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IMPROVEMENT OF THE SEMI-ANALYTICAL METHOD, FOR DETERMINING THE GEOMETRICALLY NON-LINEAR RESPONSE OF THIN STRAIGHT STRUCTURES. PART I: APPLICATION TO CLAMPED–CLAMPED AND SIMPLY SUPPORTED–CLAMPED BEAMS

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In a previous series of papers (Benamar 1990 *Ph.D. Thesis, University of Southampton*; Benamar *et al.* 1991 *Journal of Sound and Vibration* **149**, 179–195; **164**, 399–424 [1–3]) a general model based on Hamilton's principle and spectral analysis has been developed for non-linear free vibrations occurring at large displacement amplitudes of fully clamped beams and rectangular homogeneous and composite plates. The results obtained with this model corresponding to the first non-linear mode shape of a clamped–clamped (CC) beam and to the first non-linear mode shape of a CC plate are in good agreement with those obtained in previous experimental studies (Benamar *et al.* 1991 *Journal of Sound and Vibration* **149**, 179–195; **164**, 399–424 [2, 3]). More recently, this model has been re-derived (Azar *et al.* 1999 *Journal of Sound and Vibration* **224**, 377–395; submitted [4, 5]) using spectral analysis, Lagrange's equations and the harmonic balance method, and applied to obtain the non-linear steady state forced periodic response of simply supported (SS), CC, and simply supported–clamped (SSC) beams. The practical application of this approach to engineering problems necessitates the use of appropriate software in each case or use of published tables of data, obtained from numerical solution of the non-linear algebraic system, corresponding to each problem. The present work was an attempt to develop a more practical simple "multi-mode theory" based on the linearization of the non-linear algebraic equations, written on the modal basis, in the neighbourhood of each resonance. The purpose was to derive simple formulae, which are easy to use, for engineering purposes. In this paper, two models are proposed. The first is concerned with displacement amplitudes of vibration W_{max}/H, obtained at the beam centre, up to about 0·7 times the beam thickness and the second may be used for higher amplitudes W_{max}/H up to about 1·5 times the beam thickness. This new approach has been successfully used in the free vibration case to the first, second and third non-linear modes shapes of CC beams and to the first non-linear mode shape of a CSS beam. It has also been applied to obtain the non-linear steady state periodic forced response of CC and CSS beams, excited harmonically with concentrated and distributed forces.

1. INTRODUCTION

The topic of non-linear vibration of beams is of continuing interest, due to their frequent use as experimental test pieces [6–8] and because they constitute the simplest case of a continuous system, since their motion is represented by one-dimensional partial differential equations in space. Hence, they may be very useful for exploring and validating new theoretical and numerical approaches in the field of non-linear structural dynamics. As stated in reference [9], in spite of the considerable amount of research which has been carried out in the last few decades on non-linear vibrations, linear theories remain widely used in most of the practical applications, particularly in the field of modal testing. This may be attributed, among other reasons, to the fact that the various attempts to describe mathematically non-linear structural dynamic behaviour which have been developed and presented in the literature in the last few decades still appear somewhat esoteric and difficult to deal with in practical situations. This is either because the theories are quite complex and involve a variety of new concepts and predict a variety of new phenomena, unexpected within the frame of linear theories or because the associated software is not available, or is practically difficult to use. This seems to be the reason why it is sometimes thought among the scientific community working in the field of structural dynamics that non-linearity is a matter which may be dealt with only by few initiated people, or which should be considered, as stated in reference [9], as “a collective term for what we cannot accommodate or explain”. Therefore, while theoretical and experimental investigations should be continued in order to discover and describe new fascinating aspects and to develop new sophisticated analysis tools, appropriate to the non-linear world, much effort has to be directed towards studying at least some of the most important among the already known non-linear effects. These include the non-linear increase in resonant frequency and stresses with the amplitude of vibration, or the significant contributions of the higher modes in the non-linear steady state periodic forced response, which should be made reasonably easy to take into account in engineering applications, especially those related to structural safety. Such an effort will have impact on the design of high-performance structures, aircraft for example, but also will provide engineers, designers, and scientists with appropriate tools ensuring more accuracy and efficiency in current situations. The aim of this paper is the presentation of a contribution to this effort. The non-linear free and forced response of CC and CSS beams at large vibration amplitudes is taken as a first example illustrating this new approach. The application to plates will be presented later.

Previous experimental and theoretical studies have shown that the fundamental and higher mode shapes of beams are clearly dependent on the amplitude of vibration. High values of increase of beam curvatures were noticed near the clamps of structures with constrained ends, causing a highly non-linear increase in bending strain with increasing deflection, instead of the linear rate of increase predicted by the linear theory [7]. It has also been shown that such a non-linear effect may have a significant effect on the structural fatigue life [8].

In a previous series of works [1–3, 10–13], a model based on Hamilton’s principle and spectral analysis has been developed for non-linear free vibrations of thin straight structures, such as beams, homogeneous and composite plates, and shells. In this model, the non-linear free vibration problem has been reduced to the solution of a set of non-linear algebraic equations, which has been performed numerically using appropriate algorithms in order to obtain a set of non-linear mode shapes for the structure considered in each case, with the corresponding amplitude-dependent non-linear frequencies. Experimental non-linear CC beam first mode shape data obtained in reference [7] for a non-dimensional vibration displacement amplitude at the beam centre of $W_{max}/H \approx 2{\cdot}04$, and results obtained from solution of the set of non-linear algebraic equations corresponding to the same non-dimensional amplitude have shown that the curve for the measured normalized mode shape is well above the normalized theoretical linear mode shape but very close to that of the normalized non-linear theoretical mode calculated using the non-linear model mentioned above [2]. More recently, this model has been re-derived, using spectral analysis and Lagrange's equations, and has been extended to the non-linear steady state periodic forces response, leading to a set of coupled partial derivative equations which has been considered as a multi-dimensional form of the very well-known Duffing equation. Assuming harmonic motion and applying the harmonic balance method, the above set has been transformed into a set of non-linear algebraic equations with a rightside term corresponding to the generalized forces depending on the type of excitation force (concentrated or distributed) and its domain of application. This approach has been applied to determine numerically a multi-mode steady state periodic forced response of SS, CC, and SSC beams [4, 5], and the validity of the results obtained has been established via a careful comparison with other approaches and with experimental measurements [1, 4, 5, 13]. The main features of the approach presented above are from reference [1].

1) It is not subject to the practical limitation of weak non-linearity in its formulation, as was the case for some models for non-linear vibration based on the perturbation procedure developed previously.†
2) Its formulation is quite simple and does not contain any incremental procedure as in some finite element approaches.
3) Periodic solutions can be obtained directly with any desired accuracy as solutions of the set of non-linear algebraic equations.
4) Once the most significant contributing functions are known, engineering applications can be made easily using data tables or rapid computer programs using only a small number of appropriate functions.
5) Once the contributions of the functions are calculated, the resulting strains and stresses can be obtained directly, using the analytical expressions for the derivatives of the basic functions.
6) Also, this method makes the non-linear effects appear not only via the amplitude frequency dependence of the displacement amplitude, but also via the dependence of the deflection shape on the amplitude of vibration. This allows quantitative estimate of the non-linear stresses in sensible regions of the structure to be obtained, which is of crucial importance in the fatigue life prediction of structures working in a severe environment.

However, although the works mentioned above made it quite easy to calculate the non-linear mode shapes, the non-linear frequencies and the non-linear bending stress patterns of the structures considered, via the numerical solution, of a small set of non-linear algebraic equations (5 for beams [2], 8 for fully clamped plates [3], 11 for shells [12], 17 for CCCSS plates [15], 17 for fully clamped composite plates [16]), it was thought that investigations could be directed towards a further step in the development of a sort of non-linear modal analysis theory. This should allow explicit and easy calculation of the non-linear free and steady state periodic forced response of thin straight structures, in terms of their classical mass and rigidity matrices, the non-linear geometrical rigidity tensor introduced in the above model, and the amplitude of vibration. It was also hoped that such an attempt could provide users with simple formulae, ready to use for engineering purposes,

†This limitation has been overcome to some extent in the asymptotic numerical model presented in reference [14].

which would be, in their corresponding intervals of validity, much more practical than the published tables of data, obtained from the numerical solution of a non-linear algebraic system, which necessitates the use of appropriate software in each case. Also, these new formulae could be interesting from the analytical point of view, since they may be implemented in further theoretical works, investigating other non-linear effects, such as the non-linear response harmonic distortion spatial distribution, the internal resonance, fatigue life predicting models, etc.

The purpose of this paper is the presentation of the results of this investigation and discussion of the range of validity of the simple formulae proposed for both the non-linear free and steady state periodic forced response of CC and CSS beams. The plate case will be presented later.

In the next section, a review of the theory mentioned above and some numerical results obtained by solving the non-linear algebraic equations are presented. The third section is concerned with the new approach for free vibration analysis. The theory and numerical results for the first three non-linear modes of CC beams and the first non-linear mode of SSC beams are presented. In the fourth section, the theoretical formulation for the forced case is derived and applied for a harmonic concentrated excitation force. In each section, the results obtained by the new approach are discussed, to determine accurately the limit of validity of each formulation, via comparison with previous known results.

2. REVIEW OF THE BASIC THEORY FOR DETERMINATION OF THE NON-LINEAR MODE SHAPES AND RESONANCE FREQUENCIES OF BEAMS AT LARGE VIBRATION AMPLITUDES

Since the objective of the present paper is to present an improvement of the theoretical model for non-linear free and forced vibrations developed in references [1, 2, 4, 5], one starts by presenting in this section a brief review of the theory, in order to make to easy for the reader to understand the notation and the analytical developments presented in the next sections.

Consider transverse vibrations of a beam having the geometrical and material characteristics (S, L, H, I, E, ρ) defined in the notation list. The total beam strain energy can be written as the sum of the strain energy due to bending denoted as V_b, plus the axial strain energy due to the axial load induced by large deflection V_a.

V_b, V_a and the kinetic energy T are given by [1, 2]

$$V_b = \tfrac{1}{2}\int_0^L EI\left(\frac{\partial^2 W}{\partial x^2}\right)^2 \mathrm{d}x, \qquad V_a = \frac{ES}{8L}\left(\int_0^L \left(\frac{\partial W}{\partial x}\right)^2\right)^2, \qquad T = \tfrac{1}{2}\rho S\int_0^L \left(\frac{\partial W}{\partial t}\right)^2 \mathrm{d}x, \tag{1–3}$$

in which W is the beam transverse displacement. Using a generalized parameterization and the usual summation convention used in reference [2], the transverse displacement can be written as

$$W(x, t) = q_i(t) w_i(x). \tag{4}$$

Substituting W in the expressions for V_b, V_a, T and rearranging leads to

$$V_b = \tfrac{1}{2}\, q_i q_j k_{ij}, \quad V_{\mathrm{a}} = \tfrac{1}{2}\, q_i q_j q_k q_l b_{ijkl}, \quad T = \tfrac{1}{2}\, \dot{q}_i \dot{q}_j m_{ij}, \tag{5–7}$$

where m_{ij}, k_{ij} and b_{ijkl} are defined in references [1, 2] as

$$m_{ij} = \rho S\int_0^L w_i(x) w_j(x)\,\mathrm{d}x, \quad k_{ij} = \int_0^L \left(\frac{\partial^2 w_i}{\partial x^2}\right)\left(\frac{\partial^2 w_j}{\partial x^2}\right)\mathrm{d}x, \tag{8, 9}$$

$$b_{ijkl} = \frac{ES}{4L}\int_0^L \left(\frac{\partial w_i}{\partial x}\right)\left(\frac{\partial w_j}{\partial x}\right)\mathrm{d}x \int_0^L \left(\frac{\partial w_k}{\partial x}\right)\left(\frac{w_l}{\partial x}\right)\mathrm{d}x. \tag{10}$$

The dynamic behaviour of the structure may be obtained by Lagrange's equations for a conservative system, which leads to

$$(\partial/\partial t)\,(\partial T/\partial \dot{q}_r) + \partial T/\partial q_r - \partial V/\partial q_r = 0, \quad r = 1, \ldots, n. \tag{11}$$

Replacing in this equation T and $V = (V_a + V_b)$ by their expressions given above, leads to the following set of coupled Duffing equations:

$$\ddot{q}_i m_{ir} + q_i k_{ir} + 2 q_i q_j q_k b_{ijkr} = 0, \qquad r = 1, \ldots, n, \tag{12}$$

which can be written in matrix form as

$$[\mathbf{M}]\{\ddot{\mathbf{q}}\} + [\mathbf{K}]\{\mathbf{q}\} + 2[\mathbf{B}(\{\mathbf{q}\})]\{\mathbf{q}\} = \{\mathbf{0}\}, \tag{13}$$

where $[\mathbf{M}]$, $[\mathbf{K}]$, $[\mathbf{B}]$ and $\{\mathbf{q}\}$ are respectively the mass matrix, linear rigidity matrix, non-linear rigidity matrix depending on $\{\mathbf{q}\}$ and the column vector of generalized parameters $\{\mathbf{q}\}^{\mathrm{T}} = [q_1 q_2, \ldots, q_n]$.

Now assuming harmonic motion

$$q_i(t) = a_i \cos(\omega t), \tag{14}$$

substituting equation (14) into equation (13) and applying the harmonic balance method leads to

$$2([\mathbf{K}] - \omega^2[\mathbf{M}])\{\mathbf{A}\} + 3[\mathbf{B}(\mathbf{A})]\{\mathbf{A}\} = \{\mathbf{0}\}, \tag{15}$$

in which $\{\mathbf{A}\}$ is the column vector of the basic functions contribution coefficients

$$\{\mathbf{A}\}^{\mathrm{T}} = \{a_1 a_2, \ldots, a_n\}.$$

To obtain non-dimensional parameters, one puts, as in reference [1],

$$w_i(x) = H w_i^*(x/L) = H w_i^*(x^*), \qquad \omega^2/\omega^{*2} = EI/\rho S L^4, \tag{16, 17}$$

$$k_{ij}/k_{ij}^* = EIH^2/L^3, \quad m_{ij}/m_{ij}^* = \rho S H^2 L, \quad b_{ijkl}/b_{ijkl}^* = EIH^2/L^3, \tag{18–20}$$

Substituting these equations into equation (15) leads to

$$([\mathbf{K}^*] - \omega^{*2}[\mathbf{M}^*])\{\mathbf{A}\} + \tfrac{3}{2}[\mathbf{B}^*(\mathbf{A})]\{\mathbf{A}\} = \{\mathbf{0}\} \tag{21}$$

which may be written also, using tensor notation, as

$$-\,\omega^{*2} a_i\, m_{ir}^* + a_i k_{ir}^* + \tfrac{3}{2}\, a_i\, a_j\, a_k\, b_{ijkr}^* = 0, \quad r = 1, \ldots, n. \tag{22}$$

Equation (22) is identical to that obtained in reference [2] for the non-linear free vibrations of beams and plates using Hamilton's principle and integration over the range $(0, 2\pi/\omega)$. These equations are a set of non-linear algebraic equations, involving the parameters m_{ij}^*, k_{ij}^* and b_{ijkl}^* which have been computed numerically by a routine called PREP. In order to obtain the numerical solution for the non-linear problem in the neighbourhood of a given mode, the contribution of this mode is chosen and those of other modes are calculated numerically using the Harwell library routine NS01A. For the first mode, the procedure consisted of fixing a_1 and calculating the higher mode contributions from the system

$$- \omega^{*2} a_i m_{ir}^* + a_i k_{ir}^* + \tfrac{3}{2} a_i a_j a_k b_{ijkr}^* = 0, \quad r > 1, \tag{23}$$

in which ω^{*2} is obtained from the principle of conservation of energy as

$$\omega^{*2} = (a_i a_j k_{ij}^* + a_i a_j a_k a_l b_{ijkl}^*)/a_i a_j m_{ij}^*. \tag{24}$$

Numerical data corresponding to the first three CC beam mode shapes have been computed and tabulated ref. [2] for a wide range of vibration amplitudes.

