Journal of Sound and Vibration (2002) **249**(5), 909–919
doi:10.1006/jsvi.2001.3916, available online at http://www.idealibrary.com on IDEAL®

NORMAL MODE LOCALIZATION FOR A TWO DEGREES-OF-FREEDOM SYSTEM WITH QUADRATIC AND CUBIC NON-LINEARITIES

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(*Received* 24 *October* 2000, *and in final form* 20 *June* 2001)

The non-similar normal modes of free oscillations of a coupled non-linear oscillator are examined. So far, the study of non-linear vibrations has been based on the assumption that the system is admissible. This requirement is satisfied when the stiffness of the springs are odd functions of their displacement. In this work, a two-degrees-of-freedom tuned system is considered with stiffness elements having linear, quadratic and cubic non-linearities. The potential energy function of this system is not symmetric with respect to the origin (equilibrium point) of the configuration space due to the presence of the quadratic non-linearity. Hence, the system considered is no longer admissible. A study of the balancing diagrams is performed to determine the "degenerate" and "global" similar modes of the system. Manevich–Mikhlin asymptotic methodology is used for solving the singular differential equation describing the non-similar modes and approximate analytical expressions are derived. For this system, with weak coupling, localized non-similar modes are detected in a small neighborhood of degenerate similar modes of the tuned system. Numerical integration is used to verify theoretically predicted non-similar normal modes. It is found that these modes pass periodically through a non-zero point in the configuration space.

1. INTRODUCTION

The concept of normal modes and their importance in the context of linearized systems is long known. In early 1960s, Rosenberg published a series of papers on normal modes for discrete/continuous strongly non-linear dynamical systems, the results of which were subsequently included in a review paper [1]. This work laid the foundation for further developments on various aspects of non-linear normal modes (NNMs). A good account of these may be found in references [2–4].

According to Rosenberg [1], NNMs are defined as the free motions where all co-ordinates vary equiperiodically, passing through the equilibrium point in the configuration space simultaneously and reaching their extreme values at the same instant of time. Subsequent works [5–8] showed that for non-linear systems, the number of normal modes can exceed the number of degrees of freedom. Modal curves may assume the shape of straight lines (similar NNMs) or curved lines (non-similar NNMs) in the configuration space. The curved modal lines which turn out as the solutions of a set of functional differential equations, make these equations singular at the intersection points of the maximum equipotential surface [1]. In this regard, methods were suggested [9–12] to obtain approximate (asymptotic) solutions.

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Following the basic concepts of NNMs [1], Vakakis [13] brought out the phenomenon of normal mode localization for an admissible non-linear two-degrees-of-freedom system. It was shown therein that, in contrast to similar linear systems, weak substructure coupling gives rise to mode localization even in the case of no mistuning in the anchor springs. It was also clear from that study that these localized modes turn out to be non-similar in nature. An asymptotic methodology established by Manevich and Mikhlin [12] was followed for finding approximate solution of the singular functional equation. Stability of the localized modes was also investigated analytically and numerically. In addition, other interesting results on non-localized NNMs were reported. Subsequently, Happawana and Bajaj [14] used Frobenius' method to obtain closed-form series solutions in different stages of the asymptotic expansion in the same problem to obtain better estimates for larger values of the independent co-ordinate.

In almost all the developments that have taken place so far on NNMs, it is found that the springs consist of linear and cubic non-linearity terms. Naturally, the potential energy function is symmetric with respect to the origin (equilibrium point) of the configuration space. Hence the NNMs, either similar or non-similar, pass through the origin. This situation may change if in addition a quádratic non-linear spring force is also present. Nayfeh and Nayfeh [15], in the context of a simply supported Euler–Bernoulli beam, dealt with linear, quadratic and cubic non-linearities and concluded that "vibration-in-unison" does not occur due to the presence of the quadratic non-linear terms. Hence, every point of the continuous system does not pass through zero at the same instant of time.

The NNMs of a portal frame structure was studied by Balthazer and Brasil [16] in which the restoring forces were assumed to be linear and quadratic type. The conditions for the model to execute similar NNMs were determined.

The primary goal of this work is directed towards the study of non-linear normal mode localization for weak substructure coupling in a perfectly symmetric (tuned), conservative, two-degrees-of-freedom system. The restoring force in all the spring elements is given by linear, quadratic and cubic non-linear terms. The method outlined in reference [13] will be combined with the bisection method to compute approximately the localized non-similar NNMs in the configuration space.

