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LETTERS TO THE EDITOR

PERIOD PRESERVATION AND PERIOD ADJUSTMENT IN THE NUMERICAL INTEGRATION OF THE LINEAR AND NON-LINEAR EQUATIONS OF MOTION

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1. INTRODUCTION

On examining the workings of a common parametric family of explicit numerical integrators in approximating harmonic motion it is observed [1] that the long-term energy preservation property of the scheme is independent of the phase error caused by it and that the perturbation of some parameters in the integration scheme can be advantageously used to advance or retard the computed motion so as to achieve, at least on the linear model used, a zero error in the computed period.

Herein, we propose to demonstrate that this phase variation extends to non-linear problems as well by numerically integrating the two classical non-linear [2] problems, that of the widely swinging pendulum and that of the mass–spring system with a purely cubic restoring force.

Specifically, we demonstrate that if the true period of the motion is known either theoretically or possibly by observation, then the scheme can be adjusted to exactly reproduce this period.

2. THE NUMERICAL INTEGRATOR

It predicts position x_1 and velocity y_1 at point $t + \tau$ from position x_0 and velocity y_0 at time t with

$$x_1 = x_0 + \tau \dot{x}_0 + \tfrac{1}{2}\tau^2 \ddot{x}_0, \quad y_1 = y_0 + \tfrac{1}{2}\tau(\dot{y}_0 + \alpha \dot{y}_1), \tag{1}$$

where τ is the time-step size and α the period controlling parameter. For the initial value problem $\dot{x} = -y$, $\dot{y} = x$, $x(0) = x_0$, $y(0) = y_0$, equation (1) becomes

$$x_1 = x_0 - \tau y_0 - \tfrac{1}{2}\tau^2 x_0,\; y_1 = y_0 + \tfrac{1}{2}\tau(x_0 + \alpha x_1), \tag{2}$$

associated with the characteristic equation

$$z^2 + 2(-1 + \tfrac{1}{4}\tau^2\beta)z + 1 = 0, \tag{3}$$

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where $\beta = 1 + \alpha$. For equation (2) to have a complex solution step size τ and, positive, parameter β must be related such that

$$0 < \tau^2\beta < 8. \tag{4}$$

Then, since $|z| = 1$, the two roots of equation (2) may be written as $z = \cos\theta \pm \mathrm{i}\sin\theta$ with

$$\cos\theta = 1 - \tfrac{1}{4}\tau^2\beta, \qquad \sin\theta = \tfrac{1}{4}\tau\sqrt{8\beta - \tau^2\beta^2}. \tag{5}$$

Scheme (2) reproduces the exact period $T = 2\pi$ if $\theta = \tau$ which occurs according to equation (4), if

$$\beta = 4(1 - \cos\tau)/\tau^2 \tag{6}$$

or

$$\alpha = 1 - \tfrac{1}{6}\tau^2 \tag{7}$$

if $\tau \ll 1$.

3. ENERGY CONSERVATION

The numerical integrator in equation (1) is said to be energy conserving by virtue of the fact that the roots of its characteristic equation (3) are distinct and of modulus one, and this is independent of parameter α. Then, the absolute value of the energy error is bounded for all time $t \geqslant 0$, with a bound that tends to zero as the time step tends to zero. Energy preservation does not mean that energy is conserved pointwise.

Indeed, consider $\ddot{x} + x = 0$, $x(0) = 1$, $y(0) = -\dot{x}(0) = 0$ for which we have that $x^2 + \dot{x}^2 = 1$ for all $t \geqslant 0$. Our numerical integrator with $\alpha = 1$ provides the approximations

$$x_n = x_0 \cos n\theta - y_0(1/\sqrt{1 - \tfrac{1}{4}\tau^2})\sin n\theta, \quad y_n = y_0\cos n\theta + x_0\sqrt{1 - \tfrac{1}{4}\tau^2}\sin n\theta \tag{8}$$

at time $t = n\tau$, or

$$x_n = \cos n\theta, \qquad y_n = \sqrt{1 - \tfrac{1}{4}r^2}\sin n\theta \tag{9}$$

if $x_0 = 1$, $y_0 = 0$, and

$$x_n^2 + y_n^2 = 1 - (\tfrac{1}{2}\tau\sin t)^2, \tag{10}$$

provided τ is small enough so that θ may be replaced by it. The error in the energy vanishes at $t = k\pi$, $k = 0, 1, 2, \ldots$, and is at most $\tau^2/4$ at $t = k\pi/2$ if k is odd.

4. EQUIVALENT MULTISTEP INTEGRATORS

Writing

$$x_1 = x_0 - \tau y_0 - \tfrac{1}{2}\tau^2 x_0, \quad y_1 = y_0 + \tfrac{1}{2}\tau(x_0 + x_1), \quad x_2 = x_1 - \tau y_1 - \tfrac{1}{2}\tau^2 x_1 \tag{11}$$

and eliminating y_0 and y_1 between the three above equations results in

$$\frac{1}{\tau^2}(-x_0 + 2x_1 - x_1) = x_1, \tag{12}$$

which is a central multistep difference scheme for $\ddot{x} + x = 0$.

