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ASYMPTOTIC APPROACH FOR NON-LINEAR PERIODICAL VIBRATIONS OF CONTINUOUS STRUCTURES

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An asymptotic approach for determining periodic solutions of non-linear vibration problems of continuous structures (such as rods, beams, plates, etc.) is proposed. Starting with the well-known perturbation technique, the independent displacement and frequency is expanded in a power series of a natural small parameter. It leads to infinite systems of interconnected non-linear algebraic equations governing the relationships between modes, amplitudes and frequencies. A non-trivial asymptotic technique, based on the introduction of an artificial small parameter is used to solve the equations. An advantage of the procedure is the possibility to take into account a number of vibration modes. As examples, free longitudinal vibrations of a rod and lateral vibrations of a beam under cubically non-linear restoring force are considered. Resonance interactions between different modes are investigated and asymptotic formulae for corresponding backbone curves are derived.

1. INTRODUCTION

Non-linear vibrations of continuous structures (such as rods, beams, plates, shells, etc.) have been attracting many researchers. A significant peculiarity of these problems is the phenomena of internal resonances, arising in non-linear multi-degree-of-freedom structures when natural frequencies become commensurable with each other. In general, it causes the coupling of normal modes and results in multi-mode and multi-frequency response. Simple models that describe these vibrations can involve non-linear second and fourth order partial differential equations (PDE). The rigorous proof of the existence of periodic solutions was given by Rabinovitz [1]. His work inspired additional investigations, some of them are reviewed by Brézis [2]. In contrast to ordinary differential equations, construction of periodic solutions for PDEs can lead to the well-known problem of small denominators. The Kolmogorov–Arnold–Mozer theory was originally developed to overcome small denominator problems in celestial mechanics. Since then it has been extended for a wide range of multi-degree-of-freedom structures (see, for example, references [3,4]). Another approach, based on the Newton's method, was proposed by Bourgain [5]. His procedure may be used for obtaining periodic solutions for equations on spatial domains of arbitrary

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dimension as well as quasiperiodic solutions for equations on one- and two-dimensional domains. The Ritz, Galerkin and harmonic balance methods allow the dynamic behaviour of structures with large non-linearities to be predicted correctly, as was pointed out by Ling and Wu [6], Cheung and Lau [7] and by Lewandowski [8-10].

One of the most popular analytical approaches for studying non-linear structural vibrations are perturbation methods. A detailed review of the subject can be found in the monographs of Awrejcewicz *et al.* [11], Kevorkian and Cole [12], Nayfeh and Mook [13], and Troger and Steindl [14]. The methods have returned to the Lindstedt–Poincaré procedure, which was employed by earlier astronomers and laid the foundations of the modern perturbation theory. It allows the periodic vibrations to be determined directly.

In contrast to the classical Lindstedt–Poincaré technique, the method of multiple time scales can provide more general solutions (periodic as well as quasiperiodic), which are able to treat internal resonance phenomena and to investigate stability of motions (see papers of Lau *et al.* [15], Abel-Rohman [16], Ladygina and Manevitch [17], Boertjens and Van Horssen [18–20] Lacarbonara *et al.* [21], Nayfeh *et al.* [22] and Chin *et al.* [23]). Detailed investigations on comparing the multiple time scales procedure with low order Galerkin's method are presented in references [24, 25].

Another powerful asymptotic approach is the averaging method, and its strong theoretical bases relating to PDEs were given by Mitropolsky *et al.* [26].

The method of normal forms also allows the resonance phenomena in continuous structures to be described [27–29].

As a rule, perturbation methods are effectively used for studying weakly non-linear structures. Bender *et al.* [30–32] proposed an interesting asymptotic procedure which can give the possibility of taking strong non-linear effects into account. According to this approach, an artificial small parameter δ is introduced into the exponent of non-linear term (e.g., $u^3 \to u^{1+2\delta}$), so that δ represents a measure of non-linearity of the problem. Solution is sought as formal asymptotic expansions of δ. Numerical results are obtained by setting $\delta = 1$. This method was shown to be effective for ordinary non-linear differential equations. It is supposed that it could be successfully extended to non-linear PDEs [33, 34].

Continuous structures are infinite degree of freedom. In this case, use of known methods leads to infinite systems of non-linear algebraic or ordinary differential equations. For its solution truncation procedures are applied. Many studies have been restricted by considering only a few (usually two) mode representations. However, in some instances neglecting the subsequent modes without justification is not suitable and may even cause significant errors. For example, Boertjens and Van Horssen showed [18] that for bridges under non-linear drag and lift loads caused by wind flow at least four modes have to be taken into account for some values of restoring force.

In this paper, an asymptotic approach for deriving periodical solutions of non-linear vibration problems of continuous structures is proposed. An advantage of the procedure is the possibility of taking a number of vibration modes into account. As examples, free longitudinal vibrations of a rod (the second order PDE) and lateral vibrations of a beam (the fourth order PDE) under a cubically non-linear restoring force $\alpha u + \varepsilon u^3$ are considered. Here u is a displacement, $\alpha \geqslant 0$, $\varepsilon \ll 1$. Studying free vibrations provides one with the basic knowledge of the proper characteristics of a structure, and this is the starting point for investigations of more complicated dynamical problems, such as forced vibrations, dissipation effects, etc. Initially, the well-known perturbation technique is used and the independent variable u and frequency expanded in power series of the natural small parameter ε. It leads to infinite systems of the interconnected non-linear algebraic equations governing relationships between mode amplitudes and frequencies. For its solution, a non-trivial asymptotic approach is used, based on the introduction of an artificial small parameter [35, 36]. As the result, resonance interactions between different modes are investigated and the corresponding backbone curves are evaluated.

The outline of the paper is as follows. In section 2, the proposed asymptotic technique and the study of longitudinal vibrations of the rod is introduced. In the general case $\alpha \in (0; \infty)$, there are no interactions between modes with zero initial energy up to the $O(\varepsilon)$ approximation. But for specific values of the restoring force (when $\alpha = 0$), internal resonances occur in the $O(\varepsilon^0)$ approximation for an infinite number of all odd modes. In order to better understand the behaviour of the structure near the resonance, a detuning case $\alpha \to 0$, $\alpha \neq 0$ is also examined. In section 3, the developed method is used for investigating lateral vibrations of the beam. In the $O(\varepsilon^0)$ approximation, resonances can occur between two odd modes m, n if $\alpha = m^2 n^2$. However, simultaneous interactions of more than two modes are not possible. Here, resonance coupling of the first and third modes when $\alpha = 9$ is considered. The detuning case $\alpha \to 9$, $\alpha \neq 9$ is also studied. Section 4 provides some concluding remarks.