3. THE NEW APPROACH FOR LARGE-AMPLITUDE FREE VIBRATIONS OF BEAMS

3.1. GENERAL FORMULATION

The purpose of this paper is to replace the numerical solution of the set of non-linear algebraic equations (23), necessary to obtain the beam non-linear mode shapes and resonance frequencies at large vibration amplitudes, by two equivalent simple formulations, ready to use for engineering purposes. Then, comparison of the new results with the previous ones is made in order to determine exactly the limit of validity of each formulation. Analytical details are given in this section for the first three non-linear mode shapes of a CC beam. Results for the case of a CSS beam, obtained similarly are presented in section 3.2.3.

Consider the large vibration displacements of a beam in the neighbourhood of its first resonant frequency. Following the basic functions choice adopted in reference [2], to obtain the values of the linear rigidity matrix $\mathbf{k}_{ij}^*$ and non-linear geometrical rigidity tensor b_{ijkl}^* of the first and third non-linear mode shapes of a CC beam, the first six normalized symmetric CC beam functions $w_1^*, w_3^*, \ldots, w_{11}^*$ have been used (see Appendix A). The functions w_i^* have been normalized in such a manner that the obtained mass matrix equals the identity matrix. For the second non-linear mode shape, the first six antisymmetric CC beam functions $w_2^*, w_4^*, \ldots, w_{12}^*$ have been used to determine the modal parameters k_{ij}^* and b_{ijkl}^*. In the case of the first non-linear mode shape of a CSS beam, the first six normalized CSS beam functions denoted $\phi_1^*, \phi_2^*, \ldots, \phi_6^*$, have been used to calculate the modal parameters, which are given in Appendix A.

To illustrate the main idea behind the present approach, data obtained via the solution of the non-linear algebraic system (23) previously published in reference [2] are presented here in Table 1(a). It can be seen in this table, corresponding to the CC beam first non-linear mode shape, that the contribution a_1 of the first basic function, which is the first CC beam linear mode shape, remains predominant for the whole range of vibration amplitudes considered, compared to the contributions of the other functions. So the contribution coefficient vector $\{\mathbf{A}\}$ defined in reference [2] by $\{\mathbf{A}\}^{\mathrm{T}} = [a_1, a_3, \ldots, a_{11}]$ can be written as $\{\mathbf{A}\}^{\mathrm{T}} = [a_1\, \varepsilon_3, \ldots, \varepsilon_{11}]$ in which ε_i, representing the ith basic function contribution, may be considered as small, compared to a_1, for $i = 3, 5, \ldots, 11$. Since the non-linearity parameters b_{ijkl} defined in equation (10) are of the same order of magnitude (see Appendix A), due to the above observation, some terms may be neglected in the non-linear expression $a_i a_j a_k b_{ijkr}$ in equation (23), which leads to two simple formulations, called in the remainder of this paper the first and the second formulation.

3.1.1. *The first formulation*

The first formulation is based on an approximation which consists in neglecting in the expression $a_i a_j a_k b_{ijkr}$ of equation (23) both first and second order terms with respect to ε_i, i.e., terms of the type $a_1^2\varepsilon_k b_{11kr}$ or of the type $a_1\varepsilon_j\varepsilon_k b_{1jkr}$ so that the only remaining term is $a_1^3 b^*_{111r}$ and equation (23) becomes

$$(k^*_{ir} - \omega^{*2} m^*_{ir})\varepsilon_i + \tfrac{3}{2}\, a_1^3 b^*_{111r} = 0, \qquad r = 3, 5, \ldots, 11, \tag{25}$$

in which the repeated index i is summed over the range $(1, 3, \ldots, 11)$.

Since the use of linear beam mode shapes as basic functions leads to diagonal mass and rigidity matrices, equation (25) can be written as

$$(k^*_{ir} - \omega^{*2} m^*_{ir})\varepsilon_i + \tfrac{3}{2}\, a_1^3\, b^*_{111r} = 0, \quad r = 3, 5, \ldots, 11 \tag{26}$$

in which no summation is involved. The above system permits one to obtain explicitly the modal contributions $\varepsilon_3, \varepsilon_5, \ldots, \varepsilon_{11}$ of the second and higher basic functions corresponding to a given value of the assigned first basic function contribution a_1 as follows:

$$\varepsilon_r = -\tfrac{3}{2}\, a_1^3\, b^*_{111r}/(k^*_{rr} - \omega^{*2} m^*_{rr}), \qquad r = 3, 5, \ldots, 11 \tag{27}$$

where the ε_i's, for $r > 1$, depend on the classical modal parameters m_{rr}, k_{rr}, the non-linear modal parameters b_{111r}, the assigned first function contribution a_1, and the non-linear frequency parameter ω^*. On the other hand, it is shown in reference [2] that the single-mode approach gives an accurate estimate of the non-linear frequency parameter ω^* for displacement amplitudes up to twice the beam thickness, so that ω^{*2} may be well estimated, with a percentage error below 0·84% from equation (24) by

$$\omega^{*2} = k^*_{11}/m^*_{11} + (b^*_{1111}/m^*_{11})\, a_1^2. \tag{28}$$

Substituting equation (28) into equation (27) leads to

$$\varepsilon_r = 3a_1^3 b^*_{r111}/2((k^*_{11} + a_1^2 b^*_{1111})\, m^*_{rr}/m^*_{11} - k^*_{rr}), \quad r = 3, 5, \ldots, 11. \tag{29}$$

As the normalization procedure adopted in reference [2, equation (43)] leads to a mass matrix identical to the identity matrix, m^*_{11} and m^*_{rr} are equal to 1 and equation (28) may be simplified to

$$\varepsilon_r = 3a_1^3 b^*_{r111}/(2(k^*_{11} + a_1^2 b^*_{1111} - k^*_{rr})), \quad r = 3, 5, \ldots, 11. \tag{30}$$

Expression (30) is an explicit simple formula, allowing calculation of the higher mode contributions to the first non-linear beam mode shape, as functions of the assigned first mode contribution a_1 and of the known parameters k_{rr}, m_{rr} and b_{111r} (given in Appendix A), which defines the first non-linear amplitude-dependent beam mode shape $w^*_{n11}(x, a_1)$ for a given assigned value a_1 of the first function contribution as a series involving the beam modal parameters depending on the first six symmetric CC beam functions $w^*_1, w^*_3, \ldots, w^*_{11}$

$$w^*_{n11}(x, a_1) = a_1 w^*_1(x) + \frac{3a_1^3 b^*_{3111}}{2((k^*_{11} + a_1^2 b^*_{1111}) - k^*_{33})}\, w^*_3(x)$$
$$+ \frac{3a_1^3\, b^*_{5111}}{2((k^*_{11} + a_1^2 b^*_{1111}) - k^*_{55})}\, w^*_5(x) + \cdots + \frac{3a_1^3\, b^*_{11\cdot 111}}{2((k^*_{11} + a_1^2\, b^*_{1111}) - k^*_{11\cdot 11})}\, w^*_{11}(x) \tag{31}$$

in which the predominant term, proportional to the first linear mode shape, is $a_1 w_1^*(x)$, and other terms, proportional to the higher modes $w_3^*(x), \ldots, w_{11}^*(x)$, are the corrections due to the non-linearity.

The second amplitude-dependent beam non-linear mode shape $w_{n12}^*(x, a_2)$ for a given assigned value a_2 of the second function contribution is given similarly by

$$w_{n12}^*(x, a_2) = a_2 w_2^*(x) + \frac{3a_2^3 b_{4222}^*}{2((k_{22}^* + a_2^2 b_{2222}^*) - k_{44}^*)} w_4^*(x) + \frac{3a_2^3 b_{6222}^*}{2((k_{22}^* + a_2^2 b_{2222}^*) - k_{66}^*)} w_6^*(x) + \cdots + \frac{3a_2^3 b_{12\cdot 222}^*}{2((k_{22}^* + a_2^2 b_{2222}^*) - k_{12\cdot 12}^*)} w_{12}^*(x). \tag{32}$$

The third non-linear amplitude-dependent beam mode shape $w_{n13}^*(x, a_3)$ for a given assigned value a_3 of the third function contribution may be written as

$$w_{n13}^*(x, a_3) = \frac{3a_3^3 b_{1333}^*}{2((k_{33}^* + a_3^2 b_{3333}^*) - k_{11}^*)} w_1^*(x) + a_3 w_3^*(x) + \frac{3a_3^3 b_{5333}^*}{2((k_{33}^* + a_3^2 b_{3333}^*) - k_{55}^*)} w_5^*(x) + \cdots + \frac{3a_3^3 b_{11333}^*}{2((k_{1111}^* + a_3^2 b_{3333}^*) - k_{1111}^*)} w_{11}^*(x). \tag{33}$$

To obtain, more generally, the beam rth non-linear mode shape, the contributions ε_i of the ith basic function can be obtained from

$$\varepsilon_i = -\frac{\frac{3}{2} a_r^3 b_{irrr}^*}{k_{ii}^* - \omega^{*2} m_{ii}^*} = \frac{3a_r^3 b_{irrr}^*}{2(k_{rr}^* + a_r^2 b_{rrrr}^* - k_{ii}^*)}, \quad i \neq r. \tag{34}$$

The range of validity of these expressions is quite interesting. It will be discussed in the next section for the various modes and beams considered.

3.1.2. *Second formulation*

As will be shown in section 3.2, the explicit formulae established for the various non-linear beam mode shapes considered via the first formulation developed in the above subsection yield accurate results for quite large ranges of vibration amplitudes. For higher amplitudes, a second formulation has been considered in which only second order terms of the type $\varepsilon_i \varepsilon_j a_1 b_{ij1r}$ are neglected when considering the first non-linear mode, in equation (23), rewritten here for clarity as

$$-\omega^{*2} a_i m_{ir}^* + a_i k_{ir}^* + \tfrac{3}{2} a_i a_j a_k b_{ijkr}^* = 0, \quad r = 3, 5, \ldots, 11. \tag{35}$$

Separating in the non-linear expression $a_i a_j a_k b_{ijkr}^*$ terms proportional to a_1^3, terms proportional to $a_1^2 \varepsilon_i$, and neglecting terms proportional to $a_1 \varepsilon_i \varepsilon_j$ leads to

$$a_i a_j a_k b_{ijkr}^* = a_1^3 b_{111r}^* + a_1^2 \varepsilon_i b_{11ir}^*. \tag{36}$$

After substituting and rearranging, equation (35) can be written in matrix form as

$$([\mathbf{K}_{RI}^*] - \omega^{*2} [\mathbf{M}_{RI}^*]) \{\mathbf{A}_{RI}\} + \tfrac{3}{2} [\boldsymbol{\alpha}_1^*] \{\mathbf{A}_{RI}\} = \{-\tfrac{3}{2} a_1^3 b_{i11l}^*\}, \tag{37}$$

in which $[\mathbf{K}^*_{RI}] = [\mathbf{k}^*_{ij}]$ and $[\mathbf{M}^*_{RI}] = [\mathbf{m}^*_{ij}]$ are reduced rigidity and mass matrices associated with the first non-linear mode, obtained by varying i and j in the set $(3, 5, \ldots, 11)$, $[\boldsymbol{\alpha}^*_{\mathrm{I}}]$ is a 5×5 square matrix, depending on a_1, whose general term $\boldsymbol{\alpha}^*_{ij}$ is equal to $a_1^2 b^*_{ij11}$, and $\{-\frac{3}{2} a_1^3 b^*_{i111}\}$ is a column vector representing the right side of the linear system (37) in which the reduced unknown vector is $\{\mathbf{A}_{RI}\}^{\mathrm{T}} = [\varepsilon_3, \varepsilon_5, \ldots, \varepsilon_{11}]$. The modal contributions $\varepsilon_3, \varepsilon_5, \ldots, \varepsilon_{11}$ can be obtained very easily by solving the linear system (37) of five equations and five unknowns.

To obtain the second non-linear beam mode shape, a linear system similar to (37) is written as

$$([\mathbf{K}^*_{RII}] - \omega^{*2}[\mathbf{M}^*_{RII}])\{\mathbf{A}_{RII}\} + \tfrac{3}{2}[\boldsymbol{\alpha}^*_{\mathrm{II}}]\{\mathbf{A}_{RII}\} = \{-\tfrac{3}{2} a_2^3 b^*_{i222}\}, \tag{38}$$

in which the general term of the matrix $[\boldsymbol{\alpha}^*_{\mathrm{II}}]$ is equal to $a_2^2 b^*_{ij22}$, $[\mathbf{K}^*_{RII}]$ and $[\mathbf{M}^*_{RII}]$ are reduced rigidity and mass matrices corresponding to the second mode and $\{-\frac{3}{2} a_2^3 b^*_{i222}\}$ is a column vector representing the right side of the linear system (38) in which the reduced unknown vector is $\{\mathbf{A}_{RII}\}^{\mathrm{T}} = [\varepsilon_2, \varepsilon_4, \ldots, \varepsilon_{12}]$. The modal contributions can also be obtained similarly by solving a reduced linear system of five equations and five unknowns. Higher non-linear mode shapes may be obtained in a similar manner, using appropriate reduced matrices in each case.

3.1.3. *Conclusions*

It appears from the above two subsections that the basic function contributions to the amplitude-dependent non-linear beam mode shapes may be calculated via the first formulation using simple explicit expressions involving the beam generalized parameters m_{ij}, k_{ij} and b_{ijkl}. As will be shown in section 3.2 in the light of the numerical results obtained, these simple expressions yield accurate values for the basic function contributions, for vibration amplitudes, up to about 0·7 times the beam thickness. For higher amplitudes, more accurate results may be obtained, based on the second formulation, via a solution of a reduced linear system of five equations and five unknowns for vibration amplitudes, up to about 1·5 times the beam thickness.