2. FORMULATION OF THE PROBLEM

Figure 1 shows a two-degrees-of-freedom system consisting of two unit masses and relevant spring forces. The potential and kinetic energy functions of the system are considered as

$$U = \frac{x_1^2 + x_2^2}{2} + \alpha_1 \frac{x_1^3 + x_2^3}{3} + \mu_1 \frac{x_1^4 + x_2^4}{4} + k_1 \frac{(x_1 - x_2)^2}{2} + \alpha_2 \frac{(x_1 - x_2)^3}{3} + \mu_2 \frac{(x_1 - x_2)^4}{4},$$

$$T = \frac{\dot{x}_1^2 + \dot{x}_2^2}{2}. \tag{1}$$

Use of equation (1) in conjunction with the Lagrange's method gives the following equations of motion:

$$\ddot{x}_1 + x_1 + \alpha_1 x_1^2 + \mu_1 x_1^3 + k_1(x_1 - x_2) + \alpha_2(x_1 - x_2)^2 + \mu_2(x_1 - x_2)^3 = 0, \tag{2}$$

$$\ddot{x}_2 + x_2 + \alpha_1 x_2^2 + \mu_1 x_2^3 - k_1(x_1 - x_2) - \alpha_2(x_1 - x_2)^2 - \mu_2(x_1 - x_2)^3 = 0. \tag{3}$$

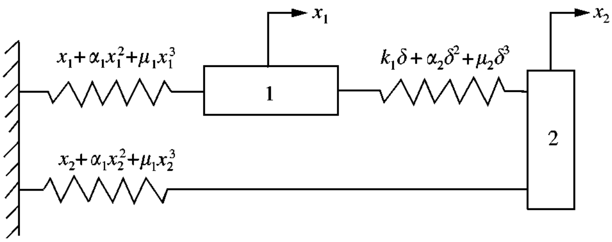

Figure 1. Two-degree-of-freedom system model.

Equations (2) and (3) are to be solved with the initial conditions $x_1(0) = X_1$, $\dot{x}_1(0) = 0$, $x_2(0) = X_2$, $\dot{x}_2(0) = 0$, where (X_1, X_2) refer to one extreme of modal oscillation, and without any loss of generality, it is assumed that $X_1 > 0$. The scalars α_1 and α_2 are the quadratic stiffness coefficients of the end and coupling-springs, respectively, while μ_1 and μ_2 are the respective cubic non-linearity coefficients; and k_1 stands for the linear coupling stiffness.

The tuned oscillator has similar normal modes given by $x_2 = cx_1$, where c is found from equations (2) and (3) by following the procedure outlined in reference [13]. The algebraic equations in this case turn out to be

$$k_1(1 - c)(1 + c) = 0, \tag{4}$$

$$(1 - c)[\alpha_1 c + \alpha_2(1 - c^2)] = 0, \tag{5}$$

and

$$(1 - c)(1 + c)[\mu_1 c + \mu_2(1 - c)^2] = 0. \tag{6}$$

These equations form the balancing diagrams for the linear, quadratic and cubic non-linear coefficients, respectively, as shown in Figure 2.

The linear balancing diagram obtained from equation (4) shows that for $k_1 \neq 0$, the modal constant c is either 1 or -1 corresponding to the symmetric and antisymmetric modes of linear vibration. In addition, a "degenerate" case exists when $k_1 = 0$. In that case, the variable c can take an arbitrary positive or negative value. Considering the quadratic balancing diagram (equation (5)), it is seen that $c = 1$ is a solution along with two additional solutions which depend on α_2/α_1 with degenerate cases as $\alpha_2/\alpha_1 \to 0$. The cubic balancing diagram, as suggested by equation (6) shows that $c = 1, -1$ are the solutions; additional solutions also exist for low values of the ratio μ_2/μ_1 (weak coupling) resulting from a bifurcation of the antisymmetric similar normal mode which degenerates as $\mu_2/\mu_1 \to 0$. Figures 2(a–c) show all the three balancing diagrams.

