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LETTERS TO THE EDITOR

ON THE BEHAVIOR OF DISSIPATIVE TIME INTEGRATION METHODS NEAR THE RESONANCE CONDITION

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Traditionally, numerical time-integration algorithms for linear dynamics are analyzed only with reference to the homogenous part of the solution [1]. Although greatly simplified and able to provide useful indications about the error evolution, this accuracy analysis is not complete. The accuracy analysis of the forcing term was performed in the past only in a few studies [2], usually limited to the investigation of the local truncation error.

A more complete approach for the accuracy analysis was presented by Preumont [3]. The time-integration operators are treated as digital recursive filters, so that the transfer functions of the discretized equations can be derived. The comparison between the numerical and the exact transfer functions provides information regarding the algorithm behavior with a forcing term and allows one to detect possible spurious resonance conditions.

This approach was adopted for the analysis of non-dissipative second and higher order algorithms in reference [4]. Higher order methods showed a better performance than traditional second order ones in the quasi-resonance condition. This analysis was performed only for conservative methods. It is well known that when a structural system is discretized by the finite-element methodology, only the vibration modes associated with the lower frequencies are meaningful [1]. Therefore, it is desirable that the time-stepping method possess high-frequency dissipation in order to damp out the inaccurate response of the high-frequency modes. The HHT-α method [1], characterized by second order accuracy, is one of the most popular dissipative algorithms for structural dynamics. However, due to its relatively low order accuracy, a non-negligible error may result in long-term simulations.

In recent years, extensive research has been conducted on dissipative higher order methods (see for example the surveys given by Fung [5, 6]). The main feature of these methods is the coupling of high-frequency dissipation and low rate of error growth, which is particularly advantageous in long-term simulations. In particular, the time discontinuous Galerkin (TDG) methodology of time discretization leads to very attractive higher order dissipative methods. Although at an increased computational cost, TDG algorithms show an improved performance over traditional second order methods [7].

In this work, the widely used HHT-α scheme and a TDG method are compared, with reference to the behavior near the resonance condition by means of the approach introduced by Preumont [3]. It will be shown that the TDG algorithm has an improved performance in such a condition compared to the HHT-α method. Therefore, the TDG algorithm should be used in analyses near the resonance condition when an algorithm with high-frequency modes dissipation is required.

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The model problem used to evaluate the numerical algorithms is the typical modal initial-value problem in first order form

$$\dot{v}(t) + 2\xi\omega v(t) + \omega^2 u(t) = f(t), \quad t \in (0, t_f], \tag{1}$$

$$\dot{u}(t) - v(t) = 0, \quad t \in (0, t_f], \tag{2}$$

$$u(0) = u_0, \qquad v(0) = v_0, \tag{3, 4}$$

where superposed dots denote differentiation with respect to time t, u is the unknown displacement, v the unknown velocity, the real non-negative parameter ω represents a natural frequency of the system, ξ the damping ratio and f is the applied force. For simplicity, we consider $\xi = 0$. Assuming that the forcing term can be expanded in a Fourier series and invoking the superposition principle [8], it is sufficient to consider a generic sinusoidal component. Setting $f(t) = e^{ivt}$, the exact response in the mode corresponding to the frequency ω is given by

$$u(t) = u_h(t) + u_p(t), \tag{5}$$

where

$$u_h(t) = u_0 \cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t), \tag{6}$$

$$u_p(t) = \frac{-\omega\cos(vt) + \omega\cos(\omega t) - i\omega\sin(vt) + iv\sin(\omega t)}{\omega(v^2 - \omega^2)} \tag{7}$$

are the homogeneous and the particular solution respectively.

