Journal of Sound and Vibration (2002) **250**(5), 849–875
doi:10.1006/jsvi.2001.3958, available online at http://www.idealibrary.com on IDEAL®

THE EFFECT OF ROTATION UPON THE NATURAL FREQUENCIES OF A MASS–SPRING SYSTEM

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(*Received* 18 *April* 2001, *and in final form* 24 *July* 2001)

When a mass–spring system vibrates it does so with frequencies characteristic of the system. If the system as a whole now undergoes a rotational motion then these characteristic frequencies will change from their non-rotational values. It is the purpose of this paper to show how these changes may be calculated for a specified system and, in particular, to investigate the role in these changes of both the system and the rotational parameters. A system of N masses linked sequentially by springs in tension is allowed to vibrate about an equilibrium configuration both radially and transversely upon a smooth turntable. If the turntable is stationary then the radial and transverse vibrations are independent of each other, provided the amplitudes of vibration are sufficiently small. There are then N natural frequencies of vibration for each mode. However, when the turntable rotates then the Coriolis effects give rise to an interaction between the two modes of vibration, and there are now $2N$ natural frequencies for the combined vibrations. If the rate of rotation is "small" then the two modes are almost separated and it is possible to discuss the "essentially radial" or "essentially transverse" mode of vibration each of which has N natural frequencies. It is these natural frequencies which are considered in this work, in particular their dependence upon the rotation rate and upon the tension in the springs (when in the static configuration). In a previous paper, it was shown that if only radial vibrations are allowed (by admitting say a guide rail) then all the natural frequencies *decrease*, with increasing rotation rate, from their static values. It is shown that the opposite is the case here in that the "essentially radial" natural frequencies *increase* with increasing rotation rate. This is due to the Coriolis interaction with the transverse vibrations. The "essentially transverse" frequencies are also found and the nature of their dependence discussed. Also included in the analysis is the effect on the frequencies of the (weak) coupling between the motion of the masses and the rotation of the turntable as a consequence of the conservation of angular momentum. In addition to treating N being finite the limiting case of an infinite number of masses is considered to determine the natural frequencies of vibration of a continuous stretched string undergoing rotation.

1. INTRODUCTION

In a recent paper, Clarke and Morgan [1], the effect of rotation upon the natural frequencies of a mass–spring system was investigated. This system was considered to be mounted upon a turntable which was free to rotate, without friction, about an axis through the centre of the turntable. The vibrations were restricted, by perhaps a guide rail, to be purely radial relative to the turntable. In this paper, this restriction is relaxed and the vibrations may be both radial and transverse. Indeed if there is any rotation present then the vibrations have to be both due to the Coriolis effects which couple the radial and transverse motions, unless artificially restricted as in reference [1]. Arnold and Maunder [2] have provided an extensive array of other instances in which the Coriolis and centripetal effects have a considerable influence.

For the present study, consideration is given to a system consisting of a sequential array of identical masses connected to their neighbours by identical springs, with the first and the last being similarly connected to the axis and to the periphery of the turntable. The latter is "wound up" and the system allowed to assume its equilibrium configuration. This occurs when all the masses are aligned along the same radial line, at positions closer to the periphery (due to the centripetal forces) then they would have been had the turntable been stationary. The rate of rotation must be restricted in order that the masses remain on the turntable, or even more restrictively, so that all the springs remain in tension. If the masses are now perturbed from this configuration, either radially, transversely or both, then they will undergo vibrational motions. It is the frequencies associated with these motions that are sought here, and in particular their dependence upon the parameters representing the rate of rotation, the tension in the springs when static, and also upon the coupling between the masses and the turntable. The latter occurs because the total angular momentum remains constant.

Even if there is no imposed rotation the turntable will oscillate if the masses are given a transverse perturbation, as this will put a torque upon the turntable causing it to rotate, albeit in an oscillatory manner.

A basic but lucid account of the vibrations of such arrays (in the absence of any rotation) was given by French [3], and that work contributes towards the initial stages of the treatment of the problem considered here.

Firstly, a single-mass system is investigated, even though it may be regarded as a special case of the N-mass system. The purpose in doing so is to "set the scene" without the obfuscation of generality. Once this problem is formulated and solved the lessons learned there are utilized in the N-mass case, which does, however, introduce difficulties of analysis not present in the simple case.

In all cases, the governing equations are derived and then linearized about the equilibrium configuration. In the single-mass case these linearized equations may be solved without any further restrictions being imposed, and expressions for the two natural frequencies associated with the system may be simply derived.

For the N-mass system the linearized equations are too complicated for a solution to be readily found without further simplification (due to the fact that they consist of $2N$-linked differential-difference equations with "variable" coefficients). It is possible that progress could be made by numerically solving the ensuing eigenvalue problem, but that approach has been eschewed in favour of using perturbation methods valid for small rotation rates compared with the smallest (static) natural frequency. The perturbation scheme used is singular and so the method of multiple scales [4] is used to overcome this difficulty.

Simultaneously with this approximation another is also made, more out of convenience than necessity. The motion of the masses and the rotation of the turntable are coupled together if the ratio of the moments of inertia (about the centre of the turntable) of the masses and of the turntable is not negligible. In this work this ratio is considered small, but not negligible, and the expressions derived for the natural frequencies of vibration are then explicit in their dependence upon this ratio, rather than implicit if this smallness is not invoked.