When linear, quadratic and cubic stiffness terms coexist (as in the present case), the permissible values of c must satisfy all the three balancing diagrams. For non-zero substructure coupling (k_1, α_2 and μ_2 non-zero) the only common value for c is 1, giving rise to a symmetric similar NNM passing through the origin of the configuration space. However, additional solutions exist as $k_1, \alpha_2, \mu_2 \to 0$ corresponding to degenerate similar modes. It will be shown later that in the neighborhood of these degenerate modes, localized, non-similar normal modes exist for the tuned non-linear system. Four such degenerate

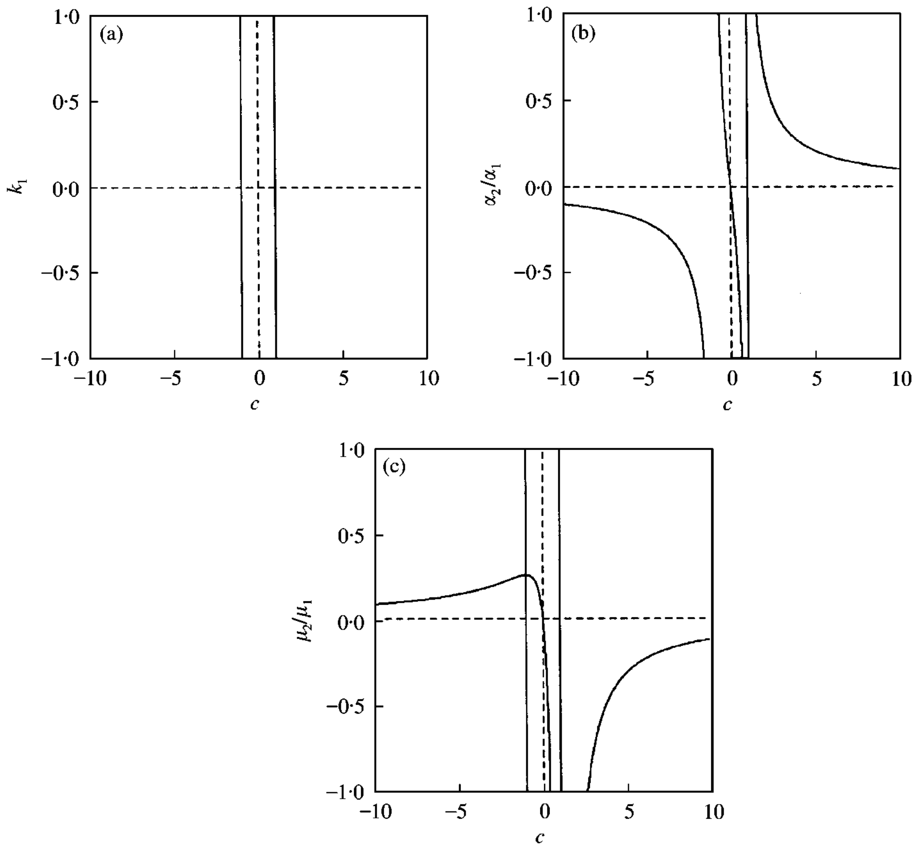

Figure 2. (a) Linear, (b) quadratic and (c) cubic balancing diagrams.

similar modes are thus identified and grouped below.

(1) $\alpha_2/\alpha_1 \to 0^+$, $\mu_2/\mu_1 \to 0^+$, $k_1 = 0$, $c = 0$,

(2) $\alpha_2/\alpha_1 \to 0^-$, $\mu_2/\mu_1 \to 0^-$, $k_1 = 0$, $c = 0$,

(3) $\alpha_2/\alpha_1 \to 0^+$, $\mu_2/\mu_1 \to 0^-$, $k_1 = 0$, $c = \infty$,

(4) $\alpha_2/\alpha_1 \to 0^-$, $\mu_2/\mu_1 \to 0^+$, $k_1 = 0$, $c = -\infty$.

These localized modes exist very close to the corresponding degenerate similar modes and describe finite motion of only mass 1 in the first two cases, and mass 2 in the latter two cases, the vibration of the other mass remaining confined to a much smaller magnitude.

Weak substructure coupling: The asymptotic methodology of Vakakis [13] will now be applied to determine the non-similar normal modes of the oscillator. Since weak substructure coupling is assumed, the linear, quadratic and cubic terms of the coupling stiffness are assumed to be small, as given below

$$k_1 = \varepsilon K_1, \qquad \alpha_2 = \varepsilon L_2, \qquad \mu_2 = \varepsilon M_2, \ |\varepsilon| \ll 1, \tag{7}$$

where ε is the perturbation parameter. A non-linear NNM in the present case is assumed to have the following form:

$$x_2 = \hat{x}_2(x_1). \tag{8}$$

Since this relation must hold for every value of time, the derivatives of the co-ordinates during a non-similar normal mode can be expressed by the chain rule as