2. LONGITUDINAL VIBRATIONS OF THE ROD

2.1. THE ASYMPTOTIC PROCEDURE

Free longitudinal vibrations of a clamped–clamped rod in non-linear elastic medium are considered. Following standard textbooks [37–39], the governing equation can be written in the form

$$ESu_{,xx} = \rho S u_{,tt} + F(u), \tag{1}$$

where u is the longitudinal displacement, ρ is the density of the rod, S is the area of cross-section, E is Young's modulus, and $F(u)$ is the restoring force per unit length acting on the rod from the surrounding medium. The length of the rod equals l. Suppose that the non-linear force has a symmetric characteristic and can be expanded in a Taylor series with $F(0) = 0$: $F(u) = g_1 u + g_3 u^3 + g_5 u^5 + \cdots$. This expansion is restricted by two leading terms.

Let the dimensionless variables $\bar{x} = (\pi/l)x$, $\bar{u} = (l/S)u$, $\bar{t} = (\pi/l)(E/\rho)^{1/2} t$ be introduced Equation (1) becomes

$$\bar{u}_{,\bar{x}\bar{x}} = \bar{u}_{,\bar{t}\bar{t}} + \alpha \bar{u} + \varepsilon \bar{u}^3, \tag{2}$$

where $\alpha = g_1\, l^2/\pi^2 ES$, $\varepsilon = g_3\, S/\pi^2 E$. Restoring force is supposed to be weakly non-linear, so that $\varepsilon \ll 1$. The case of one potential well is studied, $\alpha \geqslant 0$. Next, for the simplicity, dashes in (2) are dropped and denote $\bar{u} = u$, $\bar{x} = x$, $\bar{t} = t$.

As an illustrative example consider the case of zero initial displacement, although the proposed method can be extended for different initial conditions.

Now the input boundary value problem can be formulated as follows:

$$u_{,xx} = u_{,tt} + \alpha u + \varepsilon u^3,$$

$$u(0,t) = u(\pi,t) = 0,$$

$$u(x,0) = 0, u_{,t}(x,0) = u^*(x). \tag{3}$$

In the linear case (for $\varepsilon = 0$), displacement u can be found as superposition

$$u^{lin} = \sum_{i=1}^{\infty} a_i \sin ix \sin \omega_i^{lin} t, \tag{4}$$

where $\omega_i^{lin} = \sqrt{i^2 + \alpha}$ are frequencies, a_i are amplitudes of modes. Next, stationary solutions of the non-linear problem (3) are sought. Let a change of the time scale be introduced.

$$\tau = \omega t \tag{5}$$

and represent the solution in the form of asymptotic expansions

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots, \tag{6}$$

$$\omega^2 = \omega_0^2 + \varepsilon \xi_1 + \varepsilon^2 \xi_2 + \cdots, \tag{7}$$

where $\omega_0 = \omega_1^{lin} = \sqrt{1 + \alpha}$ is the frequency of the normal mode in the linear case $\varepsilon = 0$. Substituting series (6), (7) into problem (3) and splitting it with respect to ε gives a recurrent system of linear PDEs:

$$u_{0_{,xx}} = \omega_0^2 u_{0_{,\tau\tau}} + \alpha u_0, \tag{8}$$

$$u_{1_{,xx}} - \omega_0^2 u_{1_{,\tau\tau}} - \alpha u_1 = \xi_1 u_{0_{,\tau\tau}} + u_0^3,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots \tag{9}$$

The boundary conditions can be rewritten as follows:

$$u_i(0,\tau) = u_i(\pi,\tau) = 0, \; i = 0, 1, 2, \cdots. \tag{10}$$

Solution of the boundary value problem (8), (10) yields the $O(\varepsilon^0)$ approximation:

$$u_0 = \sum_{i=1}^{\infty} a_i \sin ix \sin \frac{\omega_i^{lin}}{\omega_0} \tau = \sum_{i=1}^{\infty} a_i \sin ix \sin \Omega_i t. \tag{11}$$

In the linear case (for $\varepsilon = 0$) expression (11) coincides with representation (4). For non-linear vibrations (when $\varepsilon \neq 0$), unknown frequencies of the modes with respect to time t are

$$\Omega_i = \frac{\omega_i^{lin}}{\omega_0} \omega = \sqrt{\frac{i^2 + \alpha}{1 + \alpha}} \sqrt{(1 + \alpha) + \varepsilon \xi_1 + \varepsilon^2 \xi_2 + \cdots}, \; i = 1, 2, 3, \cdots. \tag{12}$$

The next $O(\varepsilon)$ approximation is derived from the boundary value problems (9)and (10). The terms containing $\sin ix \sin(\omega_i^{lin}/\omega_0)\tau$, $i = 1, 2, 3, \cdots$ in the right-hand side of equation (9) will produce secular terms, which should not be parts of the uniformly valid expansion (6). In order to eliminate the secular terms, coefficients of $\sin ix \sin(\omega_i^{lin}/\omega_0)\tau$ have to be zero. This condition leads to an infinite system of non-linear algebraic equations

$$a_i \xi_1 \frac{(i^2 + \alpha)}{(1 + \alpha)} = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} C_i^{(klm)} a_k a_l a_m, \; i = 1, 2, 3, \cdots. \tag{13}$$

Here coefficients $C_i^{(klm)}$ are evaluated after substituting expression (13) into the right-hand side of equation (9) and expanding it using the goniometric relation

$$\sin\beta\sin\gamma\sin\theta = \frac{1}{4}(\sin(\beta+\gamma-\theta)-\sin(\beta-\gamma-\theta)-\sin(\beta+\gamma+\theta)+\sin(\beta-\gamma+\theta)). \tag{14}$$

Solution of system (13) provides the next term ξ_1 of the frequency expansion (7) and allows relationships between the amplitudes a_i to be obtained. Having known ξ_1 and a_i the term u_1 can be determined from the boundary value problems (9) and (10). Later, the asymptotic procedure can be continued routinely for evaluating higher order approximations.

Infinite systems such as (13) may be obtained in various ways (e.g., by means of Galerkin method [6, 9, 24, 25], by multiple time scales technique [15–23] or by the averaging procedure [26]. Existence of non-trivial solutions in the case of internal resonance is shown in papers [1–5]. In practical problems, these systems are usually treated by truncation procedures, and many studies consider only a few mode representations. Meanwhile, in some instances, neglecting the subsequent modes leads to losing information of higher order internal resonances, which can produce significant errors in the solution. Due to this, solving system (13) should be started with a detailed investigation of probable mode interactions, and truncation can be allowed only for those modes which are not involved in the resonance coupling. In this paper, the asymptotic approach for analytical solution of the non-linear system (13) is proposed. This procedure gives the possibility of taking an arbitrary number of modes into account. The proposed technique is based on the introduction of an artificial small parameter [35, 36].

