Journal of Sound and Vibration (2002) **250**(3), 567–568
doi:10.1006/jsvi.2001.3781, available online at http://www.idealibrary.com on IDEAL®

LETTERS TO THE EDITOR

MORE ON GENERALIZED HARMONIC OSCILLATORS

H. Hu

Department of Mathematics and Physics, Xiangtan Polytechnic University, Xiangtan 411201, *Hunan, People's Republic of China. E-mail*: *huihu2000@21cn.com*

(*Received* 10 *January* 2001, *and in final form* 1 *May* 2001)

Mickens [1] presented a new class of non-linear oscillator equations which take the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)y, \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = -g(y)x, \tag{1}$$

where $f(x)$ and $g(x)$ are assumed to be continuous with continuous first derivatives, and also satisfy the conditions

$$f(0) > 0, \qquad g(0) > 0. \tag{2}$$

The corresponding second order, non-linear differential equation is

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{f'(x)}{f(x)}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + f(x)g\left\{\frac{1}{f(x)}\frac{\mathrm{d}x}{\mathrm{d}t}\right\}x = 0, \tag{3}$$

where $f'(x) \equiv \mathrm{d}f/\mathrm{d}x$. The main purpose of this letter is to generalize equations (1). The generalized equations have the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)y^m, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -g(y)x^n, \tag{4}$$

where m and n are positive odd numbers.

Equations (4) can be rewritten in the form of a single second order differential equation. Obviously,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = mf(x)y^{m-1}\frac{\mathrm{d}y}{\mathrm{d}t} + f'(x)y^m\frac{\mathrm{d}x}{\mathrm{d}t}. \tag{5}$$

Using the fact that

$$y = \left\{\frac{1}{f(x)}\frac{\mathrm{d}x}{\mathrm{d}t}\right\}^{1/m} \tag{6}$$

and the second equation of equations (4), equation (5) has the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{f'(x)}{f(x)}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + mf(x)g\left\{\left(\frac{1}{f(x)}\frac{\mathrm{d}x}{\mathrm{d}t}\right)^{1/m}\right\}\left(\frac{1}{f(x)}\frac{\mathrm{d}x}{\mathrm{d}t}\right)^{(m-1)/m}x^n = 0. \tag{7}$$

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This equation is a generalized form of equation (3). If $m = n = 1$, then equation (7) coincides with equation (3). If $f(x) = g(x) = 1$, and $n = 3$, then equation (7) takes the form [2]

$$\frac{d^2x}{dt^2} + x^3 = 0. \tag{8}$$

This equation cannot be obtained from equation (3).

In the (x, y) phase space [3], the trajectories of equations (4) are determined by the first order differential equation

$$\frac{dy}{dx} = -\frac{g(y)x^n}{f(x)y^m}. \tag{9}$$

The fixed points or equilibria $(\bar{x}, \bar{y})$ correspond to the simultaneous solutions of the equations

$$g(\bar{y})\bar{x}^n = 0, \qquad f(\bar{x})\bar{y}^m = 0. \tag{10}$$

Obviously, $(\bar{x}, \bar{y}) = (0, 0)$ is always a fixed point. The first integral [4] of equation (7) can be determined by integrating equation (9); doing this yields

$$K(y) + V(x) = constant, \tag{11}$$

where

$$K(y) = \int_0^y \frac{z^m\,dz}{g(z)}, \qquad V(x) = \int_0^x \frac{w^n\,dw}{f(w)}. \tag{12}$$

$K(y)$ and $V(x)$ can be taken, respectively, as generalized kinetic and potential energies [1] for the generalized harmonic oscillator described by either equations (4) or (7). Taking into account the conditions given in equations (2), one has

$$K(y) = \int_0^y \frac{z^m\,dz}{g(z)} = \frac{y^{m+1}}{(m+1)g(0)} + O(y^{m+2}) \tag{13}$$

and

$$V(x) = \int_0^x \frac{w^n\,dw}{f(w)} = \frac{x^{n-1}}{(n+1)f(0)} + O(x^{n+2}). \tag{14}$$

Substituting equations (13) and (14) into equation (11), one can see that the fixed point $(\bar{x}, \bar{y}) = (0, 0)$ is a center.

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4. H. GOLDSTEIN 1980 *Classical Mechanics*. Reading, MA: Addison-Wesley.