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TRANSVERSE VIBRATIONS OF RECTANGULAR PLATES OF GENERALIZED ANISOTROPY AND DISCONTINUOUSLY VARYING THICKNESS

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1. INTRODUCTION

The main goal of the present study is the determination of the fundamental frequency of transverse vibration of the structural system depicted in Figure 1. It was motivated by the fact that materials of generalized anisotropy are commonly used in many technological situations on the one hand, and that no results are avilable for the case of discontinuously varying thickness which, on the other hand, is of considerable engineering importance.

The fundamental eigenvalue is obtained using the optimized Rayleigh–Ritz method for all possible combinations of clamped and simply supported edges.

In order to facilitate the algorithmic procedure the edges are assumed to posses a flexibility coefficient ϕ_i ($i = 1, 2, \ldots, 4$, see Figure 1). At the final stage of the numerical procedure ϕ_i is made to approach zero for a clamped (c) edge or infinity when the edge is simply supported (ss).

2. APPROXIMATE SOLUTION OF THE PROBLEM

Assuming normal modes of vibrations and using Lekhnitskii's classical notation [1] one expresses the governing functional in the form

$$J(W) = \iint_{\bar{P}} [D_{11} W_{\bar{x}^2}^2 + 2D_{12} W_{\bar{x}^2} W_{\bar{y}^2} + D_{22} W_{\bar{y}^2}^2 + 4D_{66} W_{\bar{x}\bar{y}}^2 + 4(D_{16} W_{\bar{x}^2} + D_{26} W_{\bar{y}^2}) W_{\bar{x}\bar{y}}]$$
$$\times \mathrm{d}\bar{x}\,\mathrm{d}\bar{y} - D_{11}(a, \bar{y}) \int_0^b W_{\bar{x}^2}(a, \bar{y})\, W_{\bar{x}}(a, \bar{y})\,\mathrm{d}\bar{y} - D_{22}(\bar{x}, b) \int_0^a W_{\bar{y}^2}(\bar{x}, b)\, W_{\bar{y}}(\bar{x}, b)\,\mathrm{d}\bar{x}$$
$$+ D_{11}(0, \bar{y}) \int_0^b W_{\bar{x}^2}(0, \bar{y})\, W_{x}(0, \bar{y})\,\mathrm{d}\bar{y} + D_{22}(\bar{x}, 0) \int_0^a W_{\bar{y}^2}(\bar{x}, 0)\, W_{\bar{y}}(\bar{x}, 0)\,\mathrm{d}\bar{x}$$
$$- \rho\omega^2 \iint_{\bar{P}} h W^2\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{y}, \qquad (1)$$

subject to the boundary conditions

$$W(0, \bar{y}) = W(a, \bar{y}) = W(\bar{x}, 0) = W(\bar{x}, b) = 0, \quad W_{\bar{x}}(0, \bar{y}) = \phi_1 D_{11}(0, \bar{y}) W_{\bar{x}^2}(0, \bar{y}),$$
$$W_{\bar{x}}(a, \bar{y}) = -\phi_2 D_{11}(a, \bar{y}) W_{\bar{x}^2}(a, \bar{y}),\ W_{\bar{y}}(\bar{x}, 0) = \phi_3 D_{22}(\bar{x}, 0) W_{\bar{y}^2}(\bar{x}, 0),$$
$$W_{\bar{y}}(\bar{x}, b) = -\phi_4 D_{22}(\bar{x}, b) W_{\bar{y}^2}(\bar{x}, b), \qquad (2)$$