$$\dot{x}_2 = \hat{x}_2' \cdot \dot{x}_1, \qquad \ddot{x}_2 = \hat{x}_2'' \cdot \dot{x}_1^2 + \hat{x}_2' \cdot \ddot{x}_1, \tag{9}$$

where $(') = \mathrm{d}/\mathrm{d}x_1$ and $(\cdot) = \mathrm{d}/\mathrm{d}t$. Substituting equation (7) into equations (2) and (3), one obtains

$$\ddot{x}_1 + x_1 + \alpha_1 x_1^2 + \mu_1 x_1^3 + \varepsilon K_1(x_1 - x_2) + \varepsilon L_2(x_1 - x_2)^2 + \varepsilon M_2(x_1 - x_2)^3 = 0. \tag{10}$$

$$\ddot{x}_2 + x_2 + \alpha_1 x_2^2 + \mu_1 x_2^3 - \varepsilon K_1(x_1 - x_2) - \varepsilon L_2(x_1 - x_2)^2 - \varepsilon M_2(x_1 - x_2)^3 = 0. \tag{11}$$

Integration of equation (10) with equation (8) gives

$$\dot{x}_1^2 = -2\int_{X_1}^{x_1} \{\xi(1 + \varepsilon K_1) + \alpha_1 \xi^2 + \mu_1 \xi^3 - \varepsilon K_1 \hat{x}_2(\xi) + \varepsilon L_2[\xi - \hat{x}_2(\xi)]^2$$
$$+ \varepsilon M_2[\xi - \hat{x}_2(\xi)]^3\}\,\mathrm{d}\xi. \tag{12}$$

Finally, use of equations (9) and (12) in equation (11) yields the functional equation for the (unknown) modal function $\hat{x}_2(x_1)$. Thus,

$$-2\hat{x}_2''[(x_1^2 - X_1^2)\frac{1 + \varepsilon K_1}{2} + \alpha_1 \frac{x_1^3 - X_1^3}{3} + \mu_1 \frac{x_1^4 - X_1^4}{4}$$
$$+ \int_{X_1}^{x_1} \{\varepsilon M_2(\xi - \hat{x}_2(\xi))^3 + \varepsilon L_2(\xi - \hat{x}_2(\xi))^2 - \varepsilon K_1 \hat{x}_2(\xi)\}\,\mathrm{d}\xi]$$
$$- \hat{x}_2'\{x_1 + \alpha_1 x_1^2 + \mu_1 x_1^3 + \varepsilon K_1(x_1 - \hat{x}_2) + \varepsilon L_2(x_1 - \hat{x}_2)^2 + \varepsilon M_2(x_1 - \hat{x}_2)^3\}$$
$$+ \hat{x}_2 + \alpha_1 \hat{x}_2^2 + \mu_1 \hat{x}_2^3 - \varepsilon K_1(x_1 - \hat{x}_2) - \varepsilon L_2(x_1 - \hat{x}_2)^2 - \varepsilon M_2(x_1 - \hat{x}_2)^3 = 0. \tag{13}$$

The functional equation becomes singular at the maximum equipotential curve (i.e., when $U = U_{max}$). This happens because the coefficient of the second derivative of x_2 becomes zero at these points. As mentioned earlier, $X_1 > 0$ is one extreme value of x_1 and corresponds to the intersection of the modal curve with the maximum equipotential curve. As the function within the integral sign is not odd, the same coefficient will not be zero for $x_1 = -X_1$ for the fact that the maximum equipotential curve is not symmetric about the origin. As a result, there exists a $x_1 = X_1^* < 0$, hitherto unknown, for which the same coefficient is bound to be zero again. This value of x_1 physically relates to the other extreme point during oscillation. The asymptotic approximation to the solution will be valid in the interval (X_1^*, X_1). To guarantee that the series solution intersects the maximum equipotential curve at the points

$(x_1, x_2) = (X_1, X_2)$, one imposes the additional boundary condition

$$
\begin{aligned}
&-\hat{x}_2'(X_1)[X_1 + \alpha_1 X_1^2 + \mu_1 X_1^3 + \varepsilon K_1\{X_1 - \hat{x}_2(X_1)\} + \varepsilon L_2\{X_1 - \hat{x}_2(X_1)\}^2 \\
&\quad + \varepsilon M_2\{X_1 - \hat{x}_2(X_1)\}^3 + \hat{x}_2(X_1) + \alpha_1 \hat{x}_2(X_1)^2 + \mu_1 \hat{x}_2(X_1)^3 - \varepsilon K_1\{X_1 - \hat{x}_2(X_1)\} \\
&\quad - \varepsilon L_2\{X_1 - \hat{x}_2(X_1)\}^2 - \varepsilon M_2\{X_1 - \hat{x}_2(X_1)\}^3 = 0, \qquad (14)
\end{aligned}
$$

and a similar condition exists at the other end, i.e., at $(x_1, x_2) = (X_1^*, X_2^*)$.