Let $0 = t_0 < t_1 < \cdots < t_N = t_f$ be a partition of the time domain with step size $\Delta t = t_{n+1} - t_n$. Assuming that the numerical solution at $t = t_n$ can be calculated from the $(n-1)$th time step, one-step time-integration algorithms can be formulated as time-advance schemes

$$\mathbf{y}_{n+1} = \mathbf{A}\mathbf{y}_n + \mathbf{L}_n, \quad n = 0, \ldots, N-1, \tag{8}$$

where u_n and v_n are the numerical approximations to $u(t_n)$ and $\dot{u}(t_n)$, respectively, $\mathbf{y}_n^{\mathrm{T}}$ is the state variable vector, $\mathbf{A}$ the amplification matrix, $\mathbf{L}_n$ the load vector, and the starting value $\mathbf{y}_0$ is obtained according to the initial conditions. In particular, the load vector can be written as

$$\mathbf{L}_n = \mathbf{B}\mathbf{r}_n, \tag{9}$$

where the vector $\mathbf{r}_n$ collects the evaluations of the forcing term at some given sampling points and $\mathbf{B}$ is a matrix which does not depend on the forcing term. In the case of HHT-α methods, one obtains

$$\mathbf{y}_n^{\mathrm{T}} = [u_n, v_n, a_n], \tag{10}$$

$$\mathbf{A} = \begin{bmatrix} \dfrac{(4 + \Omega^2\alpha - 2\Omega^2\alpha^2 + \Omega^2\alpha^3)}{d} & \dfrac{4\Delta t}{d} & \dfrac{-\Delta t^2(-1 - 2\alpha + \alpha^2)}{d} \\ \dfrac{2\Omega^2(-1 + 2\alpha)}{d\Delta t} & \dfrac{(-\Omega^2 + \Omega^2\alpha + 3\Omega^2\alpha^2 + 4 + \Omega^2\alpha^3)}{d} & \dfrac{\Delta t(\Omega^2\alpha^2 + \Omega^2\alpha^3 + 2 + 4\alpha)}{d} \\ \dfrac{-4\Omega^2}{d\Delta t^2} & -4\dfrac{(1+\alpha)\Omega^2}{d\Delta t} & \dfrac{(1+\alpha)\Omega^2(-1 - 2\alpha + \alpha^2)}{d} \end{bmatrix}, \tag{11}$$

$$\mathbf{B} = \frac{1}{d}\begin{bmatrix} -\Delta t^2(-1+\alpha)^2\alpha & \Delta t^2(-1+\alpha)^2(1+\alpha) \\ 2\Delta t\alpha(-1+2\alpha) & -2\Delta t(1+\alpha)(-1+2\alpha) \\ -4\alpha & 4(1+\alpha) \end{bmatrix}, \tag{12}$$

$$\mathbf{r}_n^{\mathrm{T}} = [f_n, f_{n+1}], \tag{13}$$

where $d = (4 + \Omega^2 - \Omega^2\alpha - \Omega^2\alpha^2 + \Omega^2\alpha^3)$ and $\Omega = \omega\Delta t$.

The TDG algorithms are derived from Galerkin weighted residual formulations of equations (1)–(4), where the initial conditions (3)–(4) are enforced weakly on each time step. Within a typical time step $[t_n, t_{n+1}]$, the approximate solution is represented by trial polynomials in the variable t:

$$u^h(t) = \mathbf{N}(t)^{\mathrm{T}}\bar{\mathbf{u}}, \qquad v^h(t) = \mathbf{N}(t)^{\mathrm{T}}\bar{\mathbf{v}}, \tag{14}$$

where $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ are the vectors of unknown coefficients and $\mathbf{N}$ is the vector of time interpolation polynomials. The test functions are chosen as

$$\tilde{u}^h(t) = \mathbf{N}(t)^{\mathrm{T}}\tilde{\mathbf{u}}, \qquad \tilde{v}^h(t) = \mathbf{N}(t)^{\mathrm{T}}\tilde{\mathbf{v}}. \tag{15}$$