The artificial small parameter μ is introduced into the right-hand side of system (13) near the terms satisfying condition $(k>i)\cup(l>i)\cup(m>i)$. For example, consider three leading equations of system (13)

$$\begin{aligned} a_1\xi_1 &= C_1^{(111)}a_1^3 \\ &\quad + C_1^{(112)}a_1^2a_2 + C_1^{(122)}a_1a_2^2 + C_1^{(222)}a_2^3 \\ &\quad + C_1^{(113)}a_1^2a_3 + C_1^{(133)}a_1a_3^2 + C_1^{(123)}a_1a_2a_3 \\ &\quad + C_1^{(223)}a_2^2a_3 + C_1^{(233)}a_2a_3^2 + C_1^{(333)}a_3^3 + \cdots, \end{aligned} \tag{15}$$

$$\begin{aligned} a_2\xi_1\frac{(4+\alpha)}{(1+\alpha)} &= C_2^{(111)}a_2^3 \\ &\quad + C_2^{(112)}a_1^2a_2 + C_2^{(122)}a_1a_2^2 + C_2^{(222)}a_2^3 \\ &\quad + C_2^{(113)}a_1^2a_3 + C_2^{(133)}a_1a_3^2 + C_2^{(123)}a_1a_2a_3 \\ &\quad + C_2^{(223)}a_2^2a_3 + C_2^{(233)}a_2a_3^2 + C_2^{(333)}a_3^3 + \cdots, \end{aligned}$$

$$\begin{aligned} a_3\xi_1\frac{(9+\alpha)}{(1+\alpha)} &= C_3^{(111)}a_1^3 \\ &\quad + C_3^{(112)}a_1^2a_2 + C_3^{(122)}a_1a_2^2 + C_3^{(222)}a_2^3 \\ &\quad + C_3^{(113)}a_1^2a_3 + C_3^{(133)}a_1a_3^2 + C_3^{(123)}a_1a_2a_3 + C_3^{(223)}a_2^2a_3 \\ &\quad + C_3^{(233)}a_2a_3^2 + C_3^{(333)}a_3^3 + \cdots, \end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .$$

μ is introduced in the following way:

$$
\begin{aligned}
a_1\xi_1 = C_1^{(111)}a_1^3 \\
&+ \mu(C_1^{(112)}a_1^2a_2 + C_1^{(122)}a_1a_2^2 + C_1^{(222)}a_2^3 \\
&+ C_1^{(113)}a_1^2a_3 + C_1^{(133)}a_1a_3^2 + C_1^{(123)}a_1a_2a_3 + C_1^{(223)}a_2^2a_3 \\
&+ C_1^{(233)}a_2a_3^2 + C_1^{(333)}a_3^3 + \cdots),
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
a_2\xi_1\frac{(4+\alpha)}{(1+\alpha)} = C_2^{(111)}a_2^3 \\
&+ C_2^{(112)}a_1^2a_2 + C_2^{(122)}a_1a_2^2 + C_2^{(222)}a_2^3 \\
&+ \mu(C_2^{(113)}a_1^2a_3 + C_2^{(133)}a_1a_3^2 + C_2^{(123)}a_1a_2a_3 \\
&+ C_2^{(223)}a_2^2a_3 + C_2^{(233)}a_2a_3^2 + C_2^{(333)}a_3^3 + \cdots),
\end{aligned}
$$

$$
\begin{aligned}
a_3\frac{(9+\alpha)}{(1+\alpha)} = C_3^{(111)}a_1^3 \\
&+ C_3^{(112)}a_1^2a_2 + C_3^{(122)}a_1a_2^2 + C_3^{(222)}a_2^3 \\
&+ C_3^{(113)}a_1^2a_3 + C_3^{(133)}a_1a_3^2 + C_3^{(123)}a_1a_2a_3 \\
&+ C_3^{(223)}a_2^2a_3 + C_3^{(233)}a_2a_3^2 + C_3^{(333)}a_3^3 + \mu(\cdots), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .
\end{aligned}
$$

This perturbation is such that at $\mu = 0$ system (16) turns into a triangular form and is reduced to a recurrent equations sequence, and at $\mu = 1$ it restores to the original form (15).

Here, ξ_1 can be considered as a kind of eigenvalue, which is associated with the specific set of eigensolutions a_i. Perturbed eigensolutions of system (16) are sought near the given unperturbed eigensolutions. Unperturbed eigenvalue $\xi_1^{(0)}$ can be obtained from the first equation of system (16) when $\mu = 0$:

$$\xi_1^{(0)} = C_1^{(111)}a_1^2. \tag{17}$$

The physical sense of solution (17) is that interconnections between different modes are neglected and vibrations in only one fundamental mode with amplitude a_1 are considered. The following formal asymptotic expansions in powers of μ starting from the unperturbed eigensolution are proposed.

$$\xi_1 = \xi_1^{(0)} + \mu\xi_1^{(1)} + \mu^2\xi_1^{(2)} + \cdots, \tag{18}$$

$$a_j = a_j^{(0)} + \mu a_j^{(1)} + \mu^2 a_j^{(2)} + \cdots, j = 2, 3, 4 \cdots. \tag{19}$$

Starting values of amplitudes $a_j^{(0)}$ are calculated by substituting expression (17) into the subsequent (second, third and so on) equations of system (16) at $\mu = 0$. In the subsequent approximations by μ, representations (18), (19) allow modes resonance interactions to be taken into account and the eigenvalue ξ_1 to be defined. Calculating coefficients in expansions (18), (19) it is supposed finally that $\mu = 1$.

2.2. GENERAL CASE: $\alpha \in (0; \infty)$

Now an investigation of system (13) is considered. On the basis of expression (11) and relation (14) one could determine that only some terms $a_k a_l a_m$ with specific combinations of k, l, m contribute to the right-hand side of system (13). So,

$$C_i^{(klm)} \neq 0 \quad \text{if} \begin{cases} \pm k \pm l \pm m = \pm i, \\ \pm\sqrt{k^2+\alpha} \pm \sqrt{l^2+\alpha} \pm \sqrt{m^2+\alpha} = \pm\sqrt{i^2+\alpha}. \end{cases} \tag{20}$$

In general case $\alpha \in (0; \infty)$ system (13) can be written as follows:

$$a_i \xi_1 \frac{(i^2+\alpha)}{(1+\alpha)} = \frac{9}{16} a_i^3 + \frac{3}{4} a_i \left(\sum_{k=1}^{i-1} a_k^2 + \sum_{k=i+1}^{\infty} a_k^2 \right), \; i = 1, 2, 3, \cdots. \tag{21}$$

Infinite algebraic systems with cubic non-linearity such as (13) may have three non-trivial solutions, which describe different resonance interactions between modes. However, in the case under consideration $\alpha \in (0; \infty)$, simple numerical analysis can show that system (21) does not have real solutions describing resonance interactions: all three non-trivial solutions are imaginary. If real roots are sought by means of the approach of artificial small parameter, then it would be found that expansion (19) diverges rapidly. The only possible solutions are:

$$\xi_1 = \frac{9}{16} \frac{(1+\alpha)}{(i^2+\alpha)} a_i^2, \; i = 1, 2, 3, \cdots;$$
$$a_j = 0, \; j = 1, 2, 3, \cdots, j \neq i. \tag{22}$$