As given in reference [13], the asymptotic solution is assumed as

$$\hat{x}_2(x_1) = \hat{x}_2^{(0)}(x_1) + \hat{x}_2^{(1)}(x_1) + \cdots, \qquad (15)$$

where the order of magnitudes of the individual terms and their derivatives are

$$O(\hat{x}_2^{(0)}) = 1, \quad O(\hat{x}_2^{(1)}) = O(\hat{x}_2^{(1)'}) = O(\hat{x}_2^{(1)''}) = O(\varepsilon). \qquad (16)$$

The displacement x_1 is assumed to be finite (of $O(1)$), and higher approximations $\hat{x}_2^{(k)}$, $k \geqslant 2$ are assumed to be of $O(\varepsilon^2)$ or higher. Series (15) is now substituted in equation (13) and equations of various order of ε can be formed with the help of equation (16).

$O(1)$ approximation: The zero order approximation $\hat{x}_2^{(0)}(x_1)$ in series (15) corresponds to the similar modes, and therefore has the form

$$\hat{x}_2^{(0)}(x_1) = cx_1, \qquad (17)$$

where the modal constant c is of $O(1)$. The results are the same as those obtained earlier and are shown in the balancing diagrams (Figure 2).

$O(\varepsilon)$ approximation: The non-similar free motions resulting as perturbations of the degenerate similar normal modes will now be computed by considering $O(\varepsilon)$ terms. Taking into account the zero order approximation with $c = 0$, the following functional equation is obtained:

$$
\begin{aligned}
&-2\hat{x}_2^{(1)''}\left\{\frac{x_1^2 - X_1^2}{2} + \alpha_1 \frac{x_1^3 - X_1^3}{3} + \mu_1 \frac{x_1^4 - X_1^4}{4}\right\} - \hat{x}_2^{(1)'}(x_1 + \alpha_1 x_1^2 + \mu_1 x_1^3) + \hat{x}_2^{(1)} \\
&\quad - \varepsilon K_1 x_1 - \varepsilon L_2 x_1^2 - \varepsilon M_2 x_1^3 = 0. \qquad (18)
\end{aligned}
$$

This is the functional equation of first order that must be satisfied by $\hat{x}_2^{(1)}$. In addition to this equation, the following boundary conditions of first order must be considered (this is derived in a similar way considering the boundary orthogonality condition (14) and retaining only terms up to $O(\varepsilon)$). Thus,

$$-\hat{x}_2^{(1)'}(X_1)(X_1 + \alpha_1 X_1^2 + \mu_1 X_1^3) + \hat{x}_2^{(1)}(X_1) - \varepsilon K_1 X_1 - \varepsilon L_2 X_1^2 - \varepsilon M_2 X_1^3 = 0. \qquad (19)$$

Since, it is expected that the curved NNM will not pass through the origin of the configuration space, the solution for $\hat{x}_2^{(1)}$ is now expressed in the form

$$\hat{x}_2^{(1)} = b_{20}^{(1)} + b_{21}^{(1)}x_1 + b_{22}^{(1)}x_1^2 + b_{23}^{(1)}x_1^3 + b_{24}^{(1)}x_1^4 + O(\varepsilon x_1^5, \varepsilon^2). \qquad (20)$$

When this series solution is substituted in equation (18) and coefficients of respective powers of x_1 are set equal to zero, the following relations among the unknown constants result:

$$b_{22}^{(1)} = \frac{-3b_{20}^{(1)}}{6X_1^2 + 4\alpha_1 X_1^3 + 3\mu_1 X_1^4}, \qquad b_{23}^{(1)} = \frac{\varepsilon K_1}{6X_1^2 + 4\alpha_1 X_1^3 + 3\mu_1 X_1^4} \tag{21, 22}$$

and

$$b_{24}^{(1)} = \frac{\alpha_1 b_{21}^{(1)} + 3b_{22}^{(1)} + \varepsilon L_2}{12X_1^2 + 8\alpha_1 X_1^3 + 6\mu_1 X_1^4}. \tag{23}$$