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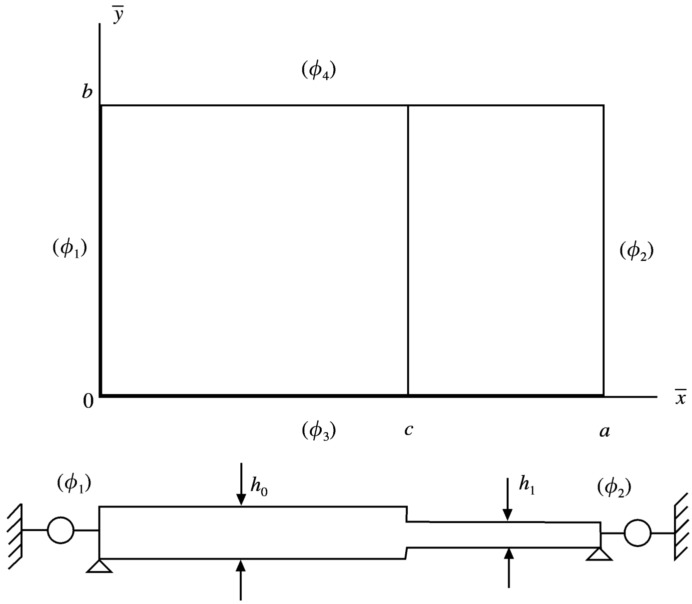


Figure 1. Mechanical system of generalized anisotropy executing transverse vibrations.

where

$$h(\bar{x}, \bar{y}) = \eta h_0 \quad \text{with } \eta = \begin{cases} 1, & 0 \leqslant \bar{x} < c, \\ \dfrac{h_1}{h_0}, & c < \bar{x} \leqslant a. \end{cases}$$

It is important to point out that equations (1) and (2) are approximate in the sense that the exact expressions of the edge normal bending moments are not used.

However, this is permissible since the end results are obtained for the situations where either the edge normal moments are zero or the edge rotations are null.

Introducing the dimensionless variables x and y,

$$\bar{x} = ax, \qquad \bar{y} = by \tag{3}$$

and substituting in equations (1) and (2) one obtains

$$\begin{aligned} \frac{a^2\lambda}{D_{110}} J(W) = {} & \iint_P \eta^3 \Bigg[W_{x^2}^2 + 2\,\frac{D_{120}}{D_{110}}\,\lambda^2\, W_{x^2} W_{y^2} + \frac{D_{220}}{D_{110}}\,\lambda^4\, W_{y^2}^2 + 4\,\frac{D_{660}}{D_{110}}\,\lambda^2\, W_{xy}^2 \\ & + 4\left(\frac{D_{160}}{D_{110}}\,\lambda\, W_{x^2} + \frac{D_{260}}{D_{110}}\,\lambda^3\, W_{y^2}\right) W_{xy} \Bigg] \mathrm{d}x\,\mathrm{d}y \\ & - \frac{\eta^3}{\lambda} \int_0^1 W_{x^2}(1, y)\, W_x(1, y)\,\mathrm{d}y - \frac{D_{220}}{D_{110}}\,\lambda^3 \int_0^1 \eta^3 W_{y^2}(x, 1)\, W_y(x, 1)\,\mathrm{d}x \\ & + \frac{1}{\lambda} \int_0^1 W_{x^2}(0, y)\, W_x(0, y)\,\mathrm{d}y + \frac{D_{220}}{D_{110}}\,\lambda^3 \int_0^1 \eta^3 W_{y^2}(x, 0)\, W_x(x, 0)\,\mathrm{d}x \\ & - \Omega^2 \iint_P \eta W^2\,\mathrm{d}x\,\mathrm{d}y, \end{aligned} \tag{4}$$

where

$$\lambda = \frac{a}{b}, \qquad \lambda_c = \frac{c}{a}, \qquad \Omega^2 = \frac{\rho h_0 a^4}{D_{110}}\,\omega^2, \qquad h(x, y) = \eta h_0, \qquad \eta = \begin{cases} 1, & 0 \leqslant x < \lambda_c, \\ \dfrac{h_1}{h_0}, & \lambda_c < x \leqslant 1. \end{cases}$$

$$W(0, y) = W(1, y) = W(x, 0) = W(x, 1) = 0,$$

$$W_x(0, y) = \phi_1' W_{x^2}(0, y), \qquad \phi_1' = \frac{\phi_1 D_{110}}{a},$$

$$W_x(1, y) = -\,\phi_2' W_{x^2}(1, y), \qquad \phi_2' = \frac{\phi_2 \eta^3 D_{110}}{a},$$

$$W_y(x, 1) = \phi_3' W_{y^2}(x, 0), \qquad \phi_3' = \frac{\phi_3 \eta^3 D_{220}}{b},$$

$$W_y(x, 1) = -\,\phi_4' W_{y^2}(x, 0), \qquad \phi_4' = \frac{\phi_4 \eta^3 D_{220}}{b}. \tag{5}$$