Relations (22) correspond to the case when in the $O(\varepsilon^0)$ approximation the rod is able to vibrate in only one ith mode ($i = 1,2,3,\cdots$). These vibrations are periodical with frequency

$$\Omega_i = \sqrt{i^2+\alpha} \left(1 + \frac{9}{32} \frac{(a_i^2)}{(i^2+\alpha)} \varepsilon \right) + O(\varepsilon^2). \tag{23}$$

The amplitude of the excited mode a_i can be evaluated from initial conditions:

$$a_i = \sqrt{\frac{2}{\pi \Omega_i^2} \int_0^{\pi} u^*(x)^2 \, dx}. \tag{24}$$

Amplitudes of all other modes equal zero: $a_j = 0$, $j = 1, 2, 3, \cdots$, $j \neq i$. In the $O(\varepsilon^0)$ approximation more than one mode cannot be excited simultaneously. The structure could be reduced to a one-degree-of-freedom oscillator [40]. In this case, mode interactions may take place only between modes with non-zero initial energy (up to $O(\varepsilon)$). So, if one starts with zero initial energy in the jth mode there will be no energy present up to $O(\varepsilon)$. This allows truncation to the modes with non-zero initial energy.

2.3. INTERNAL RESONANCES: $\alpha = 0$

When the linear part of restoring force is absent ($\alpha = 0$), extra contributions to the right-hand side of system (13) occur. System (13) has the form

$$a_1 \xi_1 = \frac{9}{16} a_1^3 + \frac{3}{4} a_1 (a_2^2 + a_3^2 + a_4^2 + a_5^2) + \frac{3}{8} (a_1 a_2 a_4 + a_2 a_3 a_4 + a_1 a_3 a_5 + a_2 a_4 a_5)$$
$$+ \frac{3}{16} (a_1^2 a_3 + a_2^2 a_3 + a_2^2 a_5 + a_3^2 a_5) + \cdots,$$

$$4a_2\xi_1 = \frac{9}{16}a_2^3 + \frac{3}{4}a_2(a_1^2 + a_3^2 + a_4^2 + a_5^2) + \frac{3}{8}(a_1a_2a_3 + a_1a_3a_4 + a_1a_2a_5 + a_1a_4a_5$$

$$+ a_3a_4a_5) + \frac{3}{16}(a_1^2a_4 + a_3^2a_4) + \cdots,$$

$$9a_3\xi_1 = \frac{1}{16}a_1^3 + \frac{9}{16}a_3^3 + \frac{3}{4}a_3(a_1^2 + a_2^2 + a_4^2 + a_5^2) + \frac{3}{8}(a_1a_2a_4 + a_2a_3a_4$$

$$+ a_1a_3a_5 + a_2a_4a_5) + \frac{3}{16}(a_1a_2^2 + a_1^2a_5 + a_4^2a_5) + \cdots, \tag{25}$$

$$16a_4\xi_1 = \frac{9}{16}a_4^3 + \frac{3}{4}a_4(a_1^2 + a_2^2 + a_3^2 + a_5^2) + \frac{3}{8}(a_1a_2a_3 + a_1a_2a_5 + a_2a_3a_5 + a_3a_4a_5)$$

$$+ \frac{3}{16}(a_2a_1^2 + a_2a_3^2) + \cdots,$$

$$25a_5\xi_1 = \frac{9}{16}a_5^3 + \frac{3}{4}a_5(a_1^2 + a_2^2 + a_3^2 + a_4^2) + \frac{3}{8}(a_1a_2a_4 + a_2a_3a_4)$$

$$+ \frac{3}{16}(a_1a_2^2 + a_1^2a_3 + a_3^2a_1 + a_3a_4^2) + \cdots,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .$$

Note that expressions (22) satisfy system (25). Besides this, system (25) may have three non-trivial solutions describing resonance interactions. In the case under consideration ($\alpha = 0$) two of them are imaginary, and only one solution is real and has got physical sense. For its evaluation an artificial small parameter μ is introduced in the following way:

$$a_1\xi_1 = \frac{9}{16}a_1^3 + \mu\left(\frac{3}{4}a_1(a_2^2 + a_3^2 + a_4^2 + a_5^2) + \frac{3}{8}(a_1a_2a_4 + a_2a_3a_4 + a_1a_3a_5 + a_2a_4a_5)\right.$$

$$\left. + \frac{3}{16}(a_1^2a_3 + a_2^2a_3 + a_2^2a_5 + a_3^2a_5) + \cdots\right),$$

$$4a_2\xi_1 = \frac{9}{16}a_2^3 + \frac{3}{4}a_1^2a_2 + \mu\left(\frac{3}{4}a_2(a_3^2 + a_4^2 + a_5^2) + \frac{3}{8}(a_1a_2a_3 + a_1a_3a_4 + a_1a_2a_5\right.$$

$$\left. + a_1a_4a_5 + a_3a_4a_5) + \frac{3}{16}(a_1^2a_4 + a_3^2a_4) + \cdots\right),$$

$$9a_3\xi_1 = \frac{1}{16}a_1^3 + \frac{9}{16}a_3^3 + \frac{3}{4}a_3(a_1^2 + a_2^2) + \frac{3}{16}a_1a_2^2 + \mu\left(\frac{3}{4}a_3(a_4^2 + a_5^2)\right.$$

$$\left. + \frac{3}{8}(a_1a_2a_4 + a_2a_3a_4 + a_1a_3a_5 + a_2a_4a_5) + \frac{3}{16}(a_1^2a_5 + a_4^2a_5) + \cdots\right), \tag{26}$$

TABLE 1

Comparison of asymptotic solution with numerical data

Variable	Asymptotic solution (27)	Numerical results
ξ_1	$0{\cdot}565376a_1^2$	$0{\cdot}565360a_1^2$
a_2	0	0
a_3	$a_3 = 1{\cdot}449 \times 10^{-2}a_1$	$a_3 = 1{\cdot}442 \times 10^{-2}a_1$
a_4	0	0
a_3	$a_5 = 2{\cdot}071 \times 10^{-4}a_1$	$a_5 = 2.049 \times 10^{-4}a_1$

$$16a_4\xi_1 = \frac{9}{16}a_4^3 + \frac{3}{4}a_4(a_1^2 + a_2^2 + a_3^2) + \frac{3}{8}a_1a_2a_3 + \frac{3}{16}(a_2a_1^2 + a_2a_3^2)$$
$$+ \mu\left(\frac{3}{4}a_4a_5^2 + \frac{3}{8}(a_1a_2a_5 + a_2a_3a_5 + a_3a_4a_5) + \cdots\right),$$
$$25a_5\xi_1 = \frac{9}{16}a_5^3 + \frac{3}{4}a_5(a_1^2 + a_2^2 + a_3^2 + a_4^2) + \frac{3}{8}(a_1a_2a_4 + a_2a_3a_4)$$
$$+ \frac{3}{16}(a_1a_2^2 + a_1^2a_3 + a_3^2a_1 + a_3a_4^2) + \cdots,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .$$

The solution is sought as asymptotic series (18), (19). Being restricted by two leading terms in expansion (18), one obtains

$$a_j = 0, j = 2,4,6,\cdots;\ a_3 = 1{\cdot}449 \times 10^{-2}\,a_1,\ a_5 = 2{\cdot}071 \times 10^{-4}\,a_1, \cdots; \tag{27}$$
$$\xi_1 = \frac{9}{16}a_1^2 + \frac{3}{4}(a_3^2 + a_5^2) + \frac{3}{8}a_3a_5 + \frac{3}{16}a_1a_3 + \frac{3}{16}\frac{a_3^2a_5}{a_1} + \cdots \approx 0{\cdot}565376a_1^2.$$

Here, the approach of an artificial small parameter yields very accurate results. In Table 1, expressions (27) are compared with numerical solutions of the non-linear system (25). The numerical data were calculated in the *Mathematica* program package by truncating system (25) to five leading equations.