The constant $b_{21}^{(1)}$ is expressed in terms of $b_{20}^{(1)}$ with the help of equations (19) and (21)–(23) as follows:

$$b_{21}^{(1)} = \frac{\begin{array}{c}\{b_{20}^{(1)} - \varepsilon X_1(K_1 + M_2X_1^2) - b_{23}^{(1)}X_1^3(2 + 3\alpha_1X_1 + 3\mu_1X_1^2)\}(12 + 8\alpha_1X_1 + 6\mu_1X_1^2) \\ - 2\varepsilon L_2X_1^2(15 + 12\alpha_1X_1 + 10\mu_1X_1^2) - b_{22}^{(1)}X_1^2\{(21 + 20\alpha_1X_1 + 18\mu_1X_1^2) \\ 4X_1(\alpha_1 + \mu_1X_1)(6 + 4\alpha_1X_1 + 3\mu_1X_1^2)\}\end{array}}{X_1^2\{\alpha_1(15 + 12\alpha_1X_1 + 10\mu_1X_1^2) + 2\mu_1X_1(6 + 4\alpha_1X_1 + 3\mu_1X_1^2)\}}. \tag{24}$$

In view of expressions (21)–(24), it is clear that determination of all other constants except $b_{23}^{(1)}$ is contingent upon finding $b_{20}^{(1)}$. The following procedure is adopted to determine the value of $b_{20}^{(1)}$.

(1) Note from equation (20) that for $x_1 = 0$, $\hat{x}_2^{(1)} = b_{20}^{(1)} = O(\varepsilon)$. Thus, an initial value of $b_{20}^{(1)}$ of $O(\varepsilon)$ is assumed. The corresponding values of $b_{21}^{(1)}$, $b_{22}^{(1)}$, $b_{23}^{(1)}$ and $b_{24}^{(1)}$ are calculated using expressions (21)–(24).
(2) Coefficients $b_{2j}^{(1)}$ so determined with initial X_1 are used to evaluate X_2 from equation (20).
(3) Substituting equation (15) in the potential energy function (1) using equation (7) and retaining terms up to $O(\varepsilon^2)$, one obtains the maximum potential energy of the system at (X_1, X_2) as

$$U_{max} = \frac{X_1^2}{2} + \alpha_1\frac{X_1^3}{3} + \mu_1\frac{X_1^4}{4} + \frac{\varepsilon K_1X_1^2}{2} + \frac{\varepsilon L_2X_1^3}{3} + \frac{\varepsilon M_2X_1^4}{4} + \frac{X_2^2}{2} - \varepsilon K_1X_1X_2$$
$$+ \varepsilon L_2X_2X_1^2 - \varepsilon M_2X_2X_1^3. \tag{25}$$

This expression must also be satisfied at the other end by (X_1^*, X_2^*). The value of X_1^* is now determined at the intersection of the modal curve with the U_{max} curve by use of the bisection method with an accuracy level of 10^{-6}.
(4) X_1^* must also satisfy the boundary orthogonality condition similar to equation (14).
(5) The value of $b_{20}^{(1)}$ is changed and steps (1)–(3) are repeated until the step (4) is satisfied to within an accuracy of 10^{-6}.
(6) A unique value of $b_{20}^{(1)}$ is thus determined and subsequently $b_{21}^{(1)}$, $b_{22}^{(1)}$, $b_{23}^{(1)}$, and $b_{24}^{(1)}$ are calculated using expressions (21)–(24). Finally, these numerical values may be substituted in equation (20) to obtain the localized non-similar NNM. Since the terms