Obviously, since only clamped or simply supported edge conditions are considered, all the simple integrals appearing in equation (1) and equation (4) are identically zero.
Assuming

$$W_a = \sum_{j=1}^{N} C_j \varphi_j(x, y) \tag{6}$$

and making use of the classical Ritz approach one obtains

$$\begin{aligned} \frac{1}{2}\frac{a^2\lambda}{D_{110}}\frac{\partial J}{\partial C_i} = \sum_{j=1}^{N} \Bigg\{ \iint_P \eta^3 \Bigg\{ & \varphi_{jx^2}\varphi_{ix^2} + \frac{D_{120}}{D_{110}}\lambda^2 \left(\varphi_{jy^2}\varphi_{ix^2} + \varphi_{jx^2}\varphi_{iy^2}\right) \\ & + \frac{D_{220}}{D_{110}}\lambda^4 \varphi_{jy^2}\varphi_{iy^2} + 4\frac{D_{660}}{D_{110}}\lambda^2 \varphi_{jxy}\varphi_{ixy} \\ & + 2\left[\frac{D_{160}}{D_{110}}\lambda(\varphi_{jxy}\varphi_{ix^2} + \varphi_{jx^2}\varphi_{ixy}) \right. \\ & \left. + \frac{D_{260}}{D_{110}}\lambda^3(\varphi_{jxy}\varphi_{iy^2} + \varphi_{jy^2}\varphi_{ixy})\right]\Bigg\} \,\mathrm{d}x\,\mathrm{d}y \\ & - \Omega^2 \iint_P \eta\varphi_j\varphi_i \,\mathrm{d}x\,\mathrm{d}y \Bigg\} C_j = 0. \end{aligned} \tag{7}$$

The following co-ordinate functions are used:

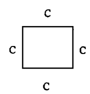
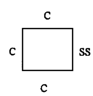
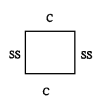
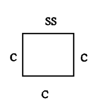
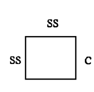
$$\varphi_j(x, y) = (x^q + \alpha_{j3}x^3 + \alpha_{j2}x^2 + \alpha_{j1}x)(y^q + \beta_{j3}y^3 + \beta_{j2}y^2 + \beta_{j1}y), \tag{8}$$

with $q = p + j - 1$ and where p is Rayleigh's optimization parameter.

Substituting equations (6) and (8) into equation (7) leads to a homogeneous system of linear equations in the C_j's. The non-triviality condition leads to a determinantal equation

TABLE 1

Frequency coefficients Ω_1 of rectangular plates of generalized anisotropy and discontinuously varying thickness for C–C–C–C, C–C–SS–C, SS–C–SS–C, C–C–C–SS *and* SS–C–C–SS *edge arrangements*

	c/a	h_1/h_0	$a/b = 3/2$	1	2/3
	0·25	0·8	43·99	29·60	23·95
		0·6	37·02	25·58	21·02
	0·50	0·8	46·81	30·65	24·34
		0·6	43·17	27·73	21·54
	0·75	0·8	49·79	31·93	25·10
		0·6	48·50	30·28	23·19
	0·25	0·8	40·91	25·43	18·91
		0·6	33·64	21·79	16·69
	0·50	0·8	43·34	26·44	19·31
		0·6	39·27	24·02	17·48
	0·75	0·8	46·53	27·87	20·17
		0·6	45·09	26·75	19·21
	0·25	0·8	39·72	22·79	14·94
		0·6	34·31	19·96	13·00
	0·50	0·8	42·32	23·96	15·63
		0·6	39·83	22·42	14·41
	0·75	0·8	44·83	25·13	16·36
		0·6	43·91	24·42	15·79
	0·25	0·8	38·53	28·10	23·54
		0·6	33·32	24·53	20·70
	0·50	0·8	40·85	29·06	23·96
		0·6	38·16	26·46	21·25
	0·75	0·8	42·30	29·87	24·66
		0·6	40·60	27·98	22·76
	0·25	0·8	36·61	24·10	18·10
		0·6	33·28	21·57	15·63
	0·50	0·8	38·57	25·05	18·82
		0·6	37·11	23·38	16·85
	0·75	0·8	39·56	25·61	19·30
		0·6	38·53	24·33	17·93