The implicit predictor–corrector scheme

$$x_1 = x_0 + \tau\dot{x}_0 + \tfrac{1}{12}\tau^2(5\ddot{x}_0 + \ddot{x}_1), \quad y_1 = y_0 + \tfrac{1}{2}\tau^2(\dot{y}_0 + \dot{y}_1) \tag{13}$$

is similarly shown to be equivalent to the Numerov method

$$\frac{1}{\tau^2}(-x_0 + 2x_1 + x_2) = \frac{1}{12}(x_0 + 10x_1 + x_2), \tag{14}$$

which is known to be $0(\tau^4)$.

5. SPRING–MASS SYSTEM WITH A CUBIC RESTORING FORCE

For oscillation starting from rest, the normalized equation of motion of the system is

$$\ddot{x} + x^3 = 0, \qquad x(0) = 1, \qquad \dot{x}(0) = 0 \tag{15}$$

and it may be integrated to yield to total energy balance equation

$$2\dot{x}^2 + x^4 = 1 \quad \text{or} \quad \dot{x} = -(\sqrt{2}/2)\sqrt{1 - x^4}, \tag{16}$$

in which the negative sign is chosen to account for the restoring nature of the force exerted by the spring on the mass. Period T of the motion is given by

$$T = 4\sqrt{2}\int_0^1 \frac{\mathrm{d}x}{\sqrt{1 - x^4}} = 4\sqrt{2}\int_0^1 \frac{1}{\sqrt{1 - x^2}}\,\frac{1}{\sqrt{1 + x^2}}\,\mathrm{d}x \tag{17}$$

that with the trigonometric substitution $x = \sin\theta$, $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$, becomes

$$T = 4\sqrt{2}\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 + \sin^2\theta}} \tag{18}$$

and we evaluate the period as $T = 7{\cdot}4163$.

Figure 1(a) depicts the phase portrait of the non-linear mass–spring system and the computed points $P(x_n, \dot{x}_n)$ marked by full hollow moons. The computation was carried out with the step size $\tau = 7{\cdot}4163/30$ and $\alpha = 0{\cdot}99303$ determined experimentally to reproduce the exact period. Figure 1(a) actually shows 180 superimposed periods with the last point marked by an asterisk. No shifts are discerned in the computed points and the asterisk appears to precisely fall at $x = 1$, $\dot{x} = 0$.

6. EXTENSIVELY SWINGING PENDULUM

Now, we consider the equation

$$\ddot{\theta} + \sin\theta = 0, \qquad \theta(0) = \pi/2, \qquad \dot{\theta}(0) = 0 \tag{19}$$

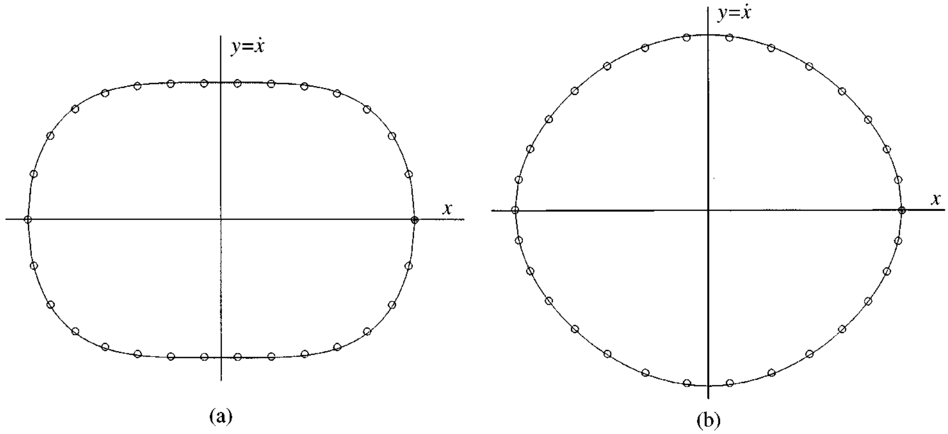

Figure 1. Computed points $P(x_n, \dot{x}_n)$ for the, (a) mass–spring system with a cubic restoring force, and (b) the widely swinging pendulum. The computation was carried out with $\tau = 7{\cdot}4163/30$ and $\alpha = 0{\cdot}99303$ over 180 periods. The last computed point is marked by an asterisk.

for swing angle θ. The variable change $\theta = \left(\frac{\pi}{2}x\right)$ transforms the equation of motion into

$$\ddot{x} + \frac{2}{\pi}\sin\left(\frac{\pi}{2}x\right) = 0, \qquad x(0) = 1, \qquad \dot{x}(0) = 0 \tag{20}$$

that can be integrated to yield the energy balance equation

$$\dot{x} = -\,(2\sqrt{2}/\pi)\sqrt{\cos\frac{\pi}{2}x}, \quad -1 \leqslant x \leqslant 1 \tag{21}$$

with the negative sign chosen to account for the restoring nature of the gravitational force acting on the pending bob.

Period T of the swinging pendulum starting its fall with zero velocity from a horizontal position is given in terms of the elliptic integral

$$T = 4\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2\sin^2\theta}}, \quad k = \sqrt{2}/2 \tag{22}$$

and we compute again $T = 7{\cdot}4163$.

Figure 1(b) shows the phase portrait of the pendulum with the computed points $P(x_n, \dot{x}_n)$. The computation was carried out with $\tau = 7{\cdot}4163/30$ and $\alpha = 0{\cdot}9903$. The figure shows 180 periods superimposed on the phase portrait with the last computed point marked by asterisk, that appears at time $t = 180 \times 7{\cdot}4163$ to be exactly where it should be.

REFERENCES

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