaries are analyzed in this paper. The methodology employed in the study is WEM, a direct variational method previously developed by the authors for boundary-value problems in one-, two- and three-dimensional domains. Also, applications to initial conditions and initial-boundary value problems have been successful. Non-linearities as well as non-conservative forces have also been addressed.

Summarizing, the algorithm is reduced to an iterative process that involves summations. Theoretically, any rectangular clamped plate of arbitrary thickness may be handled with it. In practice, as h/b gets smaller ($\rightarrow \infty$), more terms should be taken in the sums to achieve a good precision. The authors have developed another algorithm for the particular case of thin plates [2]. It is not included here but it may be shown that in the limit when $\rightarrow \infty$, the solution found with WEM is coincident with the frequency values given by the theory of Germain–Lagrange. When bending (transverse) modes are selected, one is considering subsets of the complete set of possible mode shapes. Such subsets are also complete in L_2 which assures the exactness of the frequencies and the uniform convergence of the mode shapes.

Here, only fundamental frequencies are reported but higher ones may be found without additional difficulties. The values obtained with WEM are compared with the ones obtained by Liew and co-workers using a three-dimensional Ritz formulation. Also, the authors modelled the prismatic solid using FEM. Both methods yield higher bounds.

The availability of the exact solution by means of this methodology allows to correct values found with approximate methods and, in the case of other theories, such as the widely used theory for moderately thick plates (Mindlin), it is possible to correct the shear coefficients involved there. Among the advantages, additional to the exactness of the frequencies and the uniform convergence of the modes, the systematic statement of the solution (extremizing sequence) is formally the same in *all* the problems. It is worth mentioning that as far as the differential problem is completely stated (equations and boundary and/or initial conditions), the application of WEM leads to a *pseudo-theorem of Virtual Work* in the extremizing sequences; this feature makes the method applicable to problems in which the functional does not exist in the classical sense. The authors have made use of this advantage when solving non-conservative and/or non-linear problems by stating *ad hoc* functionals that (obviously, they are not symmetrical nor positive definite) give place to the *pseudo-theorem of Virtual Work*. At present, the authors are studying rectangular prisms with other boundary conditions for which the classical solutions are not available.

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