According to expressions (27), in the $O(\varepsilon^0)$ approximation an infinite number of all odd modes are involved in resonance interactions. This provides energy transfers between odd modes, and the truncation may not be valid. The physical meaning of this phenomenon is that if the rod initially vibrates in a high mode, then low modes can be excited. This can lead to large-amplitude oscillations of the structure. Taking into account relations (27), mode frequencies can be expressed as functions of the fundamental amplitude a_1:

$$\Omega_i = i(1 + 0{\cdot}282688a_1^2\varepsilon) + O(\varepsilon^2), i = 1, 3, 5, \cdots \tag{28}$$

where a_1 is determined from the initial conditions

$$a_1 = \sqrt{\frac{2}{\pi\Omega_1^2}\int_0^\pi u^*(x)^2\,\mathrm{d}x - 9a_3^2 - 25a_5^2 - \cdots} \approx \sqrt{\frac{2}{1{\cdot}001891\pi\Omega_1^2}\int_0^\pi u^*(x)^2\,\mathrm{d}x}. \tag{29}$$

It should be pointed out that for $\alpha = 0$ the ratio of frequencies of interacting modes equals the ratio of their wave numbers:

$$\frac{\Omega_m}{\Omega_n} = \frac{m}{n}, m, n = 1, 3, 5, \cdots. \tag{30}$$

Therefore frequencies of all excited modes are commensurable with each other. In this case the $O(\varepsilon^0)$ approximation formula (11) describes periodical vibrations with the general period $T = 2\pi/\Omega_1$.

2.4. THE DETUNING CASE: $\alpha \to 0$, $\alpha \neq 0$

The behaviour of the rod near the resonance is considered below. In order to introduce detuning it is supposed that $\alpha \to 0$, but $\alpha \neq 0$. Here, the parameter α shows how far the structure is from the pure resonance state. Changing the scale of time (5), the solution of the input boundary value problem (3) as asymptotic expansions by powers of α is as follows:

$$u = u^{(0)} + \alpha u^{(1)} + \alpha^2 u^{(2)} + \cdots, \tag{31}$$

$$\omega = \xi^{(0)} + \alpha\xi^{(1)} + \alpha^2\xi^{(2)} + \cdots. \tag{32}$$

Here, each term is represented by a series

$$u^{(n)} = u^{(n,0)} + \varepsilon u^{(n,1)} + \varepsilon^2 u^{(n,2)} + \cdots, \tag{33}$$

$$\xi^{(n)} = \xi^{(n,0)} + \varepsilon\xi^{(n,1)} + \varepsilon^2\xi^{(n,2)} + \cdots, n = 0, 1, 2, \cdots, \tag{34}$$

where $\xi^{(0,0)} = 1$. Splitting problem (3) with respect to α and ε gives the recurrent sequence of equations

$$u^{(0,0)}_{,xx} - u^{(0,0)}_{,\tau\tau} = 0, \tag{35}$$

$$u^{(0,1)}_{,xx} - u^{(0,1)}_{,\tau\tau} = 2\xi^{(0,1)}u^{(0,0)}_{,\tau\tau} + (u^{(0,0)})^3, \tag{36}$$

$$u^{(1,0)}_{,xx} - u^{(1,0)}_{,\tau\tau} = 2\xi^{(1,0)}u^{(0,0)}_{,\tau\tau} + u^{(0,0)}, \tag{37}$$

$$u^{(1,1)}_{,xx} - u^{(1,1)}_{,\tau\tau} = 2\xi^{(1,0)}u^{(0,1)}_{,\tau\tau} + u^{(0,1)} + 2(\xi^{(0,1)}\xi^{(1,0)} + \xi^{(1,1)})u^{(0,0)}_{,\tau\tau}$$
$$+ 2\xi^{(0,1)}u^{(1,0)}_{,\tau\tau} + 3(u^{(0,0)})^2u^{(1,0)}, \tag{38}$$

. . . .

with boundary conditions

$$u^{(n,m)}(0,\tau) = u^{(n,m)}(\pi, \tau), m, n = 0, 1, 2, \cdots. \tag{39}$$

Solution of the boundary value problems (35) and (39) provides

$$u^{(0,0)} = \sum_{i=1}^{\infty} \bar{a}_i \sin ix \sin i\tau. \tag{40}$$

The next approximation $u^{(0,1)}$ is evaluated from the boundary value problems (36)and (39). In order to prevent secular terms in expansion (33), coefficients of $\sin ix \sin i\tau$ in the right-hand side of equation (36) have to be equated to zero. This condition leads to an infinite system of non-linear algebraic equations

$$2\bar{a}_i\xi^{(0,1)}i^2 = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} C_i^{(klm)}\bar{a}_k\bar{a}_l\bar{a}_m,\ i = 1, 2, 3, \cdots, \tag{41}$$

the right-hand side of which is identical to the right-hand side of system (25). System (41) gives the solution

$$\bar{a}_j = 0, j = 2, 4, 6, \cdots;\ \bar{a}_3 = 1{\cdot}449\times 10^{-2}\bar{a}_1,\ \bar{a}_5 = 2{\cdot}071\times 10^{-4}\,\bar{a}_1, \cdots; \tag{42}$$

$$\xi^{(0,1)} = \frac{9}{32}\bar{a}_1^2 + \frac{3}{8}(\bar{a}_3^2 + \bar{a}_5^2) + \frac{3}{16}\bar{a}_3\bar{a}_5 + \frac{3}{32}\bar{a}_1\bar{a}_3 + \frac{3}{32}\frac{\bar{a}_3^2\bar{a}_5}{\bar{a}_1} + \cdots \approx 0{\cdot}282688\bar{a}_1^2.$$