Notice that the trial functions are defined by means of the same time interpolants for the displacement and the velocity, since the analysis reported by Hulbert [7] has shown that this choice leads to optimal accuracy and stability. The TDG formulation with reference to the undamped modal problem is stated as follows: find u^h and v^h such that

$$\int_{t_n}^{t_{n+1}} \tilde{u}^h(\dot{v}^h + \omega^2 u^h - f)\,\mathrm{d}t + \int_{t_n}^{t_{n+1}} \tilde{v}^h(\dot{u}^h - v^h)\,\mathrm{d}t$$

$$+ \tilde{v}^h|_{t_n}(u^h|_{t_n} - u_n) + \tilde{u}^h|_{t_n}(v^h|_{t_n} - v_n) = 0 \tag{16}$$

for all $\tilde{u}^h$ and $\tilde{v}^h$. Enforcing this condition, one obtains a system of linear algebraic equations in the unknowns $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$, which can be written as

$$\omega^2\boldsymbol{\Psi}_1\bar{\mathbf{u}} + \boldsymbol{\Psi}_2\bar{\mathbf{v}} = v_n\mathbf{N}|_{t_n} + \int_{t_n}^{t_{n+1}} \mathbf{N}f\,\mathrm{d}t,$$

$$\boldsymbol{\Psi}_2\bar{\mathbf{u}} - \boldsymbol{\Psi}_1\bar{\mathbf{v}} = u_n\mathbf{N}|_{t_n}, \tag{17}$$

where

$$\boldsymbol{\Psi}_1 = \int_{t_n}^{t_{n+1}} \mathbf{N}\mathbf{N}^{\mathrm{T}}\,\mathrm{d}t, \qquad \boldsymbol{\Psi}_2 = (\mathbf{N}\mathbf{N}^{\mathrm{T}})|_{t_n} + \int_{t_n}^{t_{n+1}} \mathbf{N}\dot{\mathbf{N}}^{\mathrm{T}}\,\mathrm{d}t. \tag{18, 19}$$

Once the system is solved, a time-stepping method can be defined by setting

$$u_{n+1} = u^h(t_{n+1}), \quad v_{n+1} = v^h(t_{n+1}), \quad n = 0, \ldots, N-1. \tag{20}$$

The method considered here is based on the use of linear Lagrangian polynomials as time-interpolation functions and the Simpson rule to evaluate the integral depending on the

forcing term. The amplification matrix is

$$\mathbf{A} = \frac{1}{d}\begin{bmatrix} -2(7\Omega^2 - 18) & -2\Delta t(\Omega^2 - 18) \\ 2\Omega^2(\Omega^2 - 18)/\Delta t & -2(7\Omega^2 - 18) \end{bmatrix}, \tag{21}$$

where $d = (\Omega^4 + 4\Omega^2 + 36)$, $\Omega = \omega\Delta t$ and the state variable vector is $\mathbf{y}_n^{\mathrm{T}} = [u_n, v_n]$. The load term is defined by

$$\mathbf{B} = \frac{\Delta t}{d}\begin{bmatrix} -\frac{1}{3}\Delta t(\Omega^2 - 18) & \frac{2}{3}\Delta t(\Omega^2 + 18) & \frac{2}{3}\Omega^2\Delta t \\ -\frac{1}{3}(7\Omega^2 - 18) & -\frac{4}{3}(\Omega^2 - 18) & \frac{1}{3}(5\Omega^2 + 18) \end{bmatrix}, \tag{22}$$

$$\mathbf{r}_n^{\mathrm{T}} = [f_n,\ f_{n+1/2}, f_{n+1}], \tag{23}$$

where $f_n, f_{n+1/2}, f_{n+1}$ are the evaluations of the forcing term at $t_n, (t_n + \Delta t/2)$ and t_{n+1} respectively. Since the initial conditions are enforced weakly, the numerical solution can be discontinuous between time steps, hence the denomination of TDG algorithms.

As far as the classical accuracy analysis is concerned, the HHT-α methods with $-\frac{1}{3} \leqslant \alpha \leqslant 0$ are unconditionally stable and second order accurate, while the TDG algorithm described by equations (21) and (22) is unconditionally stable and third order accurate.

