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LETTERS TO THE EDITOR

NON-INVERTIBLE TEMPORAL TRANSFORMATIONS AND POWER SERIES PERIODIC SOLUTIONS

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1. INTRODUCTION

Harmonic transformation of time in combination with the power series method was used by Ince [1] for the investigation of periodic motions. ("Ince algebraization".) For example, by introducing the new variables

$$\tau = \sin t \quad \text{and} \quad x(t) = X(\tau(t)), \tag{1}$$

in the Mathieu equation, $\ddot{x} + (a + b\cos 2t)\ x = 0$, one obtains the equation $(1-\tau^2)X''_{\tau^2} - \tau X'_\tau + (a + b - 2b\tau^2)\ X = 0$, which admits periodic solutions in terms of the power series with respect to the new temporal argument $|\tau| \leqslant 1$ [2]. Transformation (1) with the power series methods were employed for non-linear vibrating systems as well [3, 4]. Non-harmonic time transformations dealing with Jacobian functions can also be found in the literature [5]. A non-smooth (sawtooth sine) oscillating time and the power series form solutions were considered in reference [6] for oscillators with strongly non-linear characteristics. As it is known, however, such transformations of time are restricted by special cases and *cannot be applied to any periodic regime*. From the mathematical point of view, it is caused by the fact that an inverse transformation does not exist over the whole period of motion.

In the present work, different versions of periodic time are introduced in such a way that the corresponding transformations become valid for any periodic motion. This result is achieved by the special complexification of the co-ordinates. In addition, the Lie series form solutions are suggested for the case of sawtooth time.

2. SINE-TRANSFORM OF TIME

Let us start from a generalization of change of variables (1). For the sake of compactness, notations

$$\tau = \tau(\varphi) \equiv \sin\varphi \quad \text{and} \quad e = e(\varphi) \equiv \cos\varphi, \tag{2}$$

will be used below.

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Proposition 1. *Any sufficiently smooth periodic function $x(\varphi)$, whose period is normalized as $T = 2\pi$, can be represented as*

$$x(\varphi) = X(\tau(\varphi)) + Y(\tau(\varphi))e(\varphi), \tag{3}$$

where the components $X\ (\tau(\varphi))$ and $Y\ (\tau(\varphi))$ are of the power series form with respect to $\tau(\varphi)$.

Proof. Let us consider some periodic function, whose period is 2π, by showing explicitly the terms with odd and even wave numbers in the corresponding Fourier series

$$x(\varphi) = \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_{2n} \cos 2n\varphi + A_{2n-1} \cos (2n-1)\varphi]$$
$$+ \sum_{n=1}^{\infty} [B_{2n} \sin 2n\varphi + B_{2n-1} \sin (2n-1)\varphi]. \tag{4}$$

By using notations (2), one can rewrite the tabulated expressions [7, formula No. 1.332] in the form

$$\cos 2n\varphi = \sum_{i=0}^{n} a_{2i}\tau^{2i}, \qquad \cos (2n-1)\varphi = \left(\sum_{i=1}^{n} a_{2i-1}\tau^{2i-2}\right)e,$$
$$\sin 2n\varphi = \left(\sum_{i=1}^{n} b_{2i-2}\tau^{2i-1}\right)e, \qquad \sin (2n-1)\varphi = \sum_{i=1}^{n} b_{2i-1}\tau^{2i-1}, \tag{5}$$

where the coefficients are listed in the referenced table. Substituting expressions (5) into series (4) and reordering the terms, gives representation (3) and thus, completes the proof. □

As seen from identities (5), the second component in representation (3) is due to the odd cosine-waves and even sine-waves of the Fourier expansion.

It is important to note that combination (3) possesses special algebraic properties which should be taken into account while introducing the new temporal argument τ into the differential equation of motion. Namely, *differentiation, integration or any sufficiently smooth function of representation* (3) *gives an element of the same two-component structure as* equation (3). These properties are simply dictated by the fact that none of the operations listed above will destroy the periodicity of the function, and hence, Proposition 1 can be applied to the result of the operations as well. For example, relations

$$(X + Ye)'_\varphi = X_{der} + Y_{der}e \quad \text{and} \quad f(X + Ye) = X_f + Y_f e \tag{6}$$

hold.

Practically, the new components on the rigid-hand sides of the above relations can be obtained by taking into account the trigonometric identities $\tau'(\varphi) = e(\varphi)$, $e'(\varphi) = -\tau(\varphi)$ and

$$e^2 = 1 - \tau^2. \tag{7}$$

Note also that the two parts of representation (3) apparently are linearly independent, thus the whole combination becomes zero if and only if both components are zero.