3.2. PRESENTATION AND DISCUSSION OF THE NUMERICAL RESULTS OBTAINED BY THE NEW APPROACH CORRESPONDING TO THE FIRST THREE NON-LINEAR MODE SHAPES OF A CC BEAM

3.2.1. *First formulation results*

Replacing in equation (31) the CC modal parameters by their numerical values given in Appendix A leads to the following expression of the first non-linear CC beam mode shape:

$$w^*_{n11}(x, a_1) = a_1 w^*_1(x) + \frac{-1077{\cdot}21 a_1^3}{2((500{\cdot}56 + 453{\cdot}87 a_1^2) - 14617{\cdot}63)} w^*_3(x)$$

$$+ \frac{-842{\cdot}427 a_1^3}{2((500{\cdot}56 + 453{\cdot}87 a_1^2) - 89135{\cdot}40)} w^*_5(x)$$

$$+ \frac{-675{\cdot}27 a_1^3}{2((500{\cdot}56 + 453{\cdot}87 a_1^2) - 308208{\cdot}45)} w^*_7(x)$$

$$+\frac{-560{\cdot}142a_1^3}{2((500{\cdot}56+453{\cdot}87a_1^2)-793406{\cdot}25)}w_9^*(x)$$

$$+\frac{-470{\cdot}475a_1^3}{2((500{\cdot}56+453{\cdot}87a_1^2)-1691832{\cdot}35)}w_{11}^*(x). \tag{39}$$

In Table 1(b), numerical results for modal contributions to the fundamental non-linear mode shape of a CC beam, calculated here via the first formulation, i.e., equation (39), are summarized. The results given correspond to the values of $\varepsilon_3, \varepsilon_5, \dots, \varepsilon_{11}$ obtained for some assigned values of a_1 varying from 0·05 to 0·8 which correspond to a maximum non-dimensional vibration amplitude at the beam centre varying from 0·0794 to 1·225. For each solution, the corresponding values of ω_{n1}^*/ω_1^* and the curvature calculated at $x^* = 0$ are also given. Comparison between Tables 1(b) and (a) taken from reference [2] where the modal contributions have been calculated via the solution of the complete non-linear algebraic system, shows that the higher basic function contributions to the first non-linear beam mode shape obtained from the explicit expressions based on the first formulation are very close to those calculated via the solution of a non-linear algebraic system for finite amplitudes of vibration up to a displacement equal to the beam thickness (which corresponds to $a_1 \cong 0{\cdot}67$). For higher values of the vibration amplitude, slight differences start to appear and increase with the amplitude of vibration. This may be seen in Figures (1)–(5), in which contributions obtained from the first and second formulations are plotted versus the maximum non-dimensional beam vibration amplitude W_{max}/H obtained at the beam centre and compared with the exact numerical solution. To have an accurate conclusion concerning the limit of validity of the first explicit formulation in engineering applications, a criterion based on the effect of the differences appearing in the estimated contributions to physical quantities, such as the non-linear frequency and the curvature at the beam ends has been adopted. It was found, as may be seen in Tables 1(a–c), that for amplitudes up to the beam thickness, the error induced by the first formulation does not exceed 0·1% for the non-linear frequency and 3% for the curvature (and hence the non-linear bending stress) at the beam clamps. This effect is shown in Figures (6) and (7), in

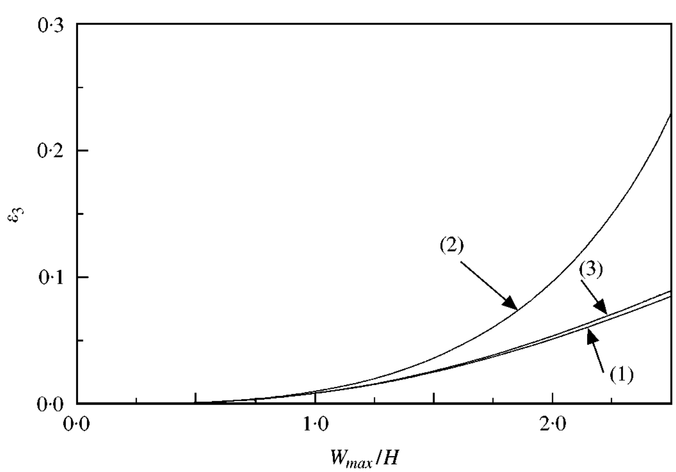

Figure 1. Comparison between the values of the modal contribution ε_3 of the first non-linear mode of the free vibration of a CC beam obtained by (1) non-linear algebraic equations; (2) first formulation and (3) second formulation.

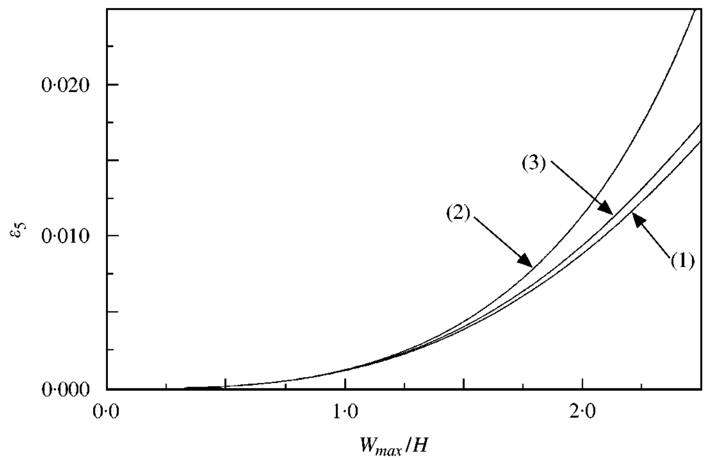

Figure 2. Comparison between the values of the modal contribution ε_5 of the first non-linear mode of the free vibration of a CC beam. Key as for Figure 1.

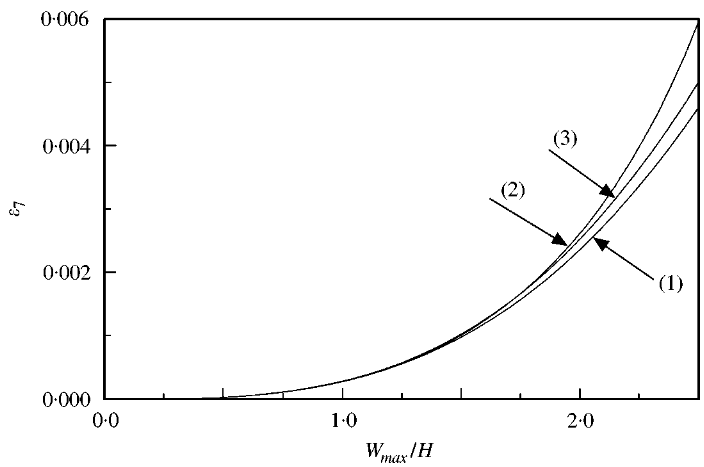

Figure 3. Comparison between the values of the modal contribution ε_7 of the first non-linear mode of the free vibration of a CC beam. Key as for Figure 1.

which the curvature and the non-linear frequency, obtained via the three approaches, are plotted for a wide range of vibration amplitudes.

In Tables 2 and 3, numerical results for modal contribution to the second and third CC beam non-linear mode shapes calculated via the different formulations are summarized. Comparison of these tables leads to the same conclusion as that given above for the fundamental non-linear mode—the higher basic functions contributions to the second and third non-linear mode shapes obtained from the explicit expressions based on the first formulation are very close to those calculated via the solution of the non-linear algebraic system for finite amplitudes of vibration up to 0·8 times the beam thickness which corresponds to $a_1 \cong 0{\cdot}5$. In Figures 8–11, the curvatures and the non-linear resonant frequency associated with the second and the third non-linear mode calculated here using the first formulation are plotted versus the maximum non-dimensional beam vibration amplitude obtained in the neighbourhood of $x^* = 0{\cdot}29$ and $0{\cdot}20$ respectively. It is noticable

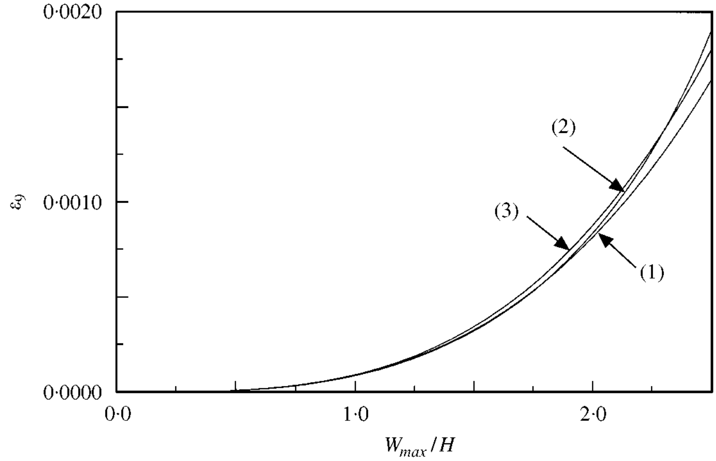

Figure 4. Comparison between the values of the modal contribution ε_9 of the first non-linear mode of the free vibration of a CC beam. Key as for Figure 1.

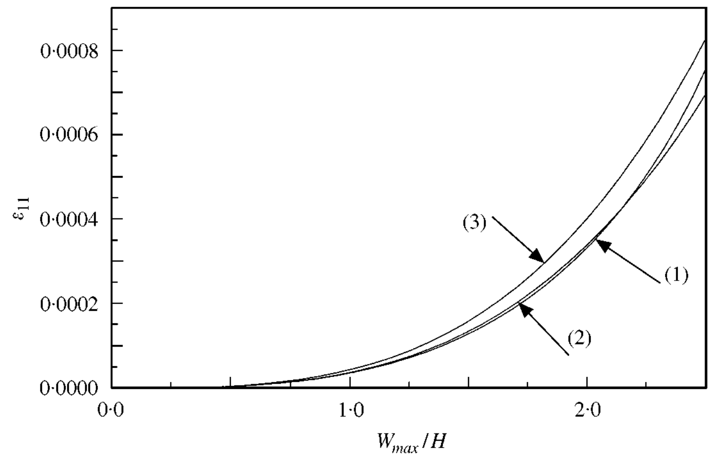

Figure 5. Comparison between the values of the modal contribution ε_{11} of the first non-linear mode of the free vibration of a CC beam. Key as for Figure 1.

that for values of vibration amplitude below 0·8 times the beam thickness, the difference between the present solution and that obtained from the exact solution of the non-linear algebraic system does not exceed 1·1% for the non-linear resonance frequency for the second non-linear mode and 1·4% for the third mode, and does not exceed 1·1% for the curvature at the beam end associated with the second mode, and 2.5% for the curvature associated with the third mode.

3.2.1.1. *Effect of various truncations of the series defined in equation* (39). As may be seen from the table of contributions of the higher modes to the first non-linear mode shape, the contribution of a basic function n decreases generally when n increases, and the contributions of the highest functions are very small for small amplitudes, but increase with amplitude. It may be concluded from these two observations that many increasing ranges of vibration amplitude may be considered to correspond to successive truncations of the series

(39) defining the non-linear mode shape. To define these ranges, one considers now the effect of various possible truncations of the series (39) on the estimated non-linear frequency and bending moments at the beam end. In Table 15 of reference [5], the values of the bending moments calculated at $x^* = 0$, for the fundamental mode shape of a CC beam and for various truncations of the series are summarized. Comparison of the percentage of error induced on the bending moments by the different models permits one to conclude that the fundamental mode shape can be approximated with a percentage error which does not exceed 2%, using the 2-D model, for amplitudes up to 0·7 times the beam thickness, so that the fundamental non-linear mode shape may be given, for this range of amplitudes of vibration, by the following expression involving only two basic functions, namely

$$W^*_{n11}(x, a_1) = a_1 w^*_1(x) + \frac{-1077{\cdot}21a_1^3}{2((500{\cdot}56 + 453{\cdot}87a_1^2) - 14617{\cdot}63)} w^*_3(x).$$

For amplitudes of vibration up to the beam thickness, the 3-D model may be used leading to the expression involving three basic functions

$$W^*_{n11}(x, a_1) = a_1 w^*_1(x) + \frac{-1077{\cdot}21a_1^3}{2((500{\cdot}56 + 453{\cdot}87a_1^2) - 14617{\cdot}63)} w^*_3(x) + \frac{-842{\cdot}427a_1^3}{2((500{\cdot}56 + 453{\cdot}87a_1^2) - 89135{\cdot}40)} w^*_5(x).$$

In Table 4, a summary is given of the errors induced in the frequency and the moment calculated at $x^* = 0$ by the different models for chosen values of the maximum non-dimensional vibration amplitude in each case.

3.2.2. *Second formulation results*

In Tables 1(c), 2(c) and 3(c), the modal contributions to the fundamental, the second and the third non-linear mode shapes of a CC beam, calculated via the second formulation are summarized. It is noticable from comparison of these tables with those obtained via the solution of the non-linear algebraic system, i.e., Tables 1(a–c), and from figures 7–11, that the corresponding intervals of validity largely exceed those obtained from the first formulation and can reach vibration amplitudes up to 2·8 times the beam thickness for the first mode, 3·2 and 2·3 times the beam thickness for the second and third modes respectively.

It is also noticable that for values of vibration amplitude below 1·5 times the beam thickness, the difference between the exact solution and that obtained by the second formulation does not exceed 0·1% for the non-linear resonance frequency of the first non-linear mode, 0·82% for the second non-linear mode and 3·2% for the third non-linear mode, and does not exceed 1·6% for the curvature at the beam end associated with the first non-linear mode, 2·25% for the second mode and 0·3% for the curvature associated with the third mode.

3.2.3. *Presentation and discussion of the numerical results corresponding to the first non-linear mode shape of a CSS beam*

In Tables 5(a–c), the modal contributions to the first non-linear mode shape of a CSS beam obtained via the solution of the non-linear algebraic system (23) are presented. It can be seen in this table that the contribution a_1 of the first basic function, which is the first CSS beam linear mode shape, remains predominant, so that the contribution coefficient vector

TABLE 1

(a) Free vibration in the first non-linear mode of a CC beam obtained numerically from solution of the non-linear algebraic equations, published in reference [2]

$\omega^{*}n1/\omega1$	$d^2w/dx^2(0)$	a_1	a_3	a_5	a_7	a_9	a_{11}
1·001	0·223911E+01	0·5000E−01	0·4765E−05	0·5944E−06	0·1374E−06	0·4423E−07	0·1749E−07
1·148	0·296695E+02	0·6000E+00	0·7213E−02	0·9921E−03	0·2382E−03	0·7824E−04	0·3131E−04
1·196	0·356782E+02	0·7000E+00	0·1097E−01	0·1556E−02	0·3783E−03	0·1250E−03	0·5025E−04
1·247	0·421030E+02	0·8000E+00	0·1564E−01	0·2289E−02	0·5642E−03	0·1879E−03	0·7581E−04
1·550	0·809348E+02	0·1300E+01	0·5144E−01	0·8920E−02	0·2378E−02	0·8263E−03	0·3422E−03
1·617	0·900628E+02	0·1400E+01	0·6073E−01	0·1088E−01	0·2950E−02	0·1035E−02	0·4310E−03
1·684	0·996355E+02	0·1500E+01	0·7058E−01	0·1306E−01	0·3599E−02	0·1275E−02	0·5341E−03

(b) *Frequency ratios of free vibration in the first non-linear mode, curvatures and modal participation of a CC beam at various amplitudes obtained by the present model (first formulation)*

		a_1†	ε_3‡	ε_5	ε_7	ε_9	ε_{11}	% error $d^2w/dx^2(0)$
0·100113E+01	0·223911E+01	0·5000E−01	0·477043E−05	0·594585E−06	0·137366E−06	0·4421847E−07	0·174043E−07	0·0
0·114787E+01	0·299574E+02	0·6000E+00	0·833921E−02	0·1029328E−02	0·2374942E−03	0·7642516E−04	0·3007762E−04	0·0
0·119525E+01	0·362745E+02	0·7000E+00	0·1329862E−01	0·1635627E−02	0·3772044E−03	0·1213693E−03	0·4776382E−04	1·7
0·124697E+01	0·432141E+02	0·8000E+00	0·1994879E−01	0·2443401E−02	0·5631821E−03	0·181185E−03	0·7130047E−04	2.8

(c) *Frequency ratios of free vibration in the first non-linear mode, curvatures and modal participation of a CC beam at various amplitudes obtained by the present model (second formulation)*

0·100113E+01	0·223912E+01	0·5000E−01	0·476706E−05	0·594929E−06	0·1375940E−06	0·4434244E−07	0·2043709E−07	0·0001
0·114798E+01	0·297389E+02	0·6000E+00	0·734947E−02	0·101316E−02	0·2436585E−03	0·8013607E−04	0·3662420E−04	0·26
0·154979E+01	0·822306E+02	0·1300E+01	0·536749E−01	0·944098E−02	0·2536978E−02	0·8859714E−03	0·4019288E−03	2·0
0·161607E+01	0·917103E+02	0·1400E+01	0·634644E−01	0·115561E−01	0·3161695E−02	0·1115430E−02	0·5063787E−03	2·23

†a_1: assigned value of the first CC beam function.
‡ε_i: contribution of the ith CC beam function to the first non-linear mode calculated via the first approximation.