Function $u^{(0,1)}$ can be represented as the harmonic expansion by τ:

$$u^{(0,1)} = \sum_{i=1}^{\infty} f_i(x)(d_i^{(1)}\sin i\tau + d_i^{(2)}\cos i\tau), \tag{43}$$

where $f_i(x)$, $d_i^{(1)}$, $d_i^{(2)}$ are some functions and coefficients. The term $u^{(1,0)}$ is determined from the boundary value problems (37) and (39). Eliminating secular terms in expansion (31), the right-hand side of equation (37) yields

$$\xi^{(1,0)} = \frac{1}{2i^2}, \tag{44}$$

where i is the mode number. Then

$$u^{(1,0)} = \sum_{i=1}^{\infty} \bar{b}_i \sin ix\,\sin i\tau. \tag{45}$$

The boundary value problems (38) and (39) allow $u^{(1,1)}$ to be evaluated. Satisfying the condition of absence of secular terms in expansion (33), the right-hand side of equation (38) gives an infinite linear system for coefficients $\bar{b}_i$ (see Appendix A). It provides $\bar{b}_i$ and $\xi^{(1,1)}$:

$$\bar{b}_j = 0, j = 2, 4, 6, \cdots;\ \bar{b}_3 = 1{\cdot}443\times 10^{-2}\bar{b}_1, \bar{b}_5 = 2{\cdot}004\times 10^{-4}\bar{b}_1, \cdots; \tag{46}$$

$$\xi^{(1,1)} \approx 0{\cdot}565352\bar{a}_1\bar{b}_1 - 0{\cdot}141344\frac{\bar{a}_1^2}{i^2}.$$

Even in the pure resonance case $\alpha = 0$ all odd modes take part in resonance interactions in the $O(\varepsilon^0)$ approximation. From expressions (42), (44) and (46), the asymptotic formula is given for unknown frequencies as:

$$\begin{aligned}\Omega_i = \sqrt{i^2+\alpha}\left(1 + 0{\cdot}282688\frac{\bar{a}_1^2 i^2}{i^2+\alpha}\varepsilon\right) + 0{\cdot}565352 i\bar{a}_1\bar{b}_1\varepsilon\alpha \\ - 0.141344\frac{\bar{a}_1^2}{i^2}\varepsilon\alpha + o(\varepsilon) + o(\alpha) + o(\varepsilon\alpha), i = 1, 3, 5\cdots.\end{aligned} \tag{47}$$

The mode amplitudes being equal to $a_i = \bar{a}_i + \alpha\bar{b}_i$, a_1 is evaluated by formula (29). For simplicity it can be assumed that $\bar{b}_i = 0$, $a_i = \bar{a}_i$. The solution correctly represents the behaviour of the structure in "limiting" cases. So, for $\varepsilon = 0$ formula (47) is in agreement with relations (23), and for $\alpha = 0$ formula (47) coincides with expression (28). It should be noted that in the detuning case the ratio of frequencies of interacting modes differs from the ratio of their wave numbers:

$$\frac{\Omega_m}{\Omega_n} = \frac{m}{n} + O(\alpha), m,n = 1, 3, 5, \cdots. \tag{48}$$

Actually, in this case the ratios of the frequencies (48) may be irrational numbers. The general solution (11) can then describe quasiperiodical motions.

3. LATERAL VIBRATIONS OF THE BEAM

In this section, the asymptotic approach is extended to study the solutions of the fourth order PDE. Free vibrations of a simply supported beam on a non-linear elastic foundation are considered. The governing equation can be written as follows [37–39]:

$$EIw_{,xxxx} - \frac{ES}{2l} w_{,xx} \int_0^l w_{,x}^2 \, dx + \rho S w_{,tt} + F(\mathrm{w}) = 0, \tag{49}$$

where w is the lateral displacement, and I is the moment of inertia of the cross section. As in the previous example, the non-linear restoring force per unit length is assumed to be in the form $F(w) = g_1 w + g_3 w^3$.

The term $(ES/2l)\, w_{,xx} \int_0^l w_{,x}^2 \, dx$ in equation (49) represents the so-called geometrical non-linearity of the beam and describes the influence of a dynamical axial force. If the ends of the beam are immovable, the axial force appears due to a change of the beam length during bending. However, it is well known [37–39] that in many practical problems when the displacements w are essentially smaller than the typical size of the cross-section, the term $(ES/2l)\, w_{,xx} \int_0^l w_{,x}^2 \, dx$ can be neglected when comparing it with the first term of equation (49) $EIw_{,xxxx}$. Here, this approximation and a geometrically linear problem are assumed without taking into account axial force. Vibrations of a geometrically non-linear beam were studied Lewandowski [9].

Introducing the dimensionless variables $\bar{x} = (\pi/l)x$, $\bar{w} = (l/S)w$, $\bar{t} = (\pi^2/l^2)\,(EI/\rho S)^{1/2} t$, equation (49) becomes

$$\bar{w}_{,\bar{x}\bar{x}\bar{x}\bar{x}} + w_{,\bar{t}\bar{t}} + \alpha\bar{w} + \varepsilon\bar{w}^3 = 0, \tag{50}$$

where $\alpha = g_1\, l^4/\pi^4 EI$, $\varepsilon = g_3\, l^2 S^2/\pi^4 EI$, if $\varepsilon \ll 1$ and $\alpha \geqslant 0$. Next these dashes in equation (50) are omitted and $\bar{u} = u$, $\bar{x} = x$, $\bar{t} = t$. The case of zero initial displacement is considered.

The input boundary value problem is formulated as follows:

$$w_{,xxxx} + w_{,tt} + \alpha w + \varepsilon w^3 = 0, \tag{51}$$

$$w(0,t) = w(\pi,t) = 0,$$

$$w(x,0) = 0, \; w_{,t}(x,0) = w^*(x).$$

For solving problem (51) the asymptotic technique, introduced in the previous section is used. All intermediate evaluations remain the same, and only final results are displayed below. In the $O(\varepsilon^0)$ approximation

$$w = \sum_{i=1}^{\infty} a_i \sin \Omega_i t \sin ix + O(\varepsilon). \tag{52}$$

In the general case for $\alpha = [0; \infty)$, $\alpha \neq m^2 n^2$, $m,n = 1,3,5,\cdots$ there are no mode interactions up to $O(\varepsilon)$. Frequencies Ω_i are determined as follows:

$$\Omega_i = \sqrt{i^4 + \alpha}\left(1 + \frac{9}{32}\frac{a_i^2}{i^4 + \alpha}\varepsilon\right) + O(\varepsilon^2),\ i = 1, 2, 3, \cdots, \tag{53}$$

where amplitudes are

$$a_i = \sqrt{\frac{2}{\pi \Omega_{\mathrm{i}}^2}\int_0^\pi w^*(x)^2 \mathrm{d}x}. \tag{54}$$

The structure can be truncated to the modes with non-zero initial energy.