For example, let us introduce the new time argument $\tau = \sin \omega t$, where ω is a frequency parameter, into the Duffing oscillator with no linear "stiffness term" [8],

$$\ddot{x} + \zeta\dot{x} + x^3 = F \sin \omega t. \tag{8}$$

Let us represent a periodic solution in the form (3). By taking into account trigonometric identities (7) on each step of the transformation, and collecting separately terms with common factor e, one obtains

$$[(1-\tau^2)X''-\tau X']\omega^2+\zeta\,[(1-\tau^2)Y'-\tau Y]\,\omega+3(1-\tau^2)XY^2+X^3=F\tau \quad (9)$$

$$[(1-\tau^2)Y''-3\tau Y'-Y]\omega^2+\zeta X'\omega+3X^2Y+(1-\tau^2)Y^3=0, \quad (10)$$

where the unknown functions must satisfy the condition of analytical continuation on the boundaries. The latter are obtained by substituting $\tau=\pm 1$ into equations (9) and (10):

$$[-\tau X'\omega^2-\tau\zeta Y\omega+X^3-F\tau]|_{\tau=\pm 1}=0, \quad (11)$$

$$[-(3\tau Y'+Y)\omega^2+\zeta X'\omega+3X^2Y]|_{\tau=\pm 1}=0. \quad (12)$$

It is seen, due to the damping term, that the above system does not admit a family of solutions on which $Y(\tau)\equiv 0$. As a result, the particular form of the representation, i.e., transformation (1), is not valid in this case. Equations (9) and (10) appear to be complicated. However, one can apply the power series methods remaining in the class of periodic solutions.

3. SAWTOOTH TIME AND LIE SERIES SOLUTIONS

Interestingly enough, the form of representation (3) remains the same in the case of sawtooth oscillating time, although the basic algebraic operation (7) is changed. Using the above notations, let us denote the sawtooth sine and its first generalized derivative as, respectively,

$$\tau=\tau(\varphi)\equiv\frac{2}{\pi}\arcsin\sin\frac{\pi\varphi}{2} \quad \text{and} \quad e=e(\varphi)=\tau'(\varphi). \quad (13)$$

The amplitude and the period are normalized such that first generalized derivative (the rectangular cosine) satisfies equality,

$$e^2=\mathbf{1} \quad (14)$$

for almost all φ. By considering functions (13), it was shown earlier [9] that *any periodic function whose period is normalized as* $T=4$ *can be represented in the form* (3), *where* $\tau(\varphi)$ *and* $e(\varphi)$ *are given by expressions* (13). The corresponding proof did not employ any Fourier series or trigonometric identities. (Moreover, the X- and Y-components of the sawtooth representation are not strongly supposed to be power series, so that the representation is also true for non-smooth and even discontinuous functions.) The basic properties of the sawtooth representation appeared to be the same as those formulated above for the sinusoidal time. The related analytical methods for the class of strongly non-linear oscillating systems can be found in reference [10]. Let us consider equation (8), assuming that the forcing function is $F\tau(t/a)$, where a is a quarter of the period. The latter assumption is not restrictive, since any periodic function can be expressed through the sawtooth sine and then expanded into the power series. Let us represent periodic solutions in the form (3), where the new temporal argument is given by expressions (13) and $\varphi=t/a$. Then, substituting representation (3) into equation (8) and considering the result as a two-component element of the algebra, one obtains the boundary value problem

$$X''a^{-2} + \zeta Y'a^{-1} + XY^2 + X^3 = F\tau, \tag{15}$$

$$Y''a^{-2} + \zeta X'a^{-1} + 3YX^2 + Y^3 = 0, \tag{16}$$

$$Y|_{\tau=\pm 1} = 0, \quad X'|_{\tau=\pm 1} = 0, \tag{17}$$

where boundary conditions (17) stay for elimination of the periodic series of Dirac functions from the first and second derivatives of the co-ordinate (the smoothness condition [10]).