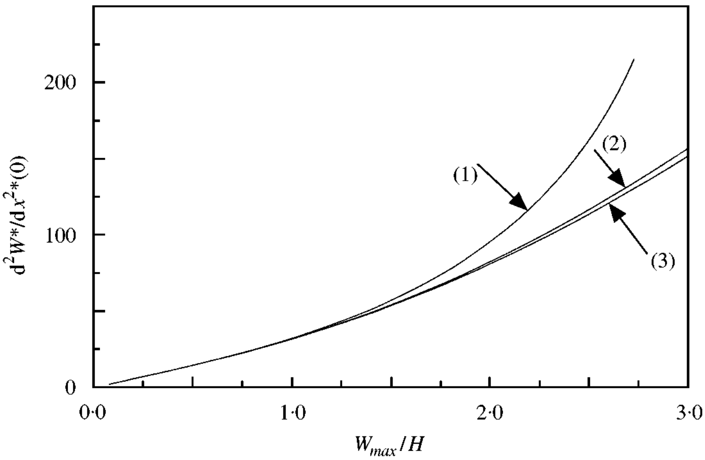

Figure 6. Comparison between values of curvature at $x^* = 0$ of the first non-linear mode shape of a CC beam. Key as for Figure 1.

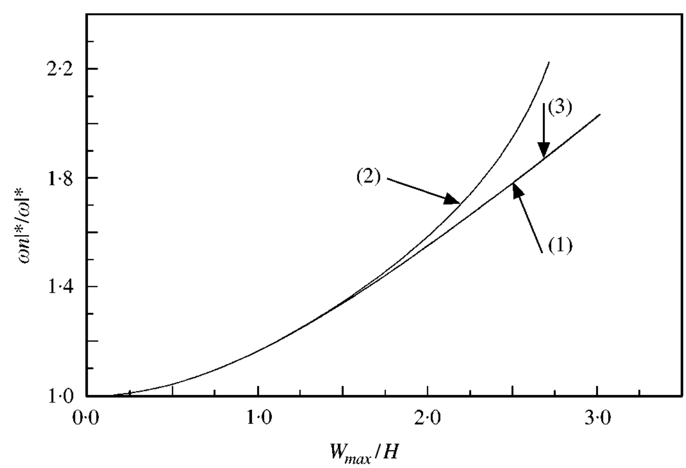

Figure 7. Comparison of frequencies for first non-linear CC beam mode shape. Key as for Figure 1.

$\{\mathbf{A}\}$ defined in reference [2] by $\{\mathbf{A}\}^T = [a_1, a_2, \dots, a_6]$ can be written as $\{\mathbf{A}\}^T = [a_1, \varepsilon_2, \dots, \varepsilon_6]$ in which ε_i, representing the ith basic function contribution, may be considered as small, compared to a_1, for $i = 2, 3, \dots, 6$. This shows that the two formulations, developed in sections 3.1.1 and 3.1.2, in the case of CC beams can be rederived in the case of CSS beams leading to two similar results. Hence, using the first formulation, the first non-linear amplitude-dependent CSS beam mode shape $\Phi^*_{n11}(x, a_1)$, for a given assigned value a_1 of the first function contribution, can be written as

$$\Phi^*_{n11}(x, a_1) = a_1\phi^*_1(x) + \frac{3a_1^3 b^*_{2111}}{2((k^*_{11} + a_1^2 b^*_{1111}) - k^*_{22})}\phi^*_2(x) + \frac{3a_1^3 b^*_{3111}}{2((k^*_{11} + a_1^2 b^*_{1111}) - k^*_{33})}\phi^*_3(x) + \cdots + \frac{3a_1^3 b^*_{6111}}{2((k^*_{11} + a_1^2 b^*_{1111}) - k^*_{66})}\phi^*_6(x). \tag{40}$$

Replacing in equation (40) the CSS modal parameters by their numerical values given in Appendix A leads to the following expression of the first non-linear mode shape of a CSS beam:

$$\Phi^*_{n11}(x, a_1) = a_1\phi^*_1(x) + \frac{-443{\cdot}3652a_1^3}{2((237{\cdot}72 + 397{\cdot}4403a_1^2) - 2496{\cdot}48)}\phi^*_2(x)$$
$$+ \frac{-392{\cdot}6061a_1^3}{2((237{\cdot}72 + 397{\cdot}4403a_1^2) - 10867{\cdot}58)}\phi^*_3(x) + \frac{-339{\cdot}7752a_1^3}{2((237{\cdot}72 + 397{\cdot}4403a_1^2) - 31780{\cdot}09)}\phi^*_4(x)$$
$$+ \frac{-295{\cdot}2315a_1^3}{2((237{\cdot}72 + 397{\cdot}4403a_1^2) - 74000{\cdot}84)}\phi^*_5(x) + \frac{-260{\cdot}0109a_1^3}{2((237{\cdot}72 + 397{\cdot}4403a_1^2) - 148634{\cdot}47)}\phi^*_6(x). \quad (41)$$

In Table 5(b), numerical results for modal contributions to the fundamental non-linear mode shape of a CSS beam, calculated via the first formulation are summarized. The results given correspond to the values of $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_6$ obtained for assigned values of a_1 varying from 0·05–1·5. For each solution, the corresponding values of ω^*_{n1}/ω^*_1, the curvature calculated at $x^* = 0$, and the maximum amplitude of vibration w_{max}/R are also given. Comparison between Tables 5(b) and (a) shows that the higher mode contributions to the first non-linear beam mode shape obtained from the explicit expressions based on the first formulation are very close to those calculated via the solution of a non-linear algebraic system for finite amplitudes of vibration up to 0·8 times the beam thickness (which corresponds to $a_1 \cong 0{\cdot}52$). For higher values of the vibration amplitude, slight differences start to appear and increase with increase of the amplitude of vibration, as may be seen in Figures 12 and 13, in which contributions obtained from the first and second formulation are plotted versus the maximum beam vibration amplitude w_{max} obtained in the neighbourhood of $x^* = 0{\cdot}58$, and compared with the exact numerical solution. It was found, as may be seen from Figures 12 and 13, that for amplitudes up to 0·8 times the beam thickness, the error induced by the first formulation does not exceed 0·35% for the non-linear frequency and 1·3% for the curvature (and hence the non-linear bending stress) at the beam clamps. The error induced in the second formulation for amplitudes up to 1·5 times the beam thickness does not exceed 0·33% for the non-linear frequency and 0·83% for the curvature at the clamped end of the beam.

4. A NEW SIMPLIFIED APPROACH TO THE NON-LINEAR STEADY STATE FORCED PERIODIC RESPONSE OF BEAMS AT LARGE VIBRATION AMPLITUDES

4.1. REVIEW OF THE THEORY AND THE SINGLE-MODE CASE

The model presented in section 2 has been recently extended to the case of non-linear forced vibration of beams [4, 5]. The authors assume that the structure is excited by the force $F(x, t)$ distributed over the range $\bar{S}$ ($\bar{S}$ is the length of the beam or a part of it); the physical force $F(x, t)$ excites the modes of the structure via a set of generalized forces $F_i(t)$ which depend on the expression for F, the excitation point for concentrated forces, the excitation length for distributed forces, and the mode considered. The generalized forces $F_i(t)$ are given by

$$F_i(t) = \int_{\bar{S}} F(x, t)w_i(x)\,\mathrm{d}x, \quad (42)$$

TABLE 2

(*a*) *Free vibration in the second non-linear mode of a CC beam obtained numerically from solution of the non-linear algebraic equations, published in reference* [2]

$\omega^* n1/\omega 1$	$d^2w/dx^2(0)$	a_2	a_4	a_6	a_8	a_{10}	a_{12}
1·002	0·617601E+01	0·5000E−01	0·1225E−04	0·2313E−05	0·6742E−06	0·2509E−06	0·1129E−06
1·072	0·388084E+02	0·3000E+00	0·2466E−02	0·4873E−03	0·1450E−03	0·5456E−04	0·2472E−04
1·124	0·534804E+02	0·4000E+00	0·5547E−02	0·1133E−02	0·3420E−03	0·1298E−03	0·5914E−04
1·187	0·694579E+02	0·5000E+00	0·1018E−01	0·2159E−02	0·6636E−03	0·2544E−03	0·1167E−03
1·421	0·126428E+03	0·8000E+00	0·3325E−01	0·7998E−02	0·2621E−02	0·1044E−02	0·4904E−03
1·510	0·148572E+03	0·9000E+01	0·4361E−01	0·1093E−01	0·3666E−02	0·1480E−02	0·7022E−03
2·101	0·312271E+03	0·1500E+01	1·2229E+00	0·3774E−01	0·1444E−01	0·6363E−02	0·3208E−02

(*b*) *Frequency ratios of free vibration in the second non-linear mode shape, curvatures and modal participation of a CC beam at various amplitudes obtained by the present model* (*first formulation*)

		a_2†	ε_4‡	ε_6‡	ε_8‡	ε_{10}‡	ε_{12}‡	% error $d^2w/dx^2(0)$
0·100208E+01	0·617602E+01	0·5000E−01	0·1228236E−04	0·231435E−05	0·674261E−06	0·2507434E−06	0·132415E−06	0·00025
0·107197E+01	0·389105E+02	0·3000E+00	0·2694511E−02	0·501543E−03	0·145801E−03	0·5418612E−04	0·286085E−04	0·28
0·112398E+01	0·538937E+02	0·4000E+00	0·6467973E−02	0·119197E−02	0·345908E−03	0·1284896E−03	0·678257E−04	0·85
0·118671E+01	0·706620E+02	0·5000E+00	0·1284211E−01	0·233598E−02	0·676371E−03	0·2510783E−03	0·132504E−03	1·9

(*c*) *Frequency ratios of free vibration in the second non-linear mode, curvatures and modal participation of a CC beam at various amplitudes obtained by the present model* (*second formulation*)

0·100208E+01	0·617595E+01	0·5000E−01	0·1224535E−04	0·231003E−05	0·6731965E−06	0·2503251E−06	0·1125662E−06	0·0001
0·142140E+01	0·127028E+03	0·8000E+00	0·3406493E−01	0·823521E−02	0·2708356E−02	0·1080743E−02	0·5519784E−03	1·0
0·151026E+01	0·149511E+03	0·9000E+00	0·4476218E−01	0·112917E−01	0·3804838E−02	0·1541429E−02	0·8046572E−03	1·33
0·210076E+01	0·316849E+03	0·1500E+01	0·126810E+00	0·393506E−01	0·1521860E−01	0·6771930E−02	0·3967705E−02	3·42

†a_2: assigned value of the second CC beam function.
‡ε_i: contribution of the *i*th CC beam function to the second non-linear mode calculated via the first approximation.

TABLE 3

(*a*) *Free vibration in the third non-linear mode of a CC beam obtained numerically from solution of the non-linear algebraic equations, published in reference* [2]

$\omega^* n1/\omega 1$	$\mathrm{d}^2 w/\mathrm{d}x^2(0)$	a_1	a_3	a_5	a_7	a_9	a_{11}
1·003	0·121085E+02	−0·3820E−04	0·5000E−01	0·1812E−04	0·4350E−05	0·1489E−05	0·6165E−06
1·039	0·494944E+02	−0·2309E−02	0·2000E+00	0·1109E−02	0·2732E−03	0·9463E−04	0·3945E−04
1·086	0·762373E+02	−0·7260E−02	0·3000E+00	0·3539E−02	0·8996E−03	0·3163E−03	0·1330E−03
1·149	0·105097E+03	−0·1572E−01	0·4000E+00	0·7798E−02	0·2062E−02	0·7393E−03	0·3145E−03
1·403	0·206864E+03	−0·6030E−01	0·7000E+00	0·3160E−02	0·9561E−02	0·3687E−02	0·1642E−02
1·504	0·245906E+03	−0·7994E−01	0·8000E+01	0·4255E−01	0·1344E−01	0·5321E−02	0·2410E−02
2·316	0·572554E+03	−0·2449E+00	0·1500E+01	0·1388E+00	0·5441E−01	0·2511E−01	0·1273E−01

(*b*) *Frequency ratios of free vibration in the first non-linear mode, curvatures and modal participation of a CC beam at various amplitudes obtained by the present model* (*first formulation*)

		ε_1†	a_3‡	ε_5†	ε_7†	ε_9†	ε_{11}†	% error $d^2w/dx^2(0)$
0·100250E+01	0·121085E+02	−0·381498E−04	0·5000E−01	0·182060E−04	0·435693E−05	0·148982E−05	0·612790E−06	0·0005
0·103929E+01	0·495555E+02	−0·226586E−02	0·2000E+00	0·119162E−02	0·280421E−03	0·955515E−04	0·392572E−04	0·1
0·108623E+01	0·766872E+02	−0·697771E−02	0·3000E+00	0·414718E−02	0·953614E−03	0·323402E−03	0·132667E−03	0·6
0·114845E+01	0·106931E+03	−0·147336E−01	0·4000E+00	0·102792E−01	0·228472E−02	0·769644E−03	0·315051E−03	1.8

(*c*) *Frequency ratios of free vibration in the third non-linear mode, curvatures and modal participation of a CC beam at various amplitudes obtained by the present model* (*second formulation*)

		ε_1§	a_3§	ε_5	ε_7	ε_9	ε_{11}	
0·100250E+01	0·121086E+02	−0·382003E−04	0·05	0·181396E−04	0·435668E−05	0·149263E−05	0·643024E−06	0·0009
0·130924E+01	0·170607E+03	−0·427742E−01	0·60	0·221210E−01	0·640552E−02	0·241027E−02	0·108201E−02	0·95
0·140323E+01	0·207296E+03	−0·603100E−01	0·70	0·318049E−01	0·963465E−02	0·372022E−02	0·169202E−02	1·4
0·150397E+01	0·246532E+03	−0·797662E−01	0·80	0·428455E−01	0·135501E−01	0·537010E−02	0·247661E−02	1·8

†ε_i: contribution of the ith CC beam function to the third non-linear mode calculated via the first approximation.
‡a_3: assigned value of the third CC beam function.
§a_3 and ε_1 as defined in Table 1(c).