These expressions are examined to see how the frequencies depend upon the rotation rate and upon the coupling between the masses and the turntable. All "essentially radial" frequencies (i.e., those that would be associated solely with radial vibrations in the absence of rotation) increase due to increasing rotation. This differs markedly from the case of purely radial vibration presented in reference [1], where the frequencies decrease with

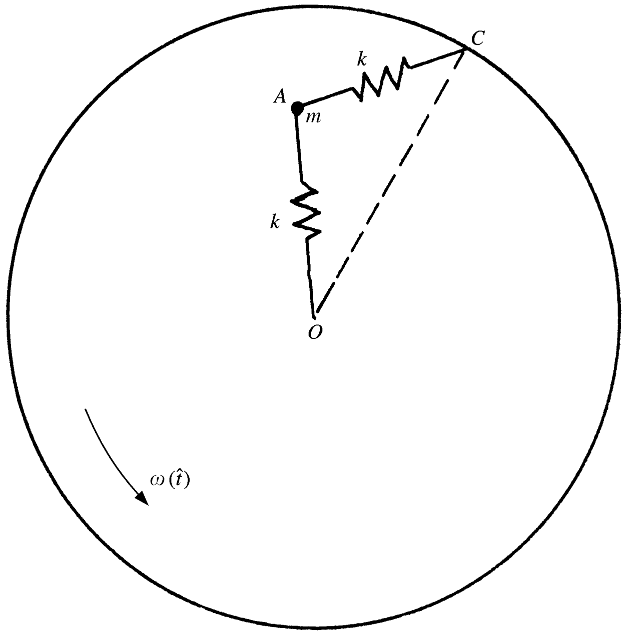

Figure 1. A single-mass system.

rotation. This difference is due to the interaction in the present case between the radial and transverse vibrations through the Coriolis effects, which are of course absent in the purely radial case. If coupling is present then the frequency increases are compounded. The level of (static) tension in the springs affects the values of the increases but not their parity. However, for the "essentially transverse" frequencies the dependence is somewhat more capricious. The details are given in section 5.

Finally in section 6 consideration is given to the case of an infinite number of infinitesimal masses connected by infinitesimal springs, i.e., $N \to \infty$, in other words a continuous elastic string. The "transverse" frequencies reduce to those familiar in the standard theory of the vibration of stretched strings, together with correction terms describing the influence of rotation and coupling.

Throughout this work, it is assumed that the motions take place in the plane of the turntable, this is it is a two-dimensional problem. It is also assumed that there is no dissipation of energy in the system, merely transfer from one form to another. Furthermore, the motions are taken to be such that the springs are always in tension (see, for instance, Appendix A for consideration of this point) and that the response of the springs to deformation is a linear one.

2. A SINGLE-MASS SYSTEM

Consider a smooth turntable of radius L which is free to rotate, without friction, in its own plane about its centre O. Upon this turntable is mounted a mass–spring system consisting (in this section) of a single mass (of mass m) connected via identical springs (of spring constant k) to both O and C, a point fixed on the circumference of the turntable. The mass is not confined to the line OC. See Figure 1.

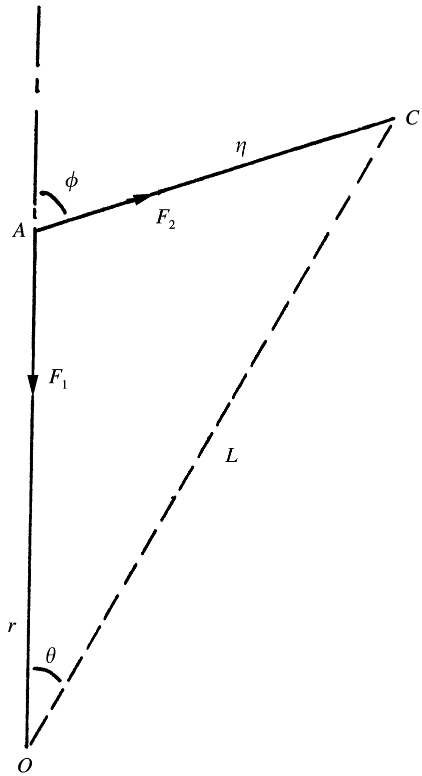

Figure 2. Showing the forces acting upon the mass and the geometric variables.

At any time $\hat{t}$ the turntable rotates with angular speed $\omega(\hat{t})$, when the mass is at distance $r(\hat{t})$ from O and $\eta(\hat{t})$ from C, the angles $\theta(\hat{t})$ and $\phi(\hat{t})$ being as in Figure 2. The relationships between these geometric variables are

$$\eta = \{L^2 + r^2 - 2Lr\cos\theta\}^{1/2}, \quad \text{and} \quad \eta\sin\phi = L\sin\theta, \tag{2.1}$$

i.e., r and θ will be taken to be the unknown variables, with η and ϕ being complementary variables given in terms of them by equations (2.1).

The mass will be subjected to forces F_1 and F_2 (see Figure 2) through the springs, these forces depending upon the extensions at that particular time. To quantify these forces, let the unstretched length of each spring be $\frac{1}{2}(1-\alpha)L$ for $0 < \alpha < 1$. Then $\frac{1}{2}\alpha L$ is the extension in the unperturbed (i.e., static) springs, and it follows that the (static) tension T in each spring is given by $T = \frac{1}{2}\alpha L k$. When the mass is in an arbitrary position (Figures 1 and 2) then F_1 is given by $F_1 = \{r - \frac{1}{2}(1-\alpha)L\}k$ and similarly for F_2.