In order to introduce the Lie series solutions, let us consider the differential equations of motion

$$\ddot{x} + f(x, \dot{x}, t) = 0. \tag{18}$$

The idea of Lie series enables one to automatically construct the power series form solutions as follows:

$$x(t) = \exp(tG)\,x_0 = \sum_{k=0}^{\infty} \frac{t^k}{k!} G^k x_0, \qquad G = \dot{x}_0 \frac{\partial}{\partial x_0} - f(x_0, \dot{x}_0, t)\frac{\partial}{\partial \dot{x}_0} + \frac{\partial}{\partial t}, \tag{19}$$

where G is a differential operator Lie associated with equation (18), and the initial conditions are $x_0 = x|_{t=0}$ and $\dot{x}_0 = \dot{x}|_{t=0}$. Note that the vector form of solution for multiple-degree-of-freedom systems ($x \in R^n$) would be the same, but the first two terms in the expression for operator G must be treated as dot products. Applying now general formula (19) to system (15) and (16), gives

$$\begin{bmatrix} X(\tau) \\ Y(\tau) \end{bmatrix} = \sum_{k=0}^{N} \frac{\tau^k}{k!} \begin{bmatrix} G^k B \\ G^k D \end{bmatrix} + O(\tau^{N+1}), \tag{20}$$

where

$$G = A\frac{\partial}{\partial B} + C\frac{\partial}{\partial D} - [a\zeta C + a^2(BD^2 + B^3) - a^2 F\tau]\frac{\partial}{\partial A}$$

$$- [a\zeta A + a^2(3DB^2 + D^3)]\frac{\partial}{\partial C} + \frac{\partial}{\partial \tau} \tag{21}$$

and notations $A = X'(0)$, $B = X(0)$, $C = Y'(0)$ and $D = Y(0)$ have been introduced.

Further substitution of solution (20) into the boundary conditions (17) gives, generally speaking, four non-linear algebraic equations. If being conducted, the corresponding analysis gives the non-linear amplitude–frequency response of the system. In many cases, the number of equations and unknowns can be reduced due to the symmetry of the differential equations and the boundary conditions. For example, in the above case, one can consider a family of solutions with odd-power terms for the X- and even-powers for the Y-component of the solution. Correspondingly, one should set $C = 0$ and $B = 0$. Such a reduced algebraic problem still remains difficult, however, if taking into account higher power terms. On the other hand, the methodology is not restricted by any assumptions about the system parameters.

4. GENERAL CASE

Let us consider now a general class of functions $\{\tau(\varphi), e(\varphi)\}$ produced by a conservative oscillator whose differential equation of motion is written in the form $\ddot{x} + \Pi'(x) = 0$, where $\Pi(x)$ is a function of the potential energy of the oscillator. In order to make the amplitude normalized to $x_{ampl.} = 1$, let us introduce the normalized potential energy function and time as $P(x) = \Pi(x)/\Pi(1)$ and $\varphi = \sqrt{2\Pi(1)}t$. As a result, the differential equation of motion and its first (energy) integral take the form

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\varphi^2} + \frac{1}{2}P'(x) = 0 \quad \text{and} \quad \left(\frac{\mathrm{d}x}{\mathrm{d}\varphi}\right)^2 = 1 - P(x) \tag{22}$$

respectively.

Let $x = \tau(\varphi)$ and $\mathrm{d}x/\mathrm{d}\varphi = e(\varphi)$ be the system co-ordinate and the velocity respectively. The co-ordinate can be determined implicitly from the energy integral as

$$\int_0^{\tau(\varphi)} \frac{\mathrm{d}s}{\sqrt{1 - P(s)}} = \varphi, \tag{23}$$

whereas expressions (22) complete the differentiation rules and the basic algebraic identity as follows:

$$\tau'(\varphi) = e(\varphi), \qquad e'(\varphi) = -\tfrac{1}{2}P'(\tau) \tag{24}$$

and

$$e^2 = 1 - P(\tau). \tag{25}$$

Now let us formulate (without proof), Proposition 2.

Proposition 2. *Any periodic function* $x(\varphi)$ *whose period is normalized to*

$$T = 4\int_0^1 \frac{\mathrm{d}s}{\sqrt{1 - P(s)}}$$

can be represented in form (3), *where the functions* $\tau(\varphi)$ *and* $e(\varphi)$ *are given by relations* (23) *and* (24), *and properties* (6) *hold.*

For example, one can transform equation (8) based on the potential energy function $P(x) = x^{2n}$. Then the boundary value problems (9)–(12) and (15)–(17) can be derived as particular cases $n = 1$ and $n \to \infty$ respectively. A physical meaning of the latter limiting case was discussed in reference [10]. This case requires a generalized treatment of the differential equations of motion though.

Let us summarize the above results. As seen from the basic algebraic expressions (7), (14) and (25), as well as the form of the transformed equations, the sawtooth version of periodic time appears to be relatively simple from the algebraic point of view such that the Lie series method can be applied directly to the transformed equations. Finally, the above-mentioned basic algebraic expressions are associated with the energy conservation of those oscillators which generate the basic functions $\{\tau(\varphi), e(\varphi)\}$.

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