TABLE 4

Analysis of the effect of various truncations of the series (39) *on the estimation of the first CC non-linear mode shape*

Series	One term	Two terms	3 terms	4 terms	Five terms
$w^*_{\mathrm{max/h}}$	0·43	0·72	1	1·6	2·74
Percentage error(ωnl)	0·015	0·044	0	0·033	–
Percentage error($M(0)$)	2	2·13	1·9	1·95	2

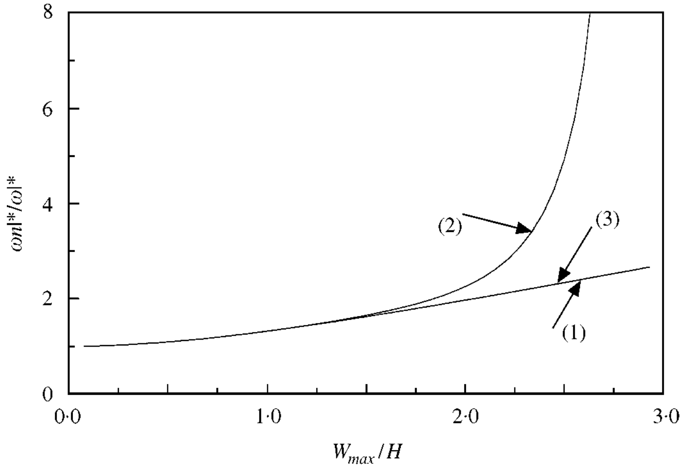

Figure 8. Comparison of frequencies for second non-linear CC beam mode shape. Key as for Figure 1.

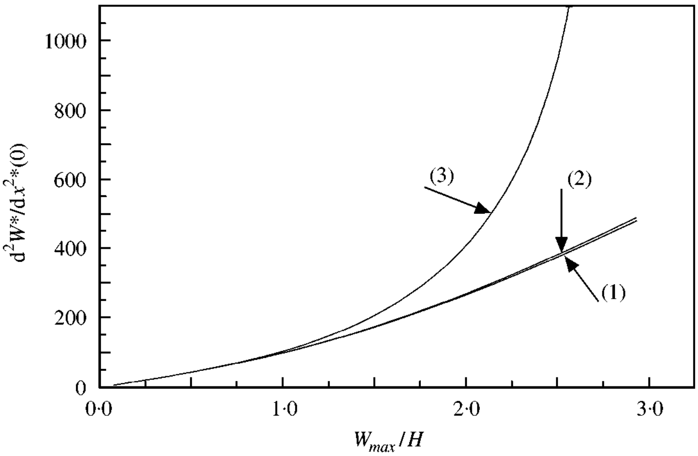

Figure 9. Comparison of values of curvature at $x^* = 0$ of the second non-linear mode shape of a CC beam. Key as for Figure 1.

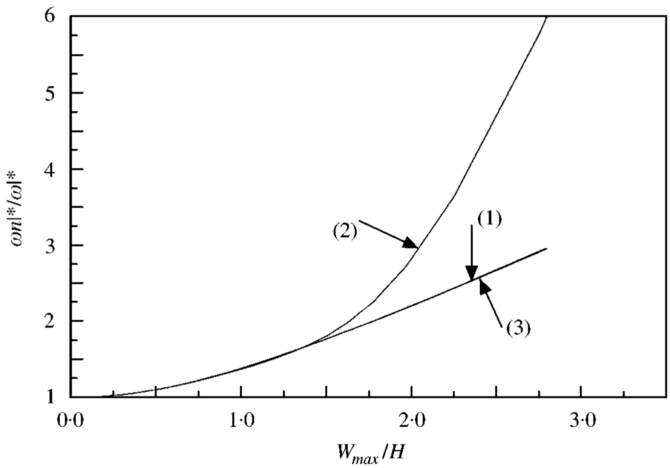

Figure 10. Comparison of frequencies for third non-linear CC beam mode shape. Key as for Figure 1.

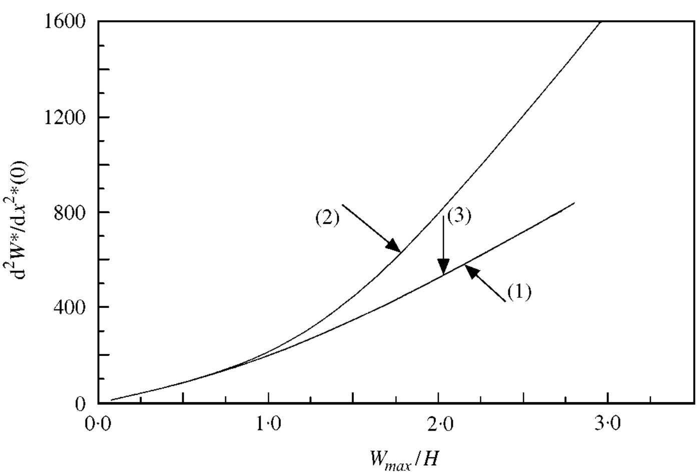

Figure 11. Comparison of values of curvature at $x^* = 0$ of the third non-linear mode shape of a CC beam. Key as for Figure 1.

in which w_i is the ith mode of the beam considered. Adding the forcing term $\{\mathbf{F}(t)\}$ to the right side of equation (13) leads to

$$[\mathbf{M}]\{\ddot{\mathbf{q}}\} + [\mathbf{K}]\{\mathbf{q}\} + 2[\mathbf{B}(\{\mathbf{q}\})]\{\mathbf{q}\} = \{\mathbf{F}(t)\}. \tag{43}$$

If only one mode is assumed, equation (43) reduces to

$$m_{11}\ddot{q}_1 + k_{11}q_1 + 2q_1^3 b_{1111} = F_1(t), \tag{44}$$

in which m_{11}, k_{11} and b_{1111} are the mass, the rigidity and the non-linearity terms corresponding to the first mode respectively.

$$\text{Assuming harmonic response } q_1(t) = a_1 \cos(\omega t), \tag{45}$$

TABLE 5

(a) Free vibration in the first non-linear mode of a CSS beam obtained numerically from solution of the non-linear algebraic equations

$\omega^*n1/\omega1$	$d^2w/dx^2(0)$	a_1	a_2	a_3	a_4	a_5	a_6
1,002087	0·154398E+01	0·5000E−01	0·1225117E−04	0·2312528E−05	0·6741729E−06	0·2509223E−06	0·1098593E−06
1,258408	0·216928E+02	0·6000E+00	0·1639346E−01	0·3623392E−02	0·1136069E−02	0·4405307E−03	0·1979033E−03
1,337087	0·264171E+02	0·7000E+00	0·2412754E−01	0·5563156E−02	0·1782364E−02	0·6999476E−03	0·3169929E−03
1,421447	0·315288E+02	0·8000E+00	0·3325070E−01	0·7997489E−02	0·2620889E−02	0·1043251E−02	0·4766001E−03
1,896606	0·626531E+02	0·1300E+01	0·9409120E−01	0·2725013E−01	0·1000441E−01	0·4283471E−02	0·2054763E−02
1,998349	0·698855E+02	0·1400E+01	0·108296E+00	0·3232574E−01	0·1211944E−01	0·5264907E−02	0·2551353E−02
2,101522	0·774096E+02	0·1500E+01	0·122922E+00	0·3772766E−01	0·1443140E−01	0·6359111E−02	0·3112962E−02

(*b*) *Non-linear free vibration in CSS beam showing modal participation with three symmetric modes and three anti-symmetric modes* (*first formulation*)

w(centre)/R	$\omega^*/\omega 1^*$	Curvature ($x = 0$)	a_1†	ε_2‡	ε_3‡	ε_4‡	ε_5‡	ε_6‡	% error $d^2W/dx^2(0)$
0·26132964E+00	0·10020880E+01	0·15439921E+01	0·500E−01	0·12282396E−04	0·23143790E−05	0·67427548E−06	0·25087025E−06	0·1098107E−06	0·00024
0·31077075E+01	0·12582715E+01	0·22410445E+02	0·600E+00	0·22650005E−01	0·40534567E−02	0·11704228E−02	0·43434081E−03	0·1899348E−03	3·7
0·36129913E+01	0·13374585E+01	0·27890296E+02	0·700E+00	0·36868148E−01	0·64686236E−02	0·18616527E−02	0·69020173E−03	0·3017147E−03	6·33
0·41118559E+01	0·14237415E+01	0·34258094E+02	0·800E+00	0·56671151E−01	0·97112937E−02	0·27842074E−02	0·10311079E−02	0·4505543E−03	9·97

(*c*) *Non-linear free vibration in CSS beam showing modal participation with three symmetric modes and three anti-symmetric modes* (*second formulation*)

0·26132967E+00	0·10020880E+01	0·15439890E+01	0·500E−01	0·12253721E−04	0·23130371E−05	0·67433109E−06	0·25097817E−06	0·1098812E−06	0·00004
0·41321387E+01	0·14213227E+01	0·31727052E+02	0·800E+00	0·34078831E−01	0·82397057E−02	0·27086732E−02	0·10802728E−02	0·4941361E−03	1·16
0·66413343E+01	0·18962301E+01	0·63549443E+02	0·130E+01	0·96940350E−01	0·28338974E−01	0·10481299E−01	0·45113644E−02	0·2172172E−02	2·97
0·71383167E+01	0·19979340E+01	0·70971429E+02	0·140E+01	0·11159476E+00	0·33630323E−01	0·12709629E−01	0·55537658E−02	0·2702730E−02	3·3
0·76341938E+01	0·21010745E+01	0·78695934E+02	0·150E+01	0·12666906E+00	0·39253258E−01	0·15142641E−01	0·67153395E−02	0·3302817E−02	3·62

a_1: assigned value of the first CSS beam function.
ε_i: contribution of the ith CSS beam function to the first non-linear mode calculated via the first approximation.

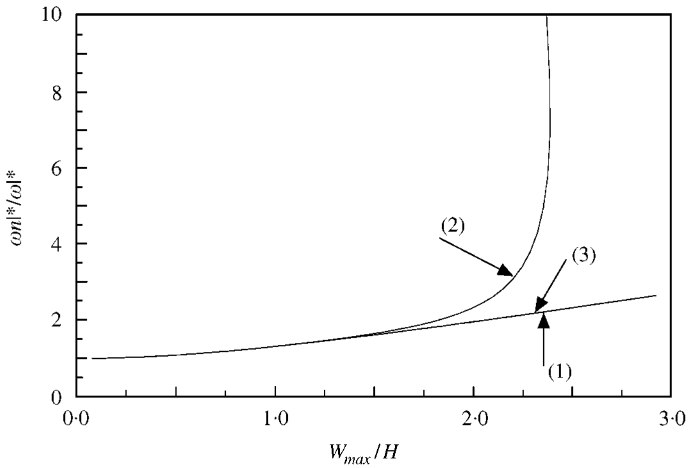

Figure 12. Comparison of frequencies for first non-linear CSS beam mode shape. Key as for Figure 1.

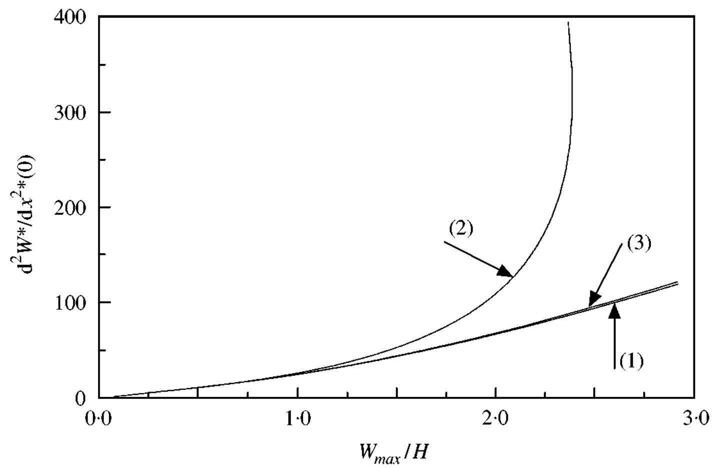

Figure 13. Comparison of values of curvature at $x^* = 0$ of the first non-linear mode shape of a CSS beam. Key as for Figure 1.

substituting equation (45) into equation (44) and applying the harmonic balance method produces

$$(k_{11} - \omega^2 m_{11})a_1 + \tfrac{3}{2} a_1^3 b_{1111} = F_1. \tag{46}$$

Using the non-dimensional parameters m_{11}^*, k_{11}^*, ω^* and b_{1111}^* defined in equations (17–20) leads to

$$(\omega^*/\omega_L^*)^2 = 1 + \tfrac{3}{2}\, b_{1111}^*\, a_1^2/k_{11}^* - F_1^*/a_1. \tag{47}$$

The dimensionless generalized forces F^{*c} and F^{*d} corresponding to the concentrated force at x_0 and the uniformity distributed force on the whole beam span are given in reference [4] as

$$F_i^{*c} = \frac{L^3 F^c}{EIH}\, w_i^*(x_0); \qquad F_i^{*d} = \frac{L^4 F^d}{EIH} \int_0^1 w_i^*(x^*)\, dx^*. \tag{48, 49}$$

In reference [4], to facilitate comparisons with previously published numerical results, the deflection function was written in the form

$$W(x, t) = R\mathscr{A}w(x)q(t). \tag{50}$$

in which the amplitude $\mathscr{A}$ is given in the one mode case by

$$\mathscr{A} = (12)^{1/2}a_1 w_1^*(1/2). \tag{51}$$

R is the beam radius of gyration defined as $R = \sqrt{I/S}$. This allows equation (47) to be written as

$$(\omega^*/\omega_1^*)^2 = 1 + \frac{b_{1111}^*}{8k_{11}^*}\mathscr{A}^2/(w_1^*(1/2))^2 - F/\mathscr{A}, \tag{52}$$

in which F is defined in the case of a distributed or a concentrated harmonic force, respectively, by

$$F^d = \mathscr{F}^d \frac{L^4}{k_{11}^* EIR} w_1^*(1/2)\int_0^L w_i^*(x^*)\,\mathrm{d}x^*, \quad F^c = \mathscr{F}^c \frac{L^3}{k_{11}^* EIR} w_1^*(1/2)w_i^*(x_0). \tag{53, 54}$$

The single mode approach consists of neglecting all the co-ordinates except the single "resonant" co-ordinate. This approach is very often used because it introduces a great simplification in the theory and the error that it introduces in the non-linear frequency remains small [4]. However, it was demonstrated in reference [5] that this simplification gives erroneous results for the estimation of non-linear bending stresses, and consequently for the estimated fatigue life of the beam, which is affected by the significant contributions of the higher modes.