The motion of the mass will be governed by Newton's law valid in an inertial frame of reference. In this frame the angular speed of the line OA is $\omega + \mathrm{d}\theta/\mathrm{d}\hat{t}$ and so the radial and transverse components of the governing equations are

$$m\left\{\frac{\mathrm{d}^2 r}{\mathrm{d}\hat{t}^2} - r\left(\omega + \frac{\mathrm{d}\theta}{\mathrm{d}\hat{t}}\right)^2\right\} = -k\left\{r - \frac{1}{2}(1-\alpha)L\right\} + k\left\{\eta - \frac{1}{2}(1-\alpha)L\right\}\cos\phi, \tag{2.2}$$

$$m\left\{r\left(\frac{\mathrm{d}\omega}{\mathrm{d}\hat{t}} + \frac{\mathrm{d}^2\theta}{\mathrm{d}\hat{t}^2}\right) + 2\frac{\mathrm{d}r}{\mathrm{d}\hat{t}}\left(\omega + \frac{\mathrm{d}\theta}{\mathrm{d}\hat{t}}\right)\right\} = -k\left\{\eta - \frac{1}{2}(1-\alpha)L\right\}\sin\phi. \tag{2.3}$$

During the motion there are no external forces or couples applied to the system, and so the total angular momentum of the system must remain constant: i.e.,

$$I\omega + mr^2(\omega + \mathrm{d}\theta/\mathrm{d}\hat{t}) = H \quad \text{(constant)}, \tag{2.4}$$

where I is the moment of inertia, about O, of the turntable.

There will be an equilibrium position in which the mass is on OC and the turntable rotates at constant speed, i.e., in this position

$$\theta = \phi = 0, \quad \omega = \Omega, \quad r = LR, \quad \eta = L(1 - R),$$

where R and Ω are constant. From equation (2.2)

$$R = 1/(2 - \mu^2), \quad \text{where } \mu^2 = m\Omega^2/k, \tag{2.5, 2.6}$$

i.e., μ is the ratio of the frequency of rotation to the "basic" frequency of vibration.

For the mass to remain upon the turntable then $\mu^2 < 1$, and more realistically for both springs to be in tension

$$\mu^2 < 2\alpha/(1 + \alpha). \tag{2.7}$$

This puts a limit upon the rotation speed if the subsequent motions are to be vibrational rather than of a "slapping" nature.

It also follows from equation (2.4) that

$$H = (I + mL^2R^2)\Omega. \tag{2.8}$$

Now to investigate small amplitude vibrations about the equilibrium position the following dimensionless variables are introduced:

$$\hat{t} = (m/k)^{1/2}\, t, r = L(R + \delta x), \theta = \delta\psi, \omega = \Omega(1 + \delta w), \tag{2.9}$$

where δ is a small dimensionless parameter. In what follows only terms of $O(\delta)$ are retained, and so

$$\eta \simeq L(1 - R - \delta x) \text{ and } \phi \sim \delta\psi/(1 - R), \tag{2.10}$$

while equations (2.2)–(2.4) become (with use of equation (2.8)) in terms of the perturbation variables

$$\mathrm{d}^2x/\mathrm{d}t^2 + (2 - \mu^2)x - \mu^2 2Rw - \mu 2R\,\mathrm{d}\psi/\mathrm{d}t = 0,$$

$$R\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2} + \left(1 - \frac{1}{2}\frac{1-\alpha}{1-R}\right)\psi + \mu R\frac{\mathrm{d}w}{\mathrm{d}t} + \mu 2\frac{\mathrm{d}x}{\mathrm{d}t} = 0,$$

$$(I + mL^2R^2)\mu R w + mL^2R^2(2\mu x + R\,\mathrm{d}\psi/\mathrm{d}t) = 0. \tag{2.11}$$

It is convenient at this stage to introduce the parameter

$$\hat{\varepsilon} = \frac{1}{4}mL^2/I, \tag{2.12}$$

which is the ratio of the moments of inertia of the mass (in its static position) and of the turntable. Also, a new "transverse" variable y is introduced by

$$y = R\psi. \tag{2.13}$$

The final equation of equation (2.11) is used to eliminate w from the first and second to achieve

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \left\{2 - \mu^2 + \frac{16\hat{\varepsilon}R^2}{1 + 4\hat{\varepsilon}R^2}\mu^2\right\}x - \frac{2\mu}{1 + 4\hat{\varepsilon}R^2}\frac{\mathrm{d}y}{\mathrm{d}t} = 0,$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{1}{R}\left\{1 - \frac{1}{2}\frac{1-\alpha}{1-R}\right\}(1 + 4\hat{\varepsilon}R^2)y + 2\mu\frac{\mathrm{d}x}{\mathrm{d}t} = 0. \tag{2.14}$$

For $\mu \neq 0$ the radial and transverse displacements are coupled by the Coriolis terms (that is the final ones in each equation) which means that there will be no purely radial and transverse frequency of vibration.

If solutions of equations (2.14) are sought for x and y both proportional to $e^{i\lambda t}$, then by standard methods it is readily found that there are two natural frequencies $\lambda_{(+)}$ and $\lambda_{(-)}$ such that

$$\begin{aligned}\lambda^2_{(\pm)} = \frac{1}{2}\Bigg\{&2 + 3\mu^2 + \frac{1}{R}(1 + 4\hat{\varepsilon}R^2)\left(1 - \frac{1}{2}\frac{1-\alpha}{1-R}\right)\\ &\pm\Bigg[\left\{2 + 3\mu^2 + \frac{1}{R}(1 + 4\hat{\varepsilon}R^2)\left(1 - \frac{1}{2}\frac{1-\alpha}{1-R}\right)\right\}^2\\ &- \frac{4}{R}(1 + 4\hat{\varepsilon}R^2)\left(1 - \frac{1}{2}\frac{1-\alpha}{1-R}\right)\left\{2 - \mu^2 + \frac{\mu^2 16\hat{\varepsilon}R^2}{1 + 4\hat{\varepsilon}R^2}\right\}\Bigg]^{1/2}.\end{aligned} \tag{2.15}$$

(Note that if $\mu^2 = 2\alpha/(1+\alpha)$, i.e., $R = \frac{1}{2}(1+\alpha)$, then $\lambda_{(-)} = 0$ and the motion is not vibrational, as commented upon after restriction (2.7).)