A more complete analysis for the accuracy of the loading term is based on the investigation of exact and numerical transfer functions [3]. For the modal initial value problem (1) with $\xi = 0$ and $f(t) = \mathrm{e}^{\mathrm{i}\nu t}$, the exact transfer function has the well-known form

$$H_u^* = \frac{1}{(\omega^2 - \nu^2)}. \tag{24}$$

An expression for the transfer function of the time advance scheme (8) can be obtained and compared with equation (24). Assuming $f(t) = \mathrm{e}^{\mathrm{i}\nu t}$, the solution at $t = t_n$ can be written as

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} H_u \\ H_v \end{pmatrix} \mathrm{e}^{\mathrm{i}\nu n\Delta t}, \tag{25}$$

where H_u and H_v are the transfer function for the numerical displacement and velocity. Using equations (8) and (25), one obtains

$$\begin{pmatrix} H_u \\ H_v \end{pmatrix} = [\mathrm{e}^{\mathrm{i}\nu\Delta t}\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{r}, \tag{26}$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{r}^{\mathrm{T}} = [1, \mathrm{e}^{\mathrm{i}\nu\Delta t}]$ for the HHT-α methods and $\mathbf{r}^{\mathrm{T}} = [1, \mathrm{e}^{\mathrm{i}\nu\Delta t/2}, \mathrm{e}^{\mathrm{i}\nu\Delta t}]$ for the TDG algorithm.

The comparison of H_u and H_v with the exact forms H_u^* and $H_v^* = \mathrm{i}\nu H_u^*$ illustrates the behavior of the algorithm as regards the forcing term.

In Figure 1 the HHT-α scheme with $\alpha = -1/3$ and the TDG method are compared assuming $\Omega/2\pi = 0{\cdot}2$. The ratio $|H_u|/|H_u^*|$ is plotted versus the ratio $\varepsilon = \nu/\omega$. It can be noted that a spurious peak occurs near the resonance condition, which corresponds to $\varepsilon = 1$. As illustrated in the picture, the width of the disturbance in the TDG algorithm is greatly reduced compared to the HHT-α scheme.

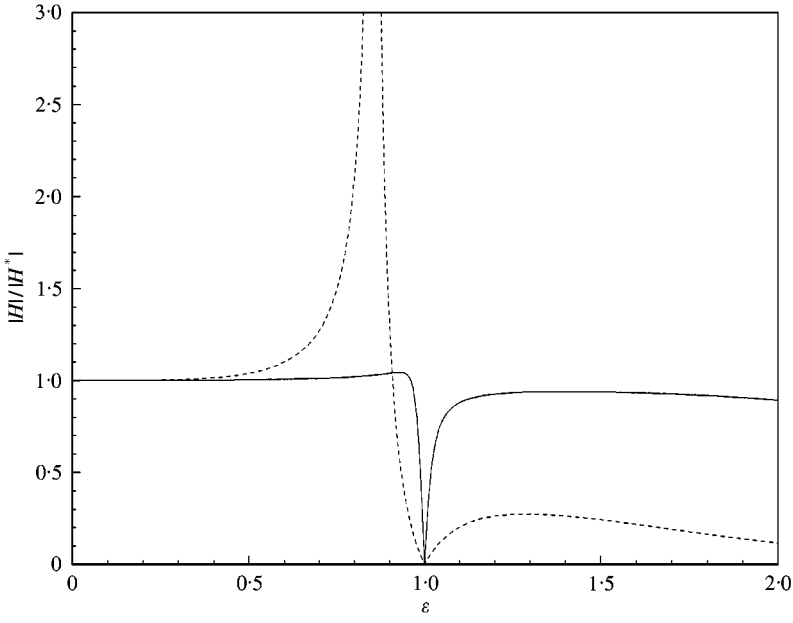

Figure 1. $|H_u|/|H_u^*|$: ----, HHT-α; ——, TDG methods.