4.2. A NEW SIMPLIFIED MULTI-MODE APPROACH

In spite of the simplicity of the single-mode approach and its ability to predict quite accurately the non-linear frequency response curve in the neighbourhood of the resonance considered, it remains insufficient because it does not give any information about the amplitude dependence of the response deflection shape, with its practically important effect on the strain and stress distributions, which are quantities of crucial importance with respect to structural safety and fatigue life prediction. In order to remedy this insufficiency, a multi-mode approach to the steady state periodic non-linear forced response was developed in reference [5], with the objective of predicting the non-linear frequency response function of the beam, not only at its maximum deflection point, i.e., the beam centre when considering the first resonance, but for the whole span of the beam. Numerical results have been obtained for various values and types of excitation force (concentrated and distributed) via the solution in each case of a set of non-linear algebraic equations. The purpose of the present section is to develop a new simplified approach allowing explicit calculation of the non-linear steady state periodic forced response of beams.

Reconsider now equation (43):

$$([\mathbf{K}^*] - \omega^{*2}[\mathbf{M}^*])\{\mathbf{A}\} + \tfrac{3}{2}[\mathbf{B}^*(\mathbf{A})]\{\mathbf{A}\} = \{\mathbf{F}^*(t)\}. \tag{55}$$

This equation appears as a generalization to the non-linear case of the classical linear forced response problem which is very well known in modal analysis theory [18], i.e.,

$$([\mathbf{K}^*] - \omega^{*2}[\mathbf{M}^*])\{\mathbf{A}\} = \{\mathbf{F}^*(t)\} \tag{56}$$

to which the correcting term $\frac{3}{2}[\mathbf{B}^*(\mathbf{A})]\{\mathbf{A}\}$, corresponding to the non-linear geometrical rigidity, is added. If the excitation is harmonic, i.e., $\{\mathbf{F}^*(t)\} = \{\mathbf{F}^*\}\sin\omega t$, the linear response calculated from equation (56) is also harmonic. As in the beam case considered here, $[\mathbf{K}]$ and $[\mathbf{M}]$ are diagonal, the response is given by the function $\mathbf{W}^*(x, t)$ as

$$W^*(x, t) = \sum_i a_i w_i^*(x) = \sum_i \frac{F_i^*}{(k_{ii}^* - \omega^{*2} m_{ii}^*)} w_i^*(x)\sin\omega t. \tag{57}$$

In the non-linear case, previous experimental and theoretical works have shown that harmonic distortion of the response occurs at large vibration amplitudes, even when the excitation is harmonic with a spatial distribution of the first and higher harmonics, so that the response may be written [1] as

$$W = \{\mathbf{A}_k\}^{\mathrm{T}}\{\mathbf{W}\}\sin k\omega t, \tag{58}$$

where $\{\mathbf{A}_k\}^{\mathrm{T}} = [a_1^k a_2^k, \ldots, a_n^k]$ is the matrix of coefficients corresponding to the kth harmonic, $\{\mathbf{W}\}^{\mathrm{T}} = [w_1\ w_2, \ldots, w_n]$ is the basic spatial function matrix, k is the number of harmonics taken into account, and the usual summation convention on the repeated index k is used. Examination of this effect would have exceeded the scope of the present work, restricted to the first harmonic distribution amplitude dependence. So, the response has been assumed to have the form

$$W(x, t) = a_i w_i(x)\sin\omega t. \tag{59}$$

Replacing the last equation in (13) and applying the harmonic balance method leads to the following system of non-linear algebraic equations:

$$([\mathbf{K}^*] - \omega^{*2}[\mathbf{M}^*])\{\mathbf{A}\} + \tfrac{3}{2}[\mathbf{B}^*(\mathbf{A})]\{\mathbf{A}\} = \{\mathbf{F}^*\} \tag{60}$$

which can be written using tensor notation as

$$a_i k_{ir}^* - \omega^{*2} a_i m_{ir}^* + \tfrac{3}{2} a_i a_j a_k b_{ijkr}^* = F_i^*, \quad i = 1, \ldots, n. \tag{61}$$

The last system is similar to that obtained in equation (23) in the free case with two differences: (1) in the free case, i varies from 2–n and the first equation is omitted, because the first contribution a_1 was assigned; (2) all of the n equations have a rightside representing the forcing term F_i^*. A simplified method for solving this system is presented in the next two subsections.

4.2.1. *First formulation*

Consider now the non-linear system (61) and apply the first formulation which, as in the case of large-amplitude free vibrations, consists in neglecting in the expression $a_i a_j a_k b_{ijkr}$ of equation (61) both first and second order terms with respect to ε_i, i.e., terms of the type $a_1^2 \varepsilon_i b_{11ir}$ or of the type $a_1 \varepsilon_i \varepsilon_j b_{1ijr}$. In the neighbourhood of the first resonance the above equation can be written as

$$(k_{ii}^* - \omega^{*2} m_{ii}^*) a_i + \tfrac{3}{2} a_1^3 b_{irrr}^* = F_i^*, \quad i = 1, \ldots, n. \tag{62}$$

This system permits one to obtain explicitly the modal contributions $a_2, \dots, a_n$ corresponding to a given value of the contribution a_1 as

$$a_i = (F_i^* - \tfrac{3}{2}\, a_1^3\, b_{i111}^*)/(k_{ii}^* - \omega^{*2} m_{ii}^*) \quad (i = 2,, \dots, n). \tag{63}$$

The first harmonic component of the non-linear steady state forced periodic response W^* is then given by

$$W_{\omega *}^*(x, t) = \left[\frac{F_1^* - \frac{3}{2}\, a_1^3 b_{1111}^*}{(k_{11}^* - \omega^{*2} m_{11}^*)}\, w_1^*(x) + \frac{F_3^* - \frac{3}{2} a_1^3\, b_{3111}^*}{(k_{33}^* - \omega^{*2} m_{33}^*)}\, w_3^*(x) + \cdots \right.$$

$$\left. + \frac{F_{11}^* - \frac{3}{2}\, a_1^3\, b_{11111}^*}{(k_{1111}^* - \omega^{*2} m_{1111}^*)}\, w_{11}^*(x) \right] \sin \omega. \tag{64}$$

Equation (64) is an extension to the non-linear case of equation (57) obtained in linear modal analysis, in which the beam total response $W_{\omega^*}^*(x, t)$ appears as the sum of the linear response $W_{\omega^* 1}^*(x, t)$ given by equation (57) and a term due to the non-linearity $W_{\omega^* n1}^*(x, t)$ given by

$$W_{\omega^* n1}^*(x, t) = -\tfrac{3}{2}\, a_1^3 \left[(b_{1111}^*/(k_{11}^* - \omega^{*2} m_{11}^*))\, w_1^*(x) + \frac{b_{3111}^*}{(k_{33}^* - \omega^{*2} m_{33}^*)}\, w_3^*(x) \right.$$

$$\left. + \cdots + (b_{11111}^*/(k_{1111}^* - \omega^{*2} m_{1111}^*))\, w_{11}^*(x) \right] \sin \omega t, \tag{65}$$

where the cubic non-linear term a_1^3 may be obtained from the excitation frequency ω^* and the excitation force F_1^* via equation (47) obtained from the single-mode approach.

4.3. SECOND FORMULATION

As in the free vibration case, the explicit formulae established via the first formulation developed in the above subsection yield accurate results for relatively small amplitudes of vibration and excitation forces. For higher amplitudes, a second formulation is considered in which only second order terms of the type $\varepsilon_i \varepsilon_j a_1 b_{ij1r}$ are neglected in equation (61) and rewritten here as

$$-\omega^{*2} a_i m_{ir}^* + a_i k_{ir}^* + \tfrac{3}{2}\, a_i a_j a_k b_{ijkr}^* = F_i, \quad r = 1, \dots, n. \tag{66}$$

Separating in the non-linear expression $a_i a_j a_k b_{ijkr}^*$ terms proportional to a_1^3, terms proportional to $a_1^2 \varepsilon_i$, and neglecting terms proportional to $a_1 \varepsilon_i \varepsilon_j$ leads to

$$a_i a_j a_k b_{ijkr}^* = a_1^3 b_{111r}^* + a_1^2 \varepsilon_i b_{11ir}^* \tag{67}$$

and after substituting and rearranging, equation (67) can be written in matrix form as

$$([\mathbf{K}_{RI}^*] - \omega^{*2} [\mathbf{M}_{RI}^*])\{\mathbf{A}_{RI}\} + \tfrac{3}{2} [\boldsymbol{\alpha}_I^*]\{\mathbf{A}_{RI}\} = \{F_i - \tfrac{3}{2}\, a_1^3 b_{i111}^*\} \tag{68}$$

in which $[\mathbf{K}_{RI}^*] = [\mathbf{k}_{ij}^*]$ and $[\mathbf{M}_{RI}^*] = [\mathbf{m}_{ij}^*]$ are reduced rigidity and mass matrices associated with the first non-linear mode, obtained by varying i and j in the set $(3, 5, \dots, 11)$, $[\boldsymbol{\alpha}_I^*]$ is a 5×5 square matrix, depending on a_1, whose general term α_{ij}^* is equal to $a_1^2 b_{ij11}^*$ and

$\{F_i - \frac{3}{2} a_1^3 b_{i111}^*\}$ is a column vector representing the right side of the linear system (68) in which the reduced unknown vector is $\{\mathbf{A}_{RI}\}^{\mathrm{T}} = [\varepsilon_3 \varepsilon_5, \ldots, \varepsilon_{11}]$. The modal contributions $\varepsilon_3, \varepsilon_5, \ldots, \varepsilon_{11}$ can be obtained quite easily by solving the linear system (68) of five equations and five unknowns.

Based on this formulation, in the neighbourhood of the rth mode shape, equation (24) becomes

$$([\mathbf{K}_{Rr}^*] - \omega^{*2}[\mathbf{M}_{Rr}^*])\{\mathbf{A}_{Rr}\} + \tfrac{3}{2}[\boldsymbol{\alpha}_R^*]\{\mathbf{A}_{Rr}\} = \{F_i - \tfrac{3}{2} a_r^3 b_{irrr}^*\}, \tag{69}$$

with

$$[\boldsymbol{\alpha}_r^*] = [a_r^2 b_{ijrr}^*]. \tag{70}$$

The modal contributions $\varepsilon_1, \ldots, \varepsilon_n$ for $i \neq j$ can be obtained quite easily by solving the matrix problem.

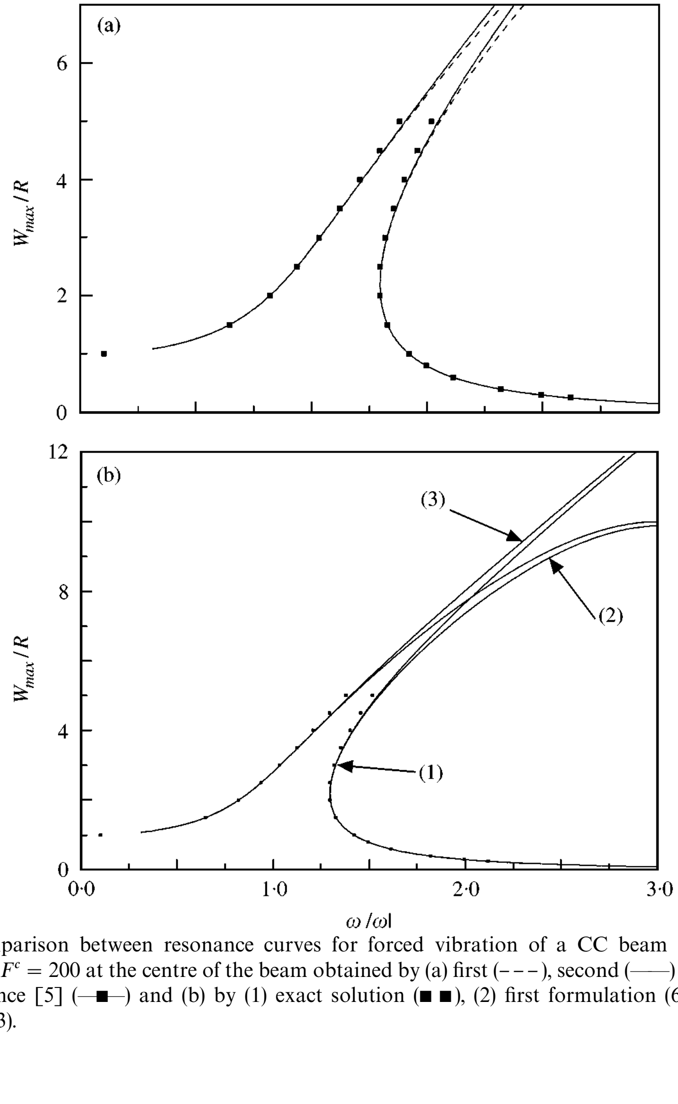

Figure 14. Comparison between resonance curves for forced vibration of a CC beam under a harmonic concentrated force $F^c = 200$ at the centre of the beam obtained by (a) first (– – –), second (——) approximations or values from reference [5] (—■—) and (b) by (1) exact solution (■ ■), (2) first formulation (6D) and (3) second formulation (6D) (3).

4.4. COMPARISON OF THE RESULTS OBTAINED BY THE FIRST AND SECOND FORMULATIONS WITH PREVIOUS RESULTS

The validation of the two formulations developed in the previous section has been made via two types of comparisons: (1) The non-linear response curves obtained by the present formulations have been compared in Figures 14–17 to that obtained previously for three values of the concentrated excitation force (corresponding to $F^c = 200$, 500 and 1000) and one value of the distributed excitation force and a range of non-dimensional vibration amplitudes up to 1·4 times the beam thickness. All curves show a very good agreement with a slight shift towards the right of the curves obtained by the first formulation for relatively

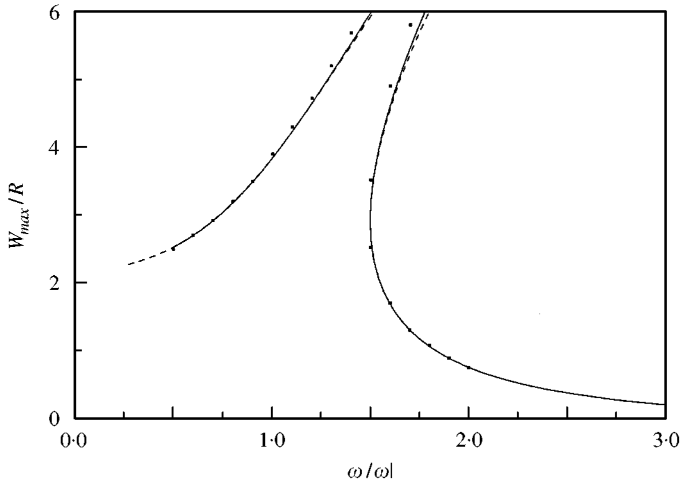

Figure 15. Comparison between resonance curves for forced vibration of a CC beam under a harmonic concentrated force $F^c = 500$ at the centre of the beam obtained by first (– – – –), second (——) approximations or exact solution from reference [5] (■ ■).