If the turntable was initially stationary, i.e., $\Omega = 0$, and so $\mu = 0$, then equations (2.14) are not coupled and

$$\lambda_{(+)} = 2^{1/2} \text{ and } \lambda_{(-)} = \{2\alpha(1 + \hat{\varepsilon})\}^{1/2}. \tag{2.16}$$

The frequency of the (truly) radial vibrations is independent of both the coupling parameter ($\hat{\varepsilon}$) and the pre-tension (α) in the springs, whereas the transverse frequency $\lambda_{(-)}$ depends upon both. The dependence upon $\hat{\varepsilon}$ is due to the fact that as the turntable is free to rotate, then it will do so if there is any transverse motion, but not if there is only radial motion.

Even though equation (2.15) gives the frequencies explicitly in terms of the parameters μ, $\hat{\varepsilon}$ and α, it is not simple to see "at a glance" how these parameters influence the frequencies. However in many practical circumstances, the parameters μ and $\hat{\varepsilon}$ will be small and the frequencies, as given by equation (2.15), may be expressed approximately as an expansion in these small parameters up to $O(\mu^2)$, $O(\hat{\varepsilon})$ and $O(\hat{\varepsilon}\mu^2)$. These are

$$\lambda_{(+)} \simeq 2^{1/2}\left\{1 + \mu^2\left[\frac{1}{4}\frac{3+\alpha}{1-\alpha} + \hat{\varepsilon}\left(\frac{\alpha}{1-\alpha}\right)^2\right]\right\},$$

$$\lambda_{(-)} \simeq (2\alpha)^{1/2}\left\{1 + \frac{1}{2}\hat{\varepsilon} - \frac{\mu^2}{4\alpha}\left[\frac{1+3\alpha}{1-\alpha} + \frac{\hat{\varepsilon}}{2(1-\alpha)^2}(1 - 10\alpha + 21\alpha^2 - 4\alpha^3)\right]\right\}. \tag{2.17}$$

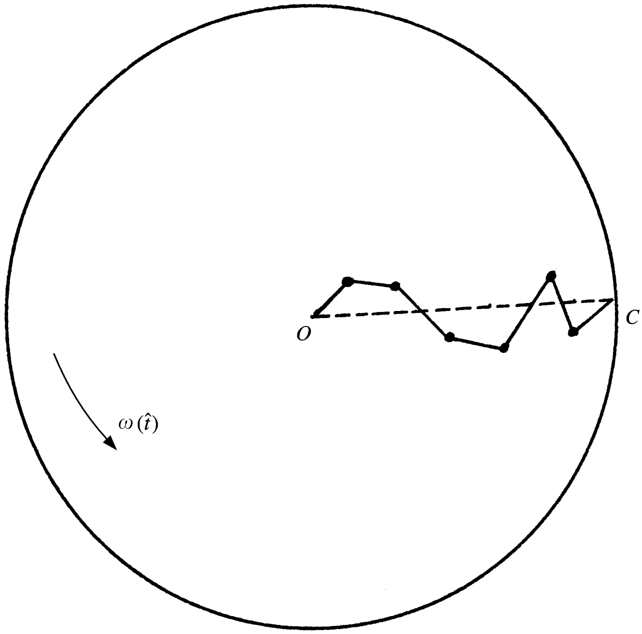

Figure 3. A six-mass system.

From equations (2.17), it may be seen that the "radial" frequency $\lambda_{(+)}$ *increases* with increasing rotation rate and the presence of coupling ($\hat{\varepsilon} > 0$) compounds this increase. It should be noted that this differs from the purely radial case considered in reference [1], where increasing rotation caused a *decrease* in the frequency. Obviously, the linking of radial and transverse modes via the Coriolis effects is responsible for this reversal of behaviour. However, for the "transverse" frequency $\lambda_{(-)}$ matters are a little more complicated. In the absence of coupling ($\hat{\varepsilon} = 0$) the frequency will decrease with increasing rotation rate, but in the presence of coupling it will increase in the absence of rotation ($\mu = 0$). If both coupling and rotation are present ($\mu > 0$, $\hat{\varepsilon} > 0$) then the above comments need to be moderated by the presence of the $O(\mu^2\hat{\varepsilon})$ term which is positive for $0{\cdot}1400 < \alpha < 0{\cdot}3769$ and negative for all other α in (0, 1); that is the response depends upon the static tension in the springs.

3. AN N-MASS SYSTEM: FORMULATION

The approach of section 2 is adapted for a system in which there is an arbitrary number (N) of identical masses, of mass m, connected sequentially by identical springs of spring constant k and natural length $(1 - \alpha)L/(N + 1)$, for $0 < \alpha < 1$, and L is again the radius of the turntable upon which the system is mounted. The masses are labelled by n for $n = 1, 2, 3, \ldots, N$, and the innermost mass $n = 1$ and the outer mass $n = N$ are connected to the centre O and the circumference at C of the turntable, respectively, by similar springs. As an example, Figure 3 displays a snapshot of a configuration, at time $\hat{t}$, of a six-mass system.