In order to get more insight, it is useful to report an analytical expression for the ratio $|H_u|/|H_u^*|$ for the TDG algorithm

$$|H_u|/|H_u^*| = \frac{1}{3}\sqrt{\frac{a}{b}}, \tag{27}$$

$$a = \Omega^4(-1+\varepsilon^2)^2[\Omega^4(-8\tau^2+4\tau+13)+\Omega^2(576\tau^3+576\tau^2 - 288\tau-216)+5184\tau^2+5184\tau+1296], \tag{28}$$

$$b = \Omega^8+\Omega^6(64\tau^4+48\tau^2-40)+\Omega^4(832\tau^4-224\tau^2+688) + \Omega^2(4608\tau^4+1152\tau^2-5760)+20736\tau^4-41472\tau^2+20736, \tag{29}$$

where $\tau = \cos(\varepsilon\Omega/2)$. Expanding in a series about $\Omega = 0$, one obtains

$$|H_u|/|H_u^*| = 1 - \frac{1}{8640}\frac{(9\varepsilon^4+35\varepsilon^3+20)\varepsilon^2}{\varepsilon^2-1}\Omega^4 + o(\Omega^6), \tag{30}$$

thus showing that the spurious disturbance is of the fourth order.

Numerical tests have been performed in order to evaluate the TDG method and to compare it with the HHT-α scheme. A single-degree-of-freedom system excited by a sinusoidal force with v close to the system frequency ω is considered. The numerical simulations have been conducted setting $v = 1$ and $\Delta t = T/20$, where $T = 2\pi/\omega$ is the free-vibration period of the system and analyzing the response up to $t_f = 20T$. The time histories of the displacement error (Figure 2), calculated as the difference between the exact quantity and the numerical one, show that the error accumulation of the HHT-α scheme is

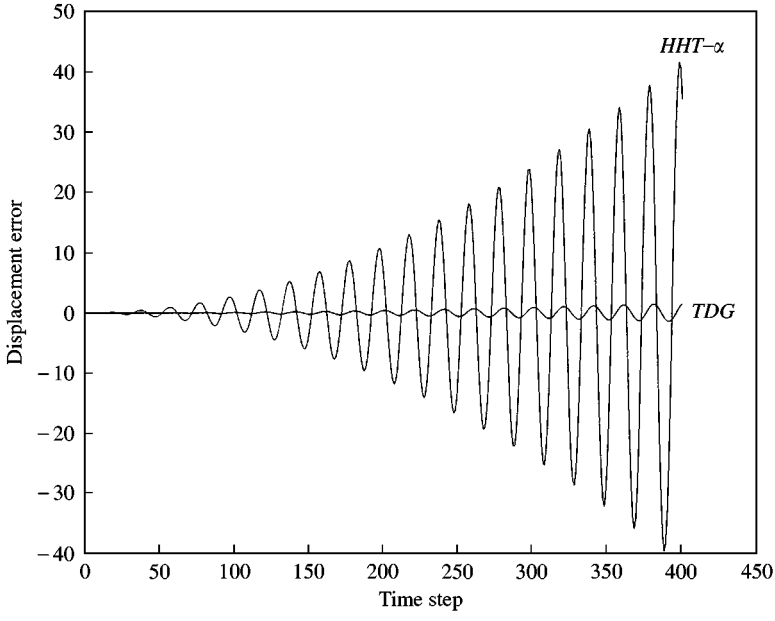

Figure 2. HHT-α and TDG methods: $\alpha = -1/3$; $\varepsilon = 0{\cdot}99$, displacement.

quite fast as the resonance condition is approached ($\varepsilon = 0{\cdot}99$). The same picture shows that the error accumulation for the TDG method is remarkably slower. As a matter of fact, the errors are at least 30 times lower than those obtained with the HHT-α scheme. Hence, the higher accuracy of the TDG method is evident.

To conclude, this analysis reveals that even in the presence of numerical dissipation, traditional algorithms such as the HHT-α scheme present spurious resonance conditions which can adversely affect the solution. On the contrary, the TDG methods show an improved performance near the resonance conditions both in the theoretical accuracy analysis and in the numerical tests. Thus, also with dissipative methods, higher order accuracy is recommended when high-quality simulations are required near the resonance condition. Although the TDG methods are more computationally expensive than traditional dissipative second order ones, effective implementation techniques have been recently developed [9]. Taking into account their remarkable computational characteristics, these algorithms appear as a valuable alternative to traditional second order disspative methods.

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