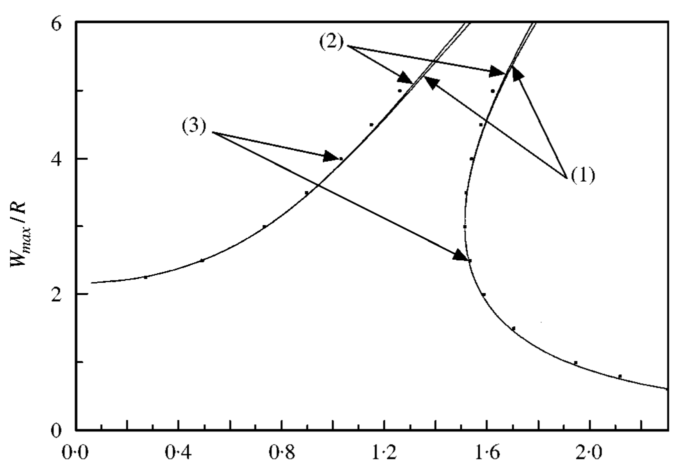

Figure 16. Comparison between resonance curves for forced vibration of a CC beam under a harmonic distributed force $F^d = 1000$ obtained by first (1) or second (2) approximations or exact solution (3) taken from reference [5].

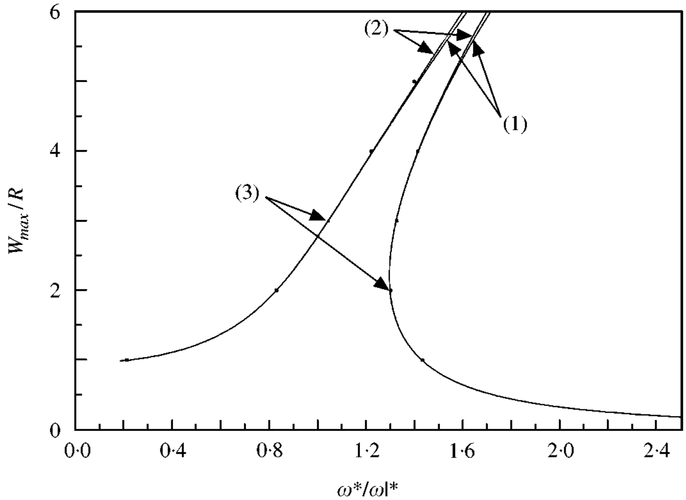

Figure 17. Comparison between resonance curves for forced vibration of a CC beam under a harmonic distributed force $F = 1$ obtained by first (1) or second (2) approximations or exact solution (3) taken from reference [17].

small amplitudes (corresponding to a harder non-linear behaviour) compared with that obtained by the second formulation. For higher amplitudes, the shift is much more pronounced (Fig. 14(b)). (2) The second way of validating the results was based on the comparison of the percentages of participation obtained here with those given in reference [5] and obtained from the numerical solution of the non-linear algebraic system in each case. Also, comparisons have been made of the curvatures obtained at $x^* = 0$ via the different approaches. It was necessary to do so because the main feature of the multi-mode approach is its ability to give information concerning the effect of non-linearity on the bending stress patterns. The moments and the percentage of participation are summarized in Tables 6–8. It can be seen in these tables that, for the range of excitation forces and amplitudes considered, the three approaches agree quite well, and that the error induced by the first formulation on the percentage contributions of higher modes becomes relatively significant for amplitudes of vibration above 0·8 times the beam thickness while the second formulation remains good for amplitudes up to 1·5 times the beam thickness.

5. GENERAL CONCLUSIONS

Considering the free vibration case, a simple approximate analytical expression for the higher mode contribution coefficients to the first three non-linear mode shapes of CC beams and to the first non-linear mode shape of CSS beam have been obtained, which coincides with the numerical solution of the non-linear algebraic system previously developed, for amplitudes up to about 0·7 of the beam thickness. For displacement amplitudes greater than the thickness, an improved formulation is presented, which leads to the exact numerical solution, via the inversion of a 5×5 matrix, which makes obtaining the non-linear mode shapes and resonance frequencies of beams very easy, for a wide range of vibration amplitudes. A similar approach has been applied to the forced response case, enabling explicit determination of the non-linear multi-mode steady state periodic forced response for finite but relatively small vibration amplitudes. The form of the explicit

TABLE 6

Frequency ratios of non-linear forced vibration of a CC beam under a harmonic concentrated force at the centre of the beam ($FORCC = 200$) showing modal participation with six symmetric modes and comparison with the single mode analysis obtained from reference [5]

W (centre)/R	$\omega^*/\omega 1^*$	a_1 (%)	a_3 (%)	a_5 (%)	a_7 (%)	a_9 (%)	a_{11} (%)	W (centre)/R	$\omega^*/\omega 1^*$ (single mode)
1·005353	0·1000000	96·39219	−2·874071	0·5100822	−0·1407441	0·569181E−01	−0·2593153E−01	1·0	0·1928962
1·500927	0·6450033	97·75072	−1·740592	0·3633680	−0·8877219E−01	0·3985901E−01	−0·1668785E−01	1·5	0·6552246
2·000284	0·8177417	98·57255	−1·020286	0·3050060	−0·5823117E−01	0·3245784E−01	−0·1146226E−01	2·0	0·8221617
2·500852	0·9343069	99·19306	−0·4468246	0·2876531	−0·3529215E−01	0·2944710E−01	−0·7718413E−02	2·5	0·9369499
3·000302	1·031101	99·58905	0·678134E−01	0·2944021	−0·1518428E−01	0·2897176E−01	−0·4573171E−02	3·0	1·033773
3·500347	1·120242	99·09354	0·5550099	0·3156096	0·4179276E−02	0·3003896E−01	−0·1619004E−02	3·5	1·123735
4·000838	1·206412	98·56756	1·025542	0·3495403	0·2357336E−01	0·3249186E−01	0·1287302E−02	4·0	1·211345
4·500776	1·291557	98·03852	1·483981	0·3935817	0·4354710E−01	0·3610522E−01	0·4260943E−02	4·5	1·298619
5·000225	1·376679	97·51002	1·931526	0·4459033	0·6442957E−01	0·4074151E−01	0·7377538E−02	5·0	1·386487
−5·000281	1·514947	−96·66279	−2·891753	−0·2879580	−0·1186933	−0·2120875E−01	−0·1759812E−01	−5·0	1·524941

TABLE 6
Continued

W(centre)/R	$\omega^*/\omega 1^*$	a_1 (%)	a_3 (%)	a_5 (%)	a_7 (%)	a_9 (%)	a_{11} (%)	W (centre)/R	$\omega^*/\omega 1^*$ (single mode)
−4·500031	1·453275	−97·05431	−2·601457	−0·2101410	−0·1045837	−0·1381919E−01	−0·1568881E−01	−4·5	1·460924
−4·000338	1·398263	−97·41199	−2·339205	−0·1347016	−0·9306238E−01	−0·6814156E−02	−0·1422664E−01	−4·0	1·404005
−3·500208	1·351558	−97·72275	−2·118571	−0·608941E−01	0·8447389E−01	−0·4549625E−04	−0·1326006E−01	−3·5	1·355963
−3·000673	1·315708	−97·92974	−1·958816	0·125113E−01	−0·7936939E−01	0·6690977E−02	−0·1287480E−01	−3·0	1·319287
−2·500924	1·294311	−97·91839	−1·887597	0·883801E−01	−0·7863952E−01	0·1377245E−01	−0·1322210E−01	−2·5	1·297718
−2·000244	1·293513	−97·74815	−1·958049	0·1729352	−0·8425936E−01	0·2195301E−01	−0·1464926E−01	−2·0	1·297573
−1·499999	1·325402	−97·29274	−2·275527	0·2805047	−0·1004443	0·3287201E−01	−0·1791381E−01	−1·5	1·331533
−1·000629	1·420321	−96·20075	−3·130726	0·4529746	−0·1391726	0·5119060E−01	−0·2518677E−01	−1·0	1·432725
−0·8007563	1·494452	−95·35081	−3·817971	0·5680137	−0·1688901	0·6365799E−01	−0·3065426E−01	−0·8	1·512764
−0·6005462	1·612104	−93·94598	−4·968149	0·7461758	−0·2171378	0·8308178E−01	−0·3946856E−01	−0·6	1·641878
−0·4007967	1·819032	−91·26941	−7·185005	1·066063	−0·3059553	0·1179550	−0·5561330E−01	−0·4	1·877914
−0·3009815	1·994473	−88·78509	−9·264190	1·347756	−0·3846030	0·1485001	−0·6986182E−01	−0·3	2·088828
−0·2503182	2·119254	−86·89849	−10·85583	1·553040	−0·4418040	0·1706226	−0·8020222E−01	−0·25	2·243612

TABLE 7

Non-linear forced vibration of a CC beam subjected to harmonic concentrated force $F = 1{\cdot}0077$ ($FORCC = 200$) showing modal participation with six symmetric modes (first formulation)

$d^2W/dx^2(0)$	W(centre)$/R$	a_1(%)	a_3(%)	a_5(%)	a_7(%)	a_9(%)	a_{11}(%)	$\omega^*/\omega 1^*$ Present model (single mode)	$A = W_{max}/R$ (single mode)
0·71908274E+01	0·10325769E+01	0·9646231E+02	−0·2824562E+01	0·4983606E+00	−0·1355028E+00	0·5476799E−01	−0·2449011E−01	0·193387	1.00017
0·11458715E+02	0·15323330E+01	0·9775797E+02	−0·1740115E+01	0·3614371E+00	−0·8586619E+01	0·3878778E−01	−0·1581464E−01	0·656072	1.50190
0·15898021E+02	0·20264197E+01	0·9854593E+02	−0·1044725E+01	0·3096403E+00	−0·5675754E−01	0·3202125E−01	−0·1092377E−01	0·822252	2.00034
0·20630068E+02	0·25206678E+01	0·9916643E+02	−0·4651104E+00	0·2971029E+00	−0·3464866E−01	0·2931494E−01	−0·7391263E−02	0·937378	2.50208
0·25658820E+02	0·30077824E+01	0·9955874E+02	0·8427740E−01	0·3082440E+00	−0·1533219E−01	0·2894958E−01	−0·4454569E−02	1·033868	3.00051
0·31120658E+02	0·34933895E+01	0·9898501E+02	0·6461930E+00	0·3338247E+00	0·3315436E−02	0·2993354E−01	−0·1720051E−02	1·124133	3.50225
0·37015231E+02	0·39702052E+01	0·9834215E+02	0·1230780E+01	0·3723589E+00	0·2172130E−01	0·3209814E−01	0·8890910E−03	1·211465	4.00069
0·43495615E+02	0·44435839E+01	0·9764004E+02	0·1858389E+01	0·4221708E+00	0·4065583E−01	0·3523323E−01	0·3504226E−02	1·299043	4.50242
0·50554703E+02	0·49063124E+01	0·9688493E+02	0·2528466E+01	0·4811537E+00	0·6015659E−01	0·3914221E−01	0·6145348E−02	1·386639	5.00086
−0·52925104E+02	−0·48331712E+01	−0·9576487E+02	−0·3804597E+01	−0·2822740E+00	−0·1149053E+00	−0·1736788E−01	−0·1598328E−01	1·52505	−5.00086

TABLE 7
Continued

$d^2W/dx^2(0)$	W(centre)$/R$	a_1(%)	a_3(%)	a_5(%)	a_7(%)	a_9(%)	a_{11}(%)	$\omega^*/\omega1^*$ Present model (single mode)	$A = W_{max}/R$ (single mode)
−0·45821852E+02	−0·43713363E+01	−0·9639898E+02	−0·3272698E+01	−0·2007412E+00	−0·1020953E+00	−0·1094803E−01	−0·1453753E−01	1·46121	−4.50242
−0·39302575E+02	−0·38987447E+01	−0·9694845E+02	−0·2819349E+01	−0·1226139E+00	−0·9150286E−01	−0·4665231E−02	−0·1341368E−01	1·40407	−4.00069
−0·33374590E+02	−0·34226053E+01	−0·9739643E+02	−0·2457706E+01	−0·4799523E−01	−0·8366226E−01	0·1505038E−02	−0·1269422E−01	1·35615	−3.50225
−0·27884134E+02	−0·29375776E+01	−0·9768063E+02	−0·2194200E+01	0·2581741E−01	−0·7904012E−01	0·7835158E−02	−0·1246982E−01	1·31931	−3.00051
−0·22831753E+02	−0·24509413E+01	−0·9773862E+02	−0·2054195E+01	0·1009626E+00	−0·7870679E−01	0·1458873E−01	−0·1292117E−01	1·29776	−2.50208
−0·18080652E+02	−0·19570791E+01	−0·9761040E+02	−0·2082723E+01	0·1851307E+00	−0·8472377E−01	0·2259752E−01	−0·1441994E−01	1·29756	−2.00034
−0·13627198E+02	−0·14632788E+01	−0·9717159E+02	−0·2382957E+01	0·2927606E+00	−0·1014852E+00	0·3345819E−01	−0·1774451E−01	1·33130	−1.50190
−0·93506799E+01	−0·96369725E+00	−0·9603598E+02	−0·3272609E+01	0·4714415E+00	−0·1423397E+00	0·5234670E−01	−0·2528117E−01	1·43267	−1.00017
−0·77086177E+01	−0·76588091E+00	−0·9513621E+02	−0·4001458E+01	0·5922172E+00	−0·1737985E+00	0·6534287E−01	−0·3096589E−01	1·51171	−0.80211
−0·60653310E+01	−0·56438090E+00	−0·9358551E+02	−0·5273890E+01	0·7870285E+00	−0·2267201E+00	0·8638899E−01	−0·4046070E−01	1·64123	−0.60076
−0·44850386E+01	−0·36551585E+00	−0·9053749E+02	−0·7806086E+01	0·1146617E+01	−0·3264054E+00	0·1251436E+00	−0·5825178E−01	1·87342	−0.40270
−0·36987735E+01	−0·26216701E+00	−0·8741314E+02	−0·1043343E+02	0·1493186E+01	−0·4226705E+00	0·1621996E+00	−0·7536718E−01	2·08780	−0.30038
−0·33355182E+01	−0·21171572E+00	−0·8500580E+02	−0·1247832E+02	0·1746378E+01	−0·4926781E+00	0·1890384E+00	−0·8777989E−01	2·24050	−0.2508

TABLE 8

Non-linear forced vibration in a CC beam subjected to a harmonic concentrated force $F = 1{\cdot}0077$ ($FORCC = 200$) showing modal participation with six symmetric modes (second formulation)