Let the jth mass have polar co-ordinates (r_j, θ_j) relative to the turntable, with the origin at the centre and θ_j is measured from OC in an anticlockwise direction. Here, $j = 1, 2, 3, \ldots, N$ and it is convenient to define, for completeness

$$r_0 = 0, \quad \theta_0 = 0, \quad r_{N+1} = L, \quad \theta_{N+1} = 0. \tag{3.1}$$

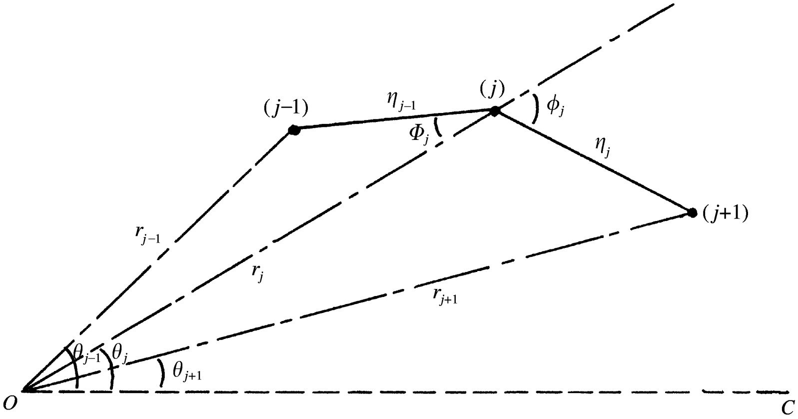

Figure 4. The *j*th mass and its neighbours.

The distance, at any time $\hat{t}$, between the jth and the $(j+1)$th mass is denoted by η_j and the angles ϕ_j and Φ_j are as shown in Figure 4.

The relationship between the neighbouring variables may be derived from the trigonometry appropriate to the two triangles in Figure 4.

$$\begin{aligned}
\eta_{j-1} &= \{r_j^2 + r_{j-1}^2 - 2r_j r_{j-1}\cos(\theta_{j-1}-\theta_j)\}^{1/2},\\
\eta_j &= \{r_{j+1}^2 + r_j^2 - 2r_{j+1} r_j\cos(\theta_j-\theta_{j+1})\}^{1/2},\\
\eta_{j-1}\sin\Phi_j &= r_{j-1}\sin(\theta_{j-1}-\theta_j),\\
\eta_j\sin\phi_j &= r_{j+1}\sin(\theta_j-\theta_{j+1}).
\end{aligned}\tag{3.2}$$

Now, upon bearing in mind that the angular speed of the jth mass in the inertial frame is $\omega + \mathrm{d}\theta_j/\mathrm{d}\hat{t}$, Newton's Law for the jth mass may be written as

$$m\left\{\frac{\mathrm{d}^2 r_j}{\mathrm{d}\hat{t}^2} - r_j\left(\omega + \frac{\mathrm{d}\theta_j}{\mathrm{d}\hat{t}}\right)^2\right\} = -k\left\{\eta_{j-1} - \frac{(1-\alpha)L}{N+1}\right\}\cos\Phi_j + k\left\{\eta_j - \frac{(1-\alpha)L}{N+1}\right\}\cos\phi_j$$

and

$$\begin{aligned}
m\left\{r_j\left(\frac{\mathrm{d}\omega}{\mathrm{d}\hat{t}} + \frac{\mathrm{d}^2\theta_j}{\mathrm{d}\hat{t}^2}\right) + 2\frac{\mathrm{d}r_j}{\mathrm{d}\hat{t}}\left(\omega + \frac{\mathrm{d}\theta_j}{\mathrm{d}\hat{t}}\right)\right\} &= k\left\{\eta_{j-1} - \frac{(1-\alpha)L}{N+1}\right\}\sin\Phi_j\\
&\quad - k\left\{\eta_j - \frac{(1-\alpha)L}{N+1}\right\}\sin\phi_j.
\end{aligned}\tag{3.3}$$

Equations (3.3) hold for $j = 1, 2 \cdots N$, with the extreme values given by equations (3.1).

In addition the total angular momentum of the system will remain constant, and so

$$I\omega + \sum_{j=1}^{N} m r_j^2\left(\omega + \frac{\mathrm{d}\theta_j}{\mathrm{d}\hat{t}}\right) = H, \tag{3.4}$$

where H is constant. For a given H this equation may be regarded as determining ω in terms of the other variables.

As in the single-mass case there will be an equilibrium configuration in which no vibrations occur. For this let

$$r_j = LR_j, \quad \omega = \Omega, \quad \theta_j = \phi_j = \Phi_j = 0 \tag{3.5}$$

and so

$$\eta_{j-1} = L(R_j - R_{j-1}), \quad \eta_j = L(R_{j+1} - R_j) \quad \text{and } R_0 = 0,\, R_{N+1} = 1. \tag{3.6}$$

Here, the R_j and Ω are independent of time.

When these forms are put into the first of equations (3.3) and (3.4) they yield that

$$-R_{j+1} + (2 - \mu^2)R_j - R_{j-1} = 0 \tag{3.7}$$

and that

$$H = (I + mL^2\sigma^2)\Omega, \tag{3.8}$$

where

$$\mu^2 = m\Omega^2/k \quad \text{and} \quad \sigma^2 = \sum_{j=1}^{N} R_j^2. \tag{3.9, 3.10}$$

The solution to the difference equation (3.7), subject to the end conditions (3.6), is (see reference [3])

$$R_j = \sin j\beta/\sin(N + 1)\beta, \tag{3.11}$$

where

$$\cos\beta = 1 - \frac{1}{2}\mu^2, \tag{3.12}$$

β being such that $(N + 1)\beta < \pi/2$.