Resonance coupling in the $O(\varepsilon^0)$ approximation can occur only for two odd modes m and n if

$$\frac{m}{n} = \sqrt{\frac{m^4 + \alpha}{n^4 + \alpha}} = \frac{\Omega_{\mathrm{m}}}{\Omega_{\mathrm{n}}},\ m,n = 1, 3, 5, \cdots. \tag{55}$$

Equation (55) determines the specific values of α which allow interactions and energy transfers between the mth and nth modes: $\alpha = m^2 n^2$. In this case, independent of the distribution of the initial energy, both the mth and nth modes can be excited. For example, when $\alpha = 9$, there is a coupling of the first and third modes, and

$$a_3 = 1{\cdot}449 \times 10^{-2} a_1, \tag{56}$$

$$\Omega_i = i\sqrt{1 + \alpha}\left(1 + \left(\frac{9}{32}a_1^2 + \frac{3}{32}a_1 a_3 + \frac{3}{8}a_3^2\right)\frac{\varepsilon}{1 + \alpha}\right) + O(\varepsilon^2),\ i = 1,3.$$

The asymptotic relations (56) may be rewritten in the form

$$\Omega_i = i\sqrt{10}(1 + 0{\cdot}0282679 a_1^2 \varepsilon) + O(\varepsilon^2),\ i = 1,3, \tag{57}$$

where the fundamental amplitude a_1 is evaluated from the initial conditions:

$$a_1 = \sqrt{\frac{2}{\pi \Omega_1^2}\int_0^\pi w^*(x)^2\,\mathrm{d}x - 9a_3^2} = \sqrt{\frac{2}{1{\cdot}001869\pi\Omega_1^2}\int_0^\pi w^*(x)^2\,\mathrm{d}x}. \tag{58}$$

In the detuning case $\alpha \to 9$, $\alpha \neq 9$, condition (55) is not satisfied strictly. The frequencies Ω_1, Ω_3 are determined as follows:

$$\begin{aligned}\Omega_i = i\Bigg(&\sqrt{1 + \alpha} + \frac{\sqrt{1 + \alpha}}{2(i^4 + \alpha)}(\alpha - 9) + 0{\cdot}282679\frac{\bar{a}_1^2}{\sqrt{1 + \alpha}}\varepsilon + 0{\cdot}565357\frac{\bar{a}_1 \bar{b}_1}{\sqrt{1 + \alpha}}(\alpha - 9)\varepsilon \\ &- 0{\cdot}141339\frac{\bar{a}_1^2}{\sqrt{1 + \alpha}(i^4 + \alpha)}(\alpha - 9)\varepsilon\Bigg) + o(\varepsilon) + o(\alpha - 9) + o(\varepsilon(\alpha - 9)),\ i = 1,3.\end{aligned} \tag{59}$$

The mode amplitudes are $a_i = \bar{a}_i + (\alpha - 9)\bar{b}_i$; here, $\bar{a}_3 = 1{\cdot}441 \times 10^{-2}\bar{a}_1$, $\bar{b}_3 = 1{\cdot}441 \times 10^{-2}\bar{b}_1$. One can suppose that $\bar{b}_i = 0$, $a_i = \bar{a}_i$. a_1 is given by expression (58). Equation (59) correctly represents "limiting" cases: for $\varepsilon = 0$ it corresponds to relation (53) and for $\alpha = 9$ it takes the form of equation (57).

4. CONCLUSIONS

In this paper, an asymptotic approach for periodical solutions of non-linear vibration problems of continuous structures is proposed. Using perturbation technique, infinite non-linear algebraic systems for mode frequencies and amplitudes were obtained. To solve this, the asymptotic approach of artificial small parameter was introduced. The procedure allows one to take into account a number of modes. Longitudinal vibrations of the rod and lateral vibrations of the beam under non-linear restoring force were studied. Internal resonance interactions between different modes were considered and asymptotic formulae for the corresponding backbone curves have been derived. The results of interest are that for specific values of parameter α energy transfers among resonance modes can lead to exciting modes of zero initial energy. From the practical point of view, this means that a small external high-frequency excitation in a high mode can cause a large low-frequency response of the structure in a low mode.

Formal asymptotic approximations were constructed. Rigorous estimations of asymptotic convergence for the perturbation technique are given by Mitropolsky *et al.* [26]. The accuracy of the approach of artificial small parameter was verified by comparison with numerical data.

The procedure developed can be used effectively in conjunction with other analytical methods. Thus, its implantation together with multi-time scale techniques would allow the study of quasiperiodic motions and forced vibrations. Different types of non-linearity can be taken into account (e.g., Rayleigh perturbation [18] $\varepsilon(w_{,t} - (1/3)\, w_{,t}^3)$, non-linear boundary conditions, etc.). The stability analysis can also be performed. Extension to structures of higher order spatial dimensions is possible.

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APPENDIX A

Infinite linear algebraic system for coefficients $\bar{b}_i$ and $\xi^{(1,1)}$is:

$$\begin{aligned}
&\frac{2}{3}(\xi^{(0,1)}\xi^{(1,0)}\bar{a}_1 + \xi^{(1,1)}\bar{a}_1 + \xi^{(0,1)}\bar{b}_1) \\
&\quad = \left(\frac{9}{16}\bar{a}_1^2 + \frac{1}{4}(\bar{a}_2^2 + \bar{a}_3^2 + \bar{a}_4^2 + \bar{a}_5^2) + \frac{1}{8}(\bar{a}_1\bar{a}_3 + \bar{a}_2\bar{a}_4 + \bar{a}_3\bar{a}_5)\right)\bar{b}_1 \\
&\qquad + \left(\frac{1}{2}\bar{a}_1\bar{a}_2 + \frac{1}{8}(\bar{a}_2\bar{a}_3 + \bar{a}_1\bar{a}_4 + \bar{a}_3\bar{a}_4 + \bar{a}_2\bar{a}_5 + \bar{a}_4\bar{a}_5)\right)\bar{b}_2 \\
&\qquad + \left(\frac{1}{16}(\bar{a}_1^2 + \bar{a}_2^2) + \frac{1}{2}\bar{a}_1\bar{a}_3 + \frac{1}{8}(\bar{a}_2\bar{a}_4 + \bar{a}_1\bar{a}_5 + \bar{a}_3\bar{a}_5)\right)\bar{b}_3 \\
&\qquad + \left(\frac{1}{2}\bar{a}_1\bar{a}_4 + \frac{1}{8}(\bar{a}_1\bar{a}_2 + \bar{a}_2\bar{a}_3 + \bar{a}_2\bar{a}_5)\right)\bar{b}_4 \\
&\qquad + \left(\frac{1}{16}(\bar{a}_2^2 + \bar{a}_3^2) + \frac{1}{2}\bar{a}_1\bar{a}_5 + \frac{1}{8}(\bar{a}_1\bar{a}_3 + \bar{a}_2\bar{a}_4)\right)\bar{b}_5 + \cdots, \\
&\frac{8}{3}(\xi^{(0,1)}\xi^{(1,0)}\bar{a}_2 + \xi^{(1,1)}\bar{a}_2 + \xi^{(0,1)}\bar{b}_2) \\
&\quad = \left(\frac{1}{2}\bar{a}_1\bar{a}_2 + \frac{1}{8}(\bar{a}_2\bar{a}_3 + \bar{a}_1\bar{a}_4 + \bar{a}_3\bar{a}_4 + \bar{a}_2\bar{a}_5 + \bar{a}_4\bar{a}_5)\right)\bar{b}_1 \\
&\qquad + \left(\frac{9}{16}\bar{a}_2^2 + \frac{1}{4}(\bar{a}_1^2 + \bar{a}_3^2 + \bar{a}_4^2 + \bar{a}_5^2) + \frac{1}{8}(\bar{a}_1\bar{a}_3 + \bar{a}_1\bar{a}_5)\right)\bar{b}_2 \\
&\qquad + \left(\frac{1}{2}\bar{a}_2\bar{a}_3 + \frac{1}{8}(\bar{a}_1\bar{a}_2 + \bar{a}_1\bar{a}_4 + \bar{a}_3\bar{a}_4 + \bar{a}_4\bar{a}_5)\right)\bar{b}_3 \\
&\qquad + \left(\frac{1}{16}(\bar{a}_1^2 + \bar{a}_3^2) + \frac{1}{2}\bar{a}_2\bar{a}_4 + \frac{1}{8}(\bar{a}_1\bar{a}_3 + \bar{a}_1\bar{a}_5 + \bar{a}_3\bar{a}_5)\right)\bar{b}_4 \\
&\qquad + \left(\frac{1}{2}\bar{a}_2\bar{a}_5 + \frac{1}{8}(\bar{a}_1\bar{a}_2 + \bar{a}_1\bar{a}_4 + \bar{a}_3\bar{a}_4)\right)\bar{b}_5 + \cdots,
\end{aligned} \tag{A1}$$