$d^2W/dx^2(0)$	W(centre)/R	a_1(%)	a_3(%)	a_5(%)	a_7(%)	a_9(%)	a_{11}(%)	$\omega^*/\omega 1^*$ Present model (single mode)	$A = W_{max}/R$ (single mode)
0·71959697E+01	0·10322228E+01	0·9649959E+02	−0·2791101E+01	0·4939064E+00	−0·1356519E+00	0·5443681E−01	−0·2531086E−01	0·193387	1.00017
0·11476029E+02	0·15315735E+01	0·9781268E+02	−0·1691889E+01	0·3549499E+00	−0·8581266E−01	0·3841341E−01	−0·1624833E−01	0·656072	1.50190
0·15923274E+02	0·20252974E+01	0·9860762E+02	−0·9922829E+00	0·3009628E+00	−0·5646319E−01	0·3161973E−01	−0·1104979E−01	0·822252	2.00034
0·20644987E+02	0·25195173E+01	0·9921761E+02	−0·4262719E+00	0·2859637E+00	−0·3401156E−01	0·2893975E−01	−0·7197631E−02	0·937378	2.50208
0·25623354E+02	0·30073875E+01	0·9957588E+02	0·8308874E−01	0·2942265E+00	−0·1421633E−01	0·2868522E−01	−0·3896686E−02	1·033868	3.00051
0·30963046E+02	0·34951635E+01	0·9907766E+02	0·5698316E+00	0·3167595E+00	0·5067776E−02	0·2993521E−01	−0·7365193E−03	1·124133	3.50225
0·36625147E+02	0·39763158E+01	0·9855149E+02	0·1037711E+01	0·3515577E+00	0·2432946E−01	0·3252382E−01	0·2379863E−02	1·211465	4.00069
0·42709426E+02	0·44572097E+01	0·9801924E+02	0·1497872E+01	0·3966158E+00	0·4438181E−01	0·3627768E−01	0·5606906E−02	1·299043	4.50242
0·49154530E+02	0·49316333E+01	0·9749015E+02	0·1945017E+01	0·4495688E+00	0·6526791E−01	0·4101890E−01	0·8970893E−02	1·386639	5.00086
−0·5096230E+02	−0·48760427E+01	−0·9665415E+02	−0·2900570E+01	−0·2858806E+00	−0·1190128E+00	−0·2115957E−01	−0·1922716E−01	1·52505	−5.00086

TABLE 8
Continued

$d^2W/dx^2(0)$	W(centre)$/R$	a_1(%)	a_3(%)	a_5(%)	a_7(%)	a_9(%)	a_{11}(%)	$\omega^*/\omega 1^*$ Present model (single mode)	$A = W_{max}/R$ (single mode)
−0·4457298E+02	−0·43995533E+01	−0·9704540E+02	−0·2610598E+01	−0·2082039E+00	−0·1049036E+00	−0·1379716E−01	−0·1709219E−01	1·46121	−4.50242
−0·3854282E+02	−0·39166561E+01	−0·9740520E+02	−0·2347002E+01	−0·1323229E+00	−0·9329576E−01	−0·6769168E−02	−0·1540276E−01	1·40407	−4.00069
−0·3293156E+02	−0·34336236E+01	−0·9771482E+02	−0·2127556E+01	−0·5862517E−01	−0·8471841E−01	−0·3374901E−04	−0·1423939E−01	1·35615	−3.50225
−0·2763910E+02	−0·29441023E+01	−0·9791601E+02	−0·1968679E+01	0·1530017E−01	−0·7960892E−01	0·6720251E−02	−0·1367953E−01	1·31931	−3.00051
−0·2270275E+02	−0·24546817E+01	−0·9790049E+02	−0·1901519E+01	0·9133406E−01	−0·7897644E−01	0·1377822E−01	−0·1389769E−01	1·29776	−2.50208
−0·1801749E+02	−0·19591204E+01	−0·9772055E+02	−0·1980395E+01	0·1769237E+00	−0·8485662E−01	0·2200302E−01	−0·1526875E−01	1·29756	−2.00034
−0·1359994E+02	−0·14643114E+01	−0·9724510E+02	−0·2315392E+01	0·2862921E+00	−0·1016081E+00	0·3301187E−01	−0·1859429E−01	1·33130	−1.50190
−0·9343334E+01	−0·96412772E+00	−0·9608090E+02	−0·3231332E+01	0·4668592E+00	−0·1425601E+00	0·5199319E−01	−0·2635034E−01	1·43267	−1.00017
−0·7706388E+01	−0·76615261E+00	−0·9517090E+02	−0·3969389E+01	0·5883584E+00	−0·1740944E+00	0·6500561E−01	−0·3224902E−01	1·51171	−0.80211
−0·6067137E+01	−0·56452862E+00	−0·9360985E+02	−0·5250993E+01	0·7838353E+00	−0·2271320E+00	0·8604513E−01	−0·4213569E−01	1·64123	−0.60076
−0·4489865E+01	−0·36557545E+00	−0·9055118E+02	−0·7792436E+01	0·1143889E+01	−0·3270146E+00	0·1247422E+00	−0·6073378E−01	1·87342	−0.40270
−0·3704903E+01	−0·26219347E+00	−0·8742071E+02	−0·1042490E+02	0·1490502E+01	−0·4234565E+00	0·1617200E+00	−0·7870883E−01	2·08780	−0.30038
−0·3342273E+01	−0·21172895E+00	−0·8501005E+02	−0·1247245E+02	0·1743604E+01	−0·4935874E+00	0·1884950E+00	−0·9180378E−01	2·24050	−0.2508

case which may appear as one more step towards the development of the "non-linear modal analysis theory" [8]. For very high amplitudes, the non-linear multi-mode steady state periodic forced response is still very easily obtained via inversion of a 5×5 square matrix and the solution coincides exactly with the numerical solution previously obtained from the non-linear algebraic system.

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APPENDIX A: BEAM FUNCTIONS

A.1. CLAMPED–CLAMPED BEAMS

The chosen basic functions w_i were the linear CC beam functions

$$w_i(x) = \frac{ch(v_i x/L) - \cos(v_i x/L)}{ch\, v_i - \cos v_i} - \frac{sh(v_i x/L) - \sin(v_i x/L)}{sh\, v_i - \sin v_i},$$

where v_i are the eigenvalue parameters for CC beam.

The values of the parameters v_i have been computed by solving numerically the transcendental equation $ch\, v_i \cos v_i = 1$ using Newton's method and are given in Table A1.

The functions w_i have been normalized in such a manner that

$$m_{ij}^* = \int_0^1 w_i^*(x^*) w_j^*(x^*)\, \mathrm{d}x^* = \delta_{ij}.$$

TABLE A1

Symmetric (a) and antisymmetric (b) eigenvalue parameters v_i for a CC beam

	(a)		(b)
1	4·73004075	2	7·85320462
3	10·99560784	4	14·13716549
5	17·27875966	6	20·42035225
7	23·56194490	8	26·70353756
9	29·84513021	10	32·98672286
11	36·12831552	12	39·26990817

The tensors k_{ij}^* and b_{ijkl}^* are defined by

$$k_{ij}^* = \int_0^1 \left(\frac{\partial^2 W_i^*}{\partial x^{*2}}\right)\left(\frac{\partial^2 W_j^*}{\partial x^{*2}}\right) \mathrm{d}x^* = \mathbf{RIG}\ (i, j),$$

$$b_{ijkl}^* = \alpha \int_0^1 \left(\frac{\partial W_i^*}{\partial x^*}\right)\left(\frac{\partial W_j^*}{\partial x^*}\right) \mathrm{d}x^* \int_0^1 \left(\frac{\partial W_k^*}{\partial x^*}\right)\left(\frac{\partial W_1^*}{\partial x^*}\right) \mathrm{d}x^* = \alpha \mathbf{RAG}\,(i, j)\ \mathbf{RAG}\ (k, l),$$

where $\alpha = SH^2/4I$ is the non-dimensional parameter characterizing the beam cross-section. For a CC beam of rectangular cross-section, corresponding to a value of $\alpha = 3$, the matrixes

RIG (i, j) and **RAG**(i, j) are given numerically (for $i, j = 1, 3, 5, 7, 9, 11$) by

$$[\mathbf{RIG}(i,j)] = \begin{bmatrix} 500{\cdot}56 & & & & & \\ & 14617{\cdot}63 & & & & \\ & & 89135{\cdot}40 & & & \\ & & & 308208{\cdot}45 & & \\ & & & & 793406{\cdot}25 & \\ & & & & & 1691832{\cdot}35 \end{bmatrix}$$

with all non-diagonal terms of [**RIG**] equaling zero, and

$$[\mathbf{RAG}(i,j) = \begin{bmatrix} 12{\cdot}30 & -9{\cdot}73 & -7{\cdot}61 & -6{\cdot}10 & -5{\cdot}06 & -4{\cdot}25 \\ -9{\cdot}73 & 98{\cdot}90 & -24{\cdot}34 & -22{\cdot}98 & -20{\cdot}85 & -18{\cdot}47 \\ -7{\cdot}61 & -24{\cdot}34 & 263{\cdot}99 & -38{\cdot}02 & -37{\cdot}95 & -35{\cdot}71 \\ -6{\cdot}10 & -22{\cdot}98 & -38{\cdot}02 & 508{\cdot}04 & -51{\cdot}22 & -51{\cdot}08 \\ -5{\cdot}06 & -20{\cdot}85 & -37{\cdot}95 & -51{\cdot}22 & 831{\cdot}05 & -62{\cdot}65 \\ -4{\cdot}25 & -18{\cdot}47 & -35{\cdot}71 & -51{\cdot}08 & -62{\cdot}65 & 1236{\cdot}08 \end{bmatrix}.$$

A.2. CLAMPED–SIMPLY SUPPORTED BEAMS

The chosen basic functions for CSS beams are CSS beam linear mode shapes. The constants v_i for a CSS beam are obtained by solving the equation $tg(v_i)$-$th(v_i) = 0$ and are given in Table A2.

TABLE A2

Odd (a) and even (b) eigenvalue parameters v_i for a CSS beam

	(a)		(b)
1	3·926602312	2	7·068582745
3	10·210176122	4	13·351768777
5	16·493361431	6	19·634954084

The tensors m^*_{ij}, k^*_{ij} and b^*_{ijkl} are defined as in the case of a CC beam. For a CSS beam of rectangular cross-section, corresponding to a value of $\alpha = 3$, the matrixes **RIG** (i, j) and **RAG**(i, j) are given numerically (for $i, j = 1, 2, 3, 4, 5, 6$) by

$$[\mathbf{RIG}(i,j)] = \begin{bmatrix} 237{\cdot}72 & & & & & \\ & 2496{\cdot}48 & & & & \\ & & 10867{\cdot}58 & & & \\ & & & 31780{\cdot}09 & & \\ & & & & 74000{\cdot}84 & \\ & & & & & 148634{\cdot}47 \end{bmatrix}$$

with all non-diagonal terms of [**RIG**] equaling zero, and

$$[\mathbf{RAG}(i,j)=\begin{bmatrix} 11{\cdot}51 & -4{\cdot}28 & -3{\cdot}79 & -3{\cdot}28 & -2{\cdot}85 & -2{\cdot}51 \\ -4{\cdot}28 & 42{\cdot}89 & -7{\cdot}81 & -7{\cdot}64 & -7{\cdot}16 & -6{\cdot}62 \\ -3{\cdot}79 & -7{\cdot}81 & 94{\cdot}03 & -11{\cdot}16 & -11{\cdot}28 & -10{\cdot}99 \\ -3{\cdot}28 & -7{\cdot}64 & -11{\cdot}16 & 164{\cdot}91 & -14{\cdot}43 & -14{\cdot}78 \\ -2{\cdot}85 & -7{\cdot}16 & -11{\cdot}28 & -14{\cdot}43 & 255{\cdot}53 & -17{\cdot}65 \\ -2{\cdot}51 & -6{\cdot}62 & -10{\cdot}99 & -14{\cdot}78 & -17{\cdot}65 & 365{\cdot}89 \end{bmatrix}.$$

APPENDIX B: DETAILS CORRESPONDING TO THE 1-D APPROACH

The single-mode approach, corresponding to the 1-D non-linear frequency response function gives [5]

$$(\omega^*/\omega_L^*)^2 = 1 + \tfrac{3}{2}\,(b_{1111}^*/k_{11}^*)\,((\mathscr{A})^2/(W_1^*(1/2)^2) - F^*/\mathscr{A}.$$

The pair of values ω^* and a_1 obtained from the equality $\mathscr{A} = W_{max}/R = a_1 w_1^*(1/2)$ are then re-injected in equation (63) to calculate the modal contributions a_i in the neighbourhood of the first mode shape.

If one looks to the expressions of non-dimensional concentrated force F^c and distributed force F^d given in reference [5], and recalls that in the second section of this paper

$$F_i^c = \frac{\mathscr{F}^c L^3}{k_{11}^* EIR}\, w_1^*(1/2)\, w_i^*(x_0), \qquad F_i^d = \frac{F^d L^4}{k_{11}^* EIR}\, w_1^*(1/2) \int_0^L w_i^*(x^*)\,\mathrm{d}x^*,$$

then one can put the non-dimensional ratio $FORCD = F^c L^3/EIR$ and the non-dimensional ratio $FORCD = F^d L^4/EIR$, then

$$F_i^c = (FORCC/k_{11}^*) w_1^*(1/2) w_i^*(x_0), \qquad F_i^d = (FORCD/k_{11}^*) w_1^*(1/2) \int_0^1 w_i^*(x)\,\mathrm{d}x^*.$$

APPENDIX C: NOMENCLATURE

V_b, V_a, V	bending, axial and total strain energy respectively
E	Young's modulus
ρ	mass per unit length
L	length of the beam
S	area of cross-section
I	second moment of area of cross-section
H	thickness of the beam
$W(x, t)$	transverse displacement at point x on the beam
T	kinetic energy
x	point co-ordinate
q_i	generalized co-ordinate $q_i(t) = a_i \sin(\omega t)$
$\{\mathbf{A}\}$	column matrix of basic functions contributions to the free or forced response $\{\mathbf{A}\}^{\mathrm{T}} = [a_1, \ldots, a_n]$
k_{ij}, m_{ij}, b_{ijkl}	general terms of the rigidity tensor, mass tensor and non-linearity tensor respectively
[**K**], [**M**], [**B**]	rigidity, mass and non-linearity matrix respectively
ϕ_i^*	the ith linear mode shape of the clamped–simply supported beam
w_i^*	the ith linear mode shape of the clamped–clamped beam

k_{11}, m_{11}, b_{1111}	rigidity, mass and non-linearity parameters corresponding to the one mode assumed, respectively.
$[a_1 \varepsilon_3, \ldots, \varepsilon_{11}]$	column matrix of the contribution coefficients to the first non-linear CC beam mode shape
$[a_2 \varepsilon_4, \ldots, \varepsilon_{12}]$	column matrix of the contribution coefficients to the second non-linear CC beam mode shape
$[\varepsilon_1 a_3, \ldots, \varepsilon_{11}]$	column matrix of the contribution coefficients to the third non-linear CC beam mode shape
$[a_1 \varepsilon_2, \ldots, \varepsilon_6]$	column matrix of the contribution coefficients to the first non-linear CSS beam mode shape
$F(x, t)$	exciting force
$\bar{S}$	range of application of the exciting force
$\{F(t)\}$	column matrix of generalized forces $F_i(t)$
$F_1(t)$	generalized force corresponding to the one mode assumed
k^*_{ij}, m^*_{ij}, b^*_{ijkl}	general terms of the non-dimensional rigidity tensor, mass tensor and non-linearity tensor respectively
ω, ω^*	frequency and non-dimensional frequency parameter respectively
$*$	star exponent indicates non-dimensional parameters
$w^*_{nli}(x, a_i)$	the ith CC beam non-linear mode shape for a given assigned value a_i of the ith function contribution
$w^*_{\omega^* 1}(x, t)$	linear term in the first harmonic component of the non-linear steady state forced periodic response
$w^*_{\omega^* nl}(x, t)$	the term due to non-linearity in the first harmonic component of the non-linear steady state forced periodic response
$w^*_{\omega^*}(x, t)$	the first harmonic component of the non-linear steady state forced periodic response