Notice that if $\mu = 0$ then $\beta = 0$ and $R_j = j/(N + 1)$, the static case in which the masses are equally spaced. Also, if μ is small then $\beta \simeq \mu$.

If there is to be a vibrational motion then all the springs have to be in tension, which implies that they will certainly have to be so when in the equilibrium configuration. This puts a restriction upon the values that μ can take, and so upon the equilibrium rotation rate Ω. This is considered in Appendix A, the outcome being that if

$$\mu < 2\sin\left\{\frac{3\alpha(N + 1)}{2(3N^2 + 3N + 1)(N + \alpha)}\right\}^{1/2}, \tag{3.13}$$

then all the springs will be in tension. This is a sufficient condition, but not a necessary one as may be judged by comparing this condition for $N = 1$ with the "exact" condition for the single-mass case given by equation (2.7).

4. AN N-MASS SYSTEM: SMALL VIBRATIONS ABOUT THE EQUILIBRIUM CONFIGURATION

If the masses are disturbed from their joint equilibrium positions they will vibrate, provided that the rotation rate is not too large. It is assumed that these vibrations are of small amplitude and so dimensionless variables are introduced to reflect this smallness as follows:

$$\hat{t} = (m/k)^{1/2}\, t, \quad r_j = L(R_j + \delta x_j), \quad \theta_j = \delta\psi_j, \quad \omega = \Omega(1 + \delta w) \tag{4.1}$$

for $\delta \ll 1$. In what follows, only those terms of $O(\delta)$ are retained, that is the governing equations are being linearized about the equilibrium configuration.

Equations (3.2) become

$$\eta_{j-1} = L\{R_j - R_{j-1} + \delta(x_j - x_{j-1})\}, \quad \eta_j = L\{R_{j+1} - R_j + \delta(x_{j+1} - x_j)\},$$

$$\Phi_j = \delta\, \frac{R_{j-1}}{R_j - R_{j-1}}(\psi_{j-1} - \psi_j), \quad \phi_j = \delta\, \frac{R_{j+1}}{R_{j+1} - R_j}(\psi_j - \psi_{j+1}). \tag{4.2}$$

When equations (4.1) and (4.2) are put into equations (3.3) and (3.4) and linearization is effected the governing equations become

$$\mathrm{d}^2 x_j/\mathrm{d}t^2 - x_{j+1} + (2 - \mu^2)x_j - x_{j-1} - 2\mu^2 R_j w - 2\mu R_j\, \mathrm{d}\psi_j/\mathrm{d}t = 0,$$

$$R_j \frac{\mathrm{d}^2\psi_j}{\mathrm{d}t^2} + \mu R_j \frac{\mathrm{d}w}{\mathrm{d}t} + 2\mu \frac{\mathrm{d}x_j}{\mathrm{d}t} - \left\{1 - \frac{(1-\alpha)/(N+1)}{R_j - R_{j-1}}\right\} R_{j-1}(\psi_{j-1} - \psi_j)$$

$$+ \left\{1 - \frac{(1-\alpha)/(N+1)}{R_{j+1} - R_j}\right\} R_{j+1}(\psi_j - \psi_{j+1}) = 0 \tag{4.3}$$

and

$$\mu w = \frac{-mL^2}{I + mL^2\sigma^2}\left\{2\mu \sum_{n=1}^{N} R_n x_n + \sum_{n=1}^{N} R_n^2 \frac{\mathrm{d}\psi_n}{\mathrm{d}t}\right\}, \tag{4.4}$$

where μ and σ are as defined in equations (3.9) and (3.10).

The new transverse variables y_j, defined by $y_j = R_j\psi_j$, are introduced, and equation (4.4) is used to eliminate w from equation (4.3) to yield

$$\frac{\mathrm{d}^2 x_j}{\mathrm{d}t^2} - x_{j+1} + (2 - \mu^2)x_j - x_{j-1} + 4\mu^2 \frac{mL^2}{I + mL^2\sigma^2} R_j \sum_{n=1}^{N} R_n x_n$$

$$- 2\mu\left\{\frac{\mathrm{d}y_j}{\mathrm{d}t} - \frac{mL^2}{I + mL^2\sigma^2} R_j \sum_{n=1}^{N} R_n \frac{\mathrm{d}y_n}{\mathrm{d}t}\right\} = 0 \tag{4.5}$$

and

$$\frac{\mathrm{d}^2 y_j}{\mathrm{d}t^2} - \frac{mL^2}{I + mL^2\sigma^2} R_j \sum_{n=1}^{N} R_n \frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} - \left\{1 - \frac{(1-\alpha)/(N+1)}{R_j - R_{j-1}}\right\}\left(y_{j-1} - \frac{R_{j-1}}{R_j} y_j\right)$$

$$+ \left\{1 - \frac{(1-\alpha)/(N+1)}{R_{j+1} - R_j}\right\}\left(\frac{R_{j+1}}{R_j} y_j - y_{j+1}\right) + 2\mu\left\{\frac{\mathrm{d}x_j}{\mathrm{d}t} - \frac{mL^2}{I + mL^2\sigma^2} R_j \sum_{n=1}^{N} R_n \frac{\mathrm{d}x_n}{\mathrm{d}t}\right\} = 0. \tag{4.6}$$