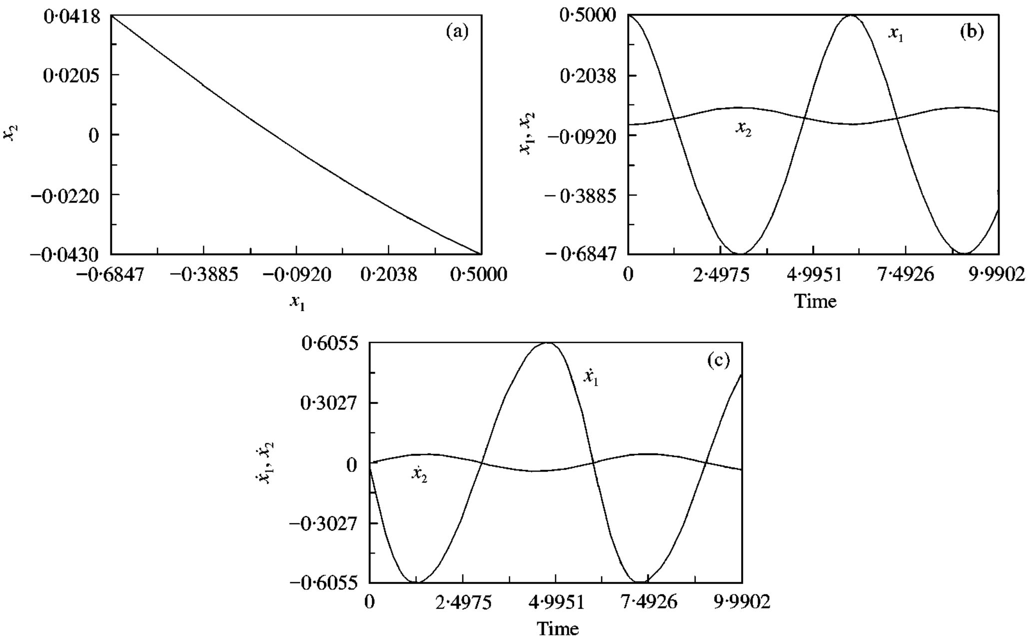

Figure 3. (a) Modal curve for the system, (b) time response for the system, (c) velocity response for the system. Initial conditions used are $x_1(0) = 0{\cdot}5$, $\dot{x}_1(0) = 0$, $x_2(0) = -\,0{\cdot}043424$, $\dot{x}_2(0) = 0$.

of order ε^2 and higher have been neglected, the solutions obtained provide an approximation of the actual solution only for small finite values of X_1.

The coefficients $b_{2j}^{(1)}$ depend explicitly on the amplitude X_1. Therefore, the modal curve depends on one of the extreme points of oscillation (or, equivalently on the total energy of the motion). This is in agreement with the prediction made in reference [1] concerning non-similar modes.

The modal relation (20) corresponds to a motion where the co-ordinate x_1 takes finite values, whereas the co-ordinate x_2 is always of $O(\varepsilon)$. Another set of localized modes can be found for the degenerate solution at $c = -\,\infty$. In this case x_2 has finite values, while x_1 is of perturbation order. To compute it, the modal curve must be expressed as

$$x_1 = \hat{x}_1(x_2) \tag{26}$$

and similar procedure with some adjustments in the equations may be repeated.

3. NUMERICAL RESULTS AND DISCUSSION

Numerical computations are performed in order to compute the predicted non-similar localized modes with weak substructure coupling. The non-linear equations of motion (10) and (11) have been numerically integrated using a fourth order Runge–Kutta algorithm and with initial conditions that are identical to the ones obtained from the foregoing method.

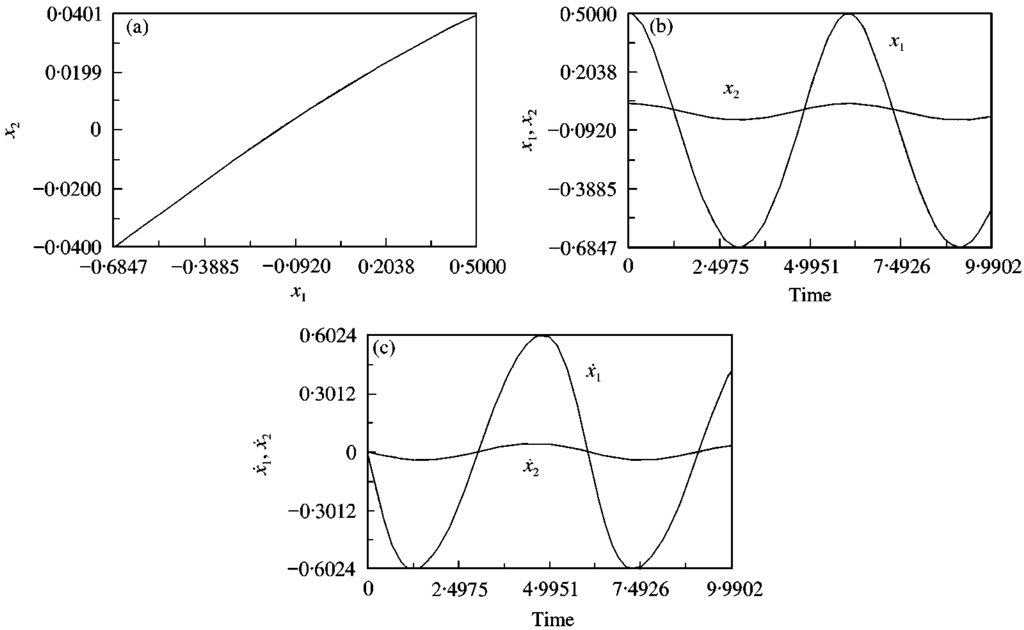

Figure 4. (a) Modal curve for the system, (b) time response for the system, (c) velocity response for the system. Initial conditions used are $x_1(0) = 0{\cdot}5$, $\dot{x}_1(0) = 0$, $x_2(0) = 0{\cdot}040125$, $\dot{x}_2(0) = 0$.