$$
\begin{aligned}
&6(\xi^{(0,1)}\xi^{(1,0)}\bar{a}_3 + \xi^{(1,1)}\bar{a}_3 + \xi^{(0,1)}\bar{b}_3) \\
&\quad = \left(\frac{1}{16}(\bar{a}_1^2 + \bar{a}_2^2) + \frac{1}{2}\bar{a}_1\bar{a}_3 + \frac{1}{8}(\bar{a}_2\bar{a}_4 + \bar{a}_1\bar{a}_5 + \bar{a}_3\bar{a}_5)\right)\bar{b}_1 \\
&\qquad + \left(\frac{1}{2}\bar{a}_2\bar{a}_3 + \frac{1}{8}(\bar{a}_1\bar{a}_2 + \bar{a}_1\bar{a}_4 + \bar{a}_3\bar{a}_4 + \bar{a}_4\bar{a}_5)\right)\bar{b}_2 \\
&\qquad + \left(\frac{9}{16}\bar{a}_3^2 + \frac{1}{4}(\bar{a}_1^2 + \bar{a}_2^2 + \bar{a}_4^2 + \bar{a}_5^2) + \frac{1}{8}(\bar{a}_2\bar{a}_4 + \bar{a}_1\bar{a}_5)\right)\bar{b}_3 \\
&\qquad + \left(\frac{1}{2}\bar{a}_3\bar{a}_4 + \frac{1}{8}(\bar{a}_1\bar{a}_2 + \bar{a}_2\bar{a}_3 + \bar{a}_2\bar{a}_5 + \bar{a}_4\bar{a}_5)\right)\bar{b}_4 \\
&\qquad + \left(\frac{1}{16}(\bar{a}_1^2 + \bar{a}_4^2) + \frac{1}{2}\bar{a}_3\bar{a}_5 + \frac{1}{8}(\bar{a}_1\bar{a}_3 + \bar{a}_2\bar{a}_4)\right)\bar{b}_5 + \cdots,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{32}{3}(\xi^{(0,1)}\xi^{(1,0)}\bar{a}_4 + \xi^{(1,1)}\bar{a}_4 + \xi^{(0,1)}\bar{b}_4) \\
&\quad = \left(\frac{1}{2}\bar{a}_1\bar{a}_4 + \frac{1}{8}(\bar{a}_1\bar{a}_2 + \bar{a}_2\bar{a}_3 + \bar{a}_2\bar{a}_5)\right)\bar{b}_1 \\
&\qquad + \left(\frac{1}{16}(\bar{a}_1^2 + \bar{a}_3^2) + \frac{1}{2}\bar{a}_2\bar{a}_4 + \frac{1}{8}(\bar{a}_1\bar{a}_3 + \bar{a}_1\bar{a}_5 + \bar{a}_3\bar{a}_5)\right)\bar{b}_2 \\
&\qquad + \left(\frac{1}{2}\bar{a}_3\bar{a}_4 + \frac{1}{8}(\bar{a}_1\bar{a}_2 + \bar{a}_2\bar{a}_3 + \bar{a}_2\bar{a}_5 + \bar{a}_4\bar{a}_5)\right)\bar{b}_3 \\
&\qquad + \left(\frac{9}{16}\bar{a}_4^2 + \frac{1}{4}(\bar{a}_1^2 + \bar{a}_2^2 + \bar{a}_3^2 + \bar{a}_5^2) + \frac{1}{8}\bar{a}_3\bar{a}_5\right)\bar{a}_4 \\
&\qquad + \left(\frac{1}{2}\bar{a}_4\bar{a}_5 + \frac{1}{8}(\bar{a}_1\bar{a}_2 + \bar{a}_2\bar{a}_3 + \bar{a}_3\bar{a}_4)\right)\bar{b}_5 + \cdots,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{50}{3}(\xi^{(0,1)}\xi^{(1,0)}\bar{a}_5 + \xi^{(1,1)}\bar{a}_5 + \xi^{(0,1)}\bar{b}_5) \\
&\quad = \left(\frac{1}{16}(\bar{a}_2^2 + \bar{a}_3^2) + \frac{1}{2}\bar{a}_1\bar{a}_5 + \frac{1}{8}(\bar{a}_1\bar{a}_3 + \bar{a}_2\bar{a}_4)\right)\bar{b}_1 \\
&\qquad + \left(\frac{1}{2}\bar{a}_2\bar{a}_5 + \frac{1}{8}(\bar{a}_1\bar{a}_2 + \bar{a}_1\bar{a}_4 + \bar{a}_3\bar{a}_4)\right)\bar{b}_2 \\
&\qquad + \left(\frac{1}{16}(\bar{a}_1^2 + \bar{a}_4^2) + \frac{1}{2}\bar{a}_3\bar{a}_5 + \frac{1}{8}(\bar{a}_1\bar{a}_3 + \bar{a}_2\bar{a}_4)\right)\bar{b}_3 \\
&\qquad + \left(\frac{1}{2}\bar{a}_4\bar{a}_5 + \frac{1}{8}(\bar{a}_1\bar{a}_2 + \bar{a}_2\bar{a}_3 + \bar{a}_3\bar{a}_4)\right)\bar{b}_4 \\
&\qquad + \left(\frac{9}{16}\bar{a}_5^2 + \frac{1}{4}(\bar{a}_1^2 + \bar{a}_2^2 + \bar{a}_3^2 + \bar{a}_4^2)\right)\bar{b}_5 + \cdots, \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .
\end{aligned}
$$