Equation (4.6) may be simplified by the artifice of multiplying that equation by R_j and summing over all j to obtain

$$\frac{mL^2}{I + mL^2\sigma^2}\left\{\sum_{n=1}^{N} R_n \frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} + 2\mu \sum_{n=1}^{N} R_n \frac{\mathrm{d}x_n}{\mathrm{d}t}\right\} + \frac{mL^2}{I}\left\{1 - \frac{(1-\alpha)/(N+1)}{1 - R_N}\right\} y_N = 0,$$

where the extreme values of R_j and y_j have been used in this derivation, and so the second equation may be written as

$$\frac{d^2 y_j}{dt^2} - \left\{1 - \frac{(1-\alpha)/(N+1)}{R_j - R_{j-1}}\right\}\left(y_{j-1} - \frac{R_{j-1}}{R_j} y_j\right) + \left\{1 - \frac{(1-\alpha)/(N+1)}{R_{j+1} - R_j}\right\}\left(\frac{R_{j+1}}{R_j} y_j - y_{j+1}\right)$$
$$+ \frac{mL^2}{I} R_j \left\{1 - \frac{(1-\alpha)/(N+1)}{1 - R_N}\right\} y_N + 2\mu \frac{dx_j}{dt} = 0. \quad (4.7)$$

The pair of equations (4.5) and (4.7), which are coupled if $\mu > 0$ then govern the vibrations of the system. In addition the extreme values x_0, x_{N+1}, y_0 and y_{N+1} all vanish.

5. THE NATURAL FREQUENCIES AT SMALL ROTATION RATE AND WITH WEAK COUPLING

Even after linearization, the governing equations remain too complicated (due to the dependence of the coefficients upon j) for a simple analytical solution for unrestricted values of the parameters μ, α and mL^2/I. However if μ is small, that is the rotation rate is much less than any natural frequency of the static system then analytical progress can be made in the form of a perturbation scheme based upon small μ. This scheme is singular but the method of multiple scales may be used to overcome this difficulty.

At the same time, the further simplifying assumption of weak coupling will be made. The notion of weak coupling is quantified by taking the ratio of the total moment of inertia of the mass system, in its static configuration, and the moment of inertia of the turntable to be small.

That is

$$\hat{\varepsilon} = mL^2\sigma_0^2/I \ll 1, \quad (5.1)$$

where

$$\sigma_0^2 = \sum_{j=1}^{N}\left(\frac{j}{N+1}\right)^2 = \frac{N(2N+1)}{6(N+1)}.$$

Then

$$mL^2/I = \hat{\varepsilon}/\sigma_0^2 = \varepsilon \ll 1$$

and

$$\frac{mL^2}{I + mL^2\sigma^2} = \frac{\varepsilon}{1 + \varepsilon\sigma^2}.$$

(Note that while $\hat{\varepsilon}$ is the parameter with direct physical interpretation, $\hat{\varepsilon}/\sigma_0^2$ occurs so frequently in what follows that setting $\varepsilon = \hat{\varepsilon}/\sigma_0^2$ is notationally convenient.) Whilst taking ε to be small is not essential for progress to be made it certainly makes it simpler and the final outcome is explicit in its dependence upon ε rather than implicit. It is likely that most practical applications will be such that both μ and ε will be small and so these simplifications will be appropriate.

In equations (4.5) and (4.7) expressions depending upon μ are expanded up to and including terms of $O(\mu^2)$. For example,

$$R_j = \frac{j}{N+1}\left\{1 + \frac{1}{6}[(N+1)^2 - j^2]\mu^2 + \cdots\right\}.$$

Also, only those terms up to $O(\varepsilon)$ are retained, though also retaining those of $O(\varepsilon\mu^2)$. Equations (4.5) and (4.7) may now be rewritten as

$$\frac{\mathrm{d}^2 x_j}{\mathrm{d}t^2} - x_{j+1} + 2x_j - x_{j-1} = 2\mu\left\{\frac{\mathrm{d}y_j}{\mathrm{d}t} - \varepsilon\frac{j}{(N+1)^2}\sum_{n=1}^{N} n\frac{\mathrm{d}y_n}{\mathrm{d}t}\right\},$$
$$+ \mu^2\left\{x_j - 4\varepsilon\frac{j}{(N+1)^2}\sum_{n=1}^{N} nx_n\right\} + \cdots \tag{5.2}$$

and

$$\frac{\mathrm{d}^2 y_j}{\mathrm{d}t^2} + \alpha\left\{-y_{j+1} + 2y_j - y_{j-1} + \varepsilon\frac{j}{N+1}y_N\right\} = -2\mu\frac{\mathrm{d}x_j}{\mathrm{d}t}$$
$$+ \mu^2\left\{\alpha y_j + \frac{(1-\alpha)}{6}[\{N+1)^2 - 3j^2 - 3j - 1\}y_{j+1}\right.$$
$$+ \{(N+1)^2 - 3j^2 + 3j - 1\}y_{j-1} - \{2(N+1)^2 - 6j^2 - 8\}y_j]$$
$$\left. - \frac{\varepsilon}{6}\frac{j}{N+1}[\alpha\{(N+1)^2 - j^2\} - (1+\alpha)N(2N+1)y_N]\right\} + \cdots \tag{5.3}$$

Solutions to equations (5.2) and (5.3) are to be sought in the form of a power series in the small parameter μ, noting that if $\mu = 0$ then the two equations are uncoupled. However, in pursuing this scheme secularity arises and so, to circumvent this, the method of multiple scales is used. To this end, the slow time variable τ is introduced by

$$\tau = \mu^2 t \tag{5.4}$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + \mu^2\frac{\partial}{\partial\tau} \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} \equiv \frac{\partial^2}{\partial t^2} + \mu^2 2\frac{\partial^2}{\partial t\partial\tau}$$

and solutions are sought in the form

$$x_j(t,\mu) = x_j^{(0)}(t,\tau) + \mu x_j^{(1)}(t,\tau) + \mu^2 x_j^{(2)}(t,\tau) + \cdots \tag{5.5}$$

and similarly for $y_j(t,\mu)$. See the comments after equation (5.19) which give the reasons for choosing the slow variable to be proportional to μ^2 rather than to μ.