whose lowest root is the fundamental frequency coefficient of the system $\Omega_1 = \sqrt{\rho h_0/D_{110}}\,\omega_1 a^2$. An optimized value of Ω_1 is obtained, minimizing it with respect to p.

3. NUMERICAL RESULTS

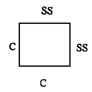
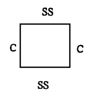
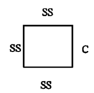
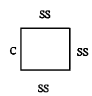
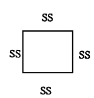
All calculations were performed for

$$\frac{D_{120}}{D_{110}} = \frac{D_{220}}{D_{110}} = \frac{D_{660}}{D_{110}} = 0{\cdot}5, \qquad \frac{D_{160}}{D_{110}} = \frac{D_{260}}{D_{110}} = \frac{1}{3}, \qquad N = 5, \qquad \lambda = \tfrac{3}{2},\ 1,\ \tfrac{2}{3},$$

$$\lambda_c = 0{\cdot}25,\ 0{\cdot}5,\ 0{\cdot}75, \qquad \eta = 0{\cdot}6,\ 0{\cdot}8.$$

TABLE 2

Frequency coefficients Ω_1 of rectangular plates of generalized anisotropy and discontinuously varying thickness for C–C–SS–SS, C–SS–C–SS, SS–SS–C–SS, C–SS–SS–SS *and* SS–SS–SS–SS *edge arrangements*

	c/a	h_1/h_0	$a/b = 3/2$	1	2/3
	0·25	0·8	34·90	23·60	18·45
		0·6	29·53	20·55	16·36
	0·50	0·8	37·06	24·57	18·86
		0·6	34·30	22·60	17·13
	0·75	0·8	38·92	25·51	19·65
		0·6	37·67	24·42	18·60
	0·25	0·8	33·77	26·31	22·82
		0·6	28·87	22·81	19·96
	0·50	0·8	35·06	26·92	23·11
		0·6	31·71	23·88	20·16
	0·75	0·8	36·44	27·86	23·92
		0·6	34·07	25·69	21·94
	0·25	0·8	30·74	21·69	17·07
		0·6	27·32	18·87	14·36
	0·50	0·8	31·93	22·42	17·75
		0·6	29·73	20·18	15·43
	0·75	0·8	33·00	23·17	18·36
		0·6	31·40	21·58	16·87
	0·25	0·8	29·90	21·68	17·63
		0·6	25·24	18·73	15·51
	0·50	0·8	31·09	22·26	17·88
		0·6	28·02	19·98	15·94
	0·75	0·8	32·48	23·17	18·75
		0·6	30·68	21·68	17·46
	0·25	0·8	27·56	18·22	13·24
		0·6	24·36	15·97	11·34
	0·50	0·8	28·69	18·89	13·79
		0·6	26·77	17·32	12·29
	0·75	0·8	29·83	19·65	14·51
		0·6	28·71	18·73	13·72

Tables 1 and 2 depict values of fundamental frequency coefficients of the system for all the possible combinations of clamped and simply supported edges. In view of the mechanical complexity of the problem and simplicity of the approach and co-ordinate functions employed, one may expect that the eigenvalues determined using the present methodology yield rather rough approximations for the problem under study. Nevertheless, the general approach may be useful for a designer's purpose.

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REFERENCES

1. S. G. Lekhnitskii 1968 *Anisotropic Plates*. New York, NY: Gordon and Breach Publishers, Inc.
2. P. A. A. Laura, L. E. Luisoni and C. P. Filipich 1977 *Journal of Sound and Vibration* **55**, 327–333. A note on the determination of the fundamental frequency of vibration of thin, rectangular plates with edges possessing different rotational flexibility coefficients.