The first simulation is done for $\alpha_1 = \mu_1 = 1$, and $k_1 = \alpha_2 = \mu_2 = 0{\cdot}005$. This falls in the category of case (1) in section 2. The degenerate similar mode exists for $c = 0$ for which the non-similar mode shows confinement of oscillation of mass 2. For $X_1 = 0{\cdot}5$, the value of X_2 is found to be $-0{\cdot}043424$ and a modal curve is obtained which can be improved by slight adjustment in the third place of decimal, as shown in Figure 3(a). It is to be observed in this figure that the modal curve does not pass through the origin of the configuration space. However, there exists a non-zero point through which both masses pass at the same instant of time as suggested by Figures 3(a) and 3(b). The displacement and velocity responses (Figures 3(b) and 3(c)) also show that the two masses are mostly out of phase, i.e., they move in opposite directions and reach their extremes at the same instant of time where the velocities become zero. The same degenerate similar mode is also approached for $k_1 = \mu_2 = 0{\cdot}005$ and $\alpha_2 = -\,0{\cdot}005$ (case (4), section 2) for which identical values have been obtained with X_1 and X_2 being replaced by each other, i.e., mass 1 is now almost stationary.

Simulation results are presented for $\alpha_1 = \mu_1 = 1$, $k_1 = \mu_2 = \alpha_2 = -\,0{\cdot}005$ (case (2), section 2) for which the entire motion is confined to mass 1. The initial values obtained are $X_1 = 0{\cdot}5$, $X_2 = 0{\cdot}043424$ which give a slightly banded modal curve. This banded nature is removed by adjustment in the third place of decimal to get the value of X_2 as 0·040125 for which the resulting modal curve is shown in Figure 4(a). This error can be removed by considering second order approximations in equation (15), the analysis of which is beyond the scope of this paper. As in the above cases, the displacement and velocity responses (Figures 4(b) and 4(c)) show that the trajectories intersect at non-zero points and the two masses are in phase. Identical values have been obtained for $k_1 = \mu_2 = -\,0{\cdot}005$ and $\alpha_2 = 0{\cdot}005$ (case (3), section 2) with the roles of X_1 and X_2 interchanged.

4. CONCLUDING REMARKS

In this study, the non-similar normal modes of free oscillation of a two-degrees-of-freedom, coupled, non-linear oscillator with quadratic and cubic non-linearities have been examined following the asymptotic methodology given in reference [13].

It has been shown analytically that for weak substructure coupling, the oscillator possesses localized non-similar modes, since the motion is approximately confined to only one of the two co-ordinates. This result has also been verified through simulation. It is to be noted that the modal trajectory passes periodically through a non-zero point, and not through the origin (the equilibrium point) of the two-dimensional space. In other words, the calculated modes are not "vibrations-in-unison" as also indicated by reference [15]. Such motions can be designated as motion in normal mode if Rosenberg's definition of normal modes [1] is relaxed. Stability of the localized modes is checked indirectly by the Runge–Kutta solutions that give bounded solution in every case.

The balancing diagrams (Figure 2) show that in the case when three stiffness terms are present, a similar normal mode corresponding to $c = 1$ (symmetric mode) always exists and is a vibration-in-unison. Same result is also obtained by applying the MMS [15] to the present discrete system. This method also yields the result that non-local, non-similar modes, not passing through the origin of the configuration plane, appear near the similar antisymmetric linear mode. However, in the limit when k_1, α_2/α_1 and $\mu_2/\mu_1 \to 0$, the degenerate non-similar normal mode could not be determined. The authors believe that slight variation of reference [15] may be needed to achieve this result.

The presence of linear and quadratic non-linear terms does not ensure global stability of the two-degrees-of-freedom conservative system, although it may be stable locally. The cubic non-linear terms are added to achieve the global stability. This can be proved very easily for the special case of the symmetric, similar normal modes.

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