When equations (5.5) and (5.4) are put into equations (5.2) and (5.3) and the coefficients of like powers of μ are compared, a sequence of problems in $x_j^{(p)}$, $y_j^{(p)}$ for $p = 0$, 1 and 2 is derived, for each of which it is also required that

$$x_0^{(p)} = x_{N+1}^{(p)} = y_0^{(p)} = y_{N+1}^{(p)} = 0. \tag{5.6}$$

(i) *The zero order problem*:

$$Dx_j^{(0)} \equiv \frac{\partial^2}{\partial t^2}x_j^{(0)} - x_{j+1}^{(0)} + 2x_j^{(0)} - x_{j-1}^{(0)} = 0,$$

$$Ey_j^{(0)} \equiv \frac{\partial^2}{\partial t^2}y_j^{(0)} + \alpha\left\{-y_{j+1}^{(0)} + 2y_j^{(0)} - y_{j-1}^{(0)} + \varepsilon\frac{j}{N+1}y_N^{(0)}\right\} = 0. \tag{5.7}$$

(ii) *The first order problem*:

$$Dx_j^{(1)} = 2\left\{\frac{\partial}{\partial t} y_j^{(0)} - \varepsilon \frac{j}{(N+1)^2} \sum_{n=1}^{N} n \frac{\partial}{\partial t} y_n^{(0)}\right\},$$

$$Ey_j^{(1)} = -2 \frac{\partial}{\partial t} x_j^{(0)}. \tag{5.8}$$

(iii) *The second order problem*:

$$Dx_j^{(2)} = -2 \frac{\partial^2}{\partial t \partial \tau} x_j^{(0)} + 2\left\{\frac{\partial}{\partial t} y_j^{(1)} - \varepsilon \frac{j}{(N+1)^2} \sum_{n=1}^{N} n \frac{\partial}{\partial t} y_n^{(1)}\right\} + x_j^{(0)} - 4\varepsilon \frac{j}{(N+1)^2} \sum_{n=1}^{N} n x_n^{(0)},$$

$$\begin{aligned} Ey_j^{(2)} = &-2 \frac{\partial^2}{\partial t \partial \tau} y_j^{(0)} - 2 \frac{\partial}{\partial t} x_j^{(1)} + \alpha y_j^{(0)} - \varepsilon \frac{j}{6(N+1)} \{\alpha[(N+1)^2 - j^2] \\ &- (1-\alpha) N(2N+1)\} y_N^{(0)} + \frac{1}{6}(1-\alpha)\{[(N+1)^2 - 3j^2 - 3j - 1] y_{j+1}^{(0)} \\ &+ [(N+1)^2 - 3j^2 + 3j - 1] y_{j-1}^{(0)} - [2(N+1)^2 - 6j^2 - 8] y_j^{(0)}\}. \end{aligned} \tag{5.9}$$

These problems are now solved sequentially, ensuring that the solutions satisfy the extreme values (5.6) and also ensuring that no secular terms arise. The solution to the zero order problem is readily seen to be

$$x_j^{(0)}(t, \tau) = \sum_{n=1}^{N} A_n^{(0)}(\tau) \sin j\gamma_n \exp\{\mathrm{i}\lambda_n t\}, \tag{5.10}$$

$$y_j^{(0)}(t, \tau) = \sum_{n=1}^{N} B_n^{(0)}(\tau) \left\{\sin j v_n - \frac{j}{N+1} \sin(N+1) v_n\right\} \exp\{\mathrm{i}\alpha^{1/2} q_n t\}, \tag{5.11}$$

where

$$\lambda_n^2 = 2(1 - \cos\gamma_n), \text{ i.e. } \lambda_n = 2\sin\left(\frac{1}{2}\gamma_n\right) \tag{5.12}$$

with

$$\gamma_n = n\pi/(N+1), \quad \text{for } n = 1, 2, 3 \cdots N \tag{5.13}$$

and where

$$q_n^2 = 2(1 - \cos v_n), \text{ i.e. } q_n = 2\sin(1/2\, v_n) \tag{5.14}$$

with the v_n being the N distinct solutions of

$$q_n^2 \sin(N+1) v_n + \varepsilon\left\{\sin N v_n - \frac{N}{N+1} \sin(N+1) v_n\right\} = 0. \tag{5.15}$$

It follows that for small ε

$$v_n = \gamma_n + \varepsilon \frac{\sin\gamma_n}{\lambda_n^2(N+1)} + \cdots \quad \text{and} \quad q_n^2 = \lambda_n^2 + \varepsilon \frac{2\sin^2\gamma_n}{\lambda_n^2(N+1)} + \cdots. \tag{5.16}$$

The "amplitudes" $A_n^{(0)}(\tau)$ and $B_n^{(0)}(\tau)$ will to some extent remain arbitrary, as no initial conditions are being prescribed, but their dependence upon τ will be found by the no-secularity condition as will be seen.

To obtain the first order solutions, the zero order solutions (5.10) and (5.11) are put into the right-hand side of the first order equations (5.8) to obtain

$$Dx_j^{(1)} = \sum_{n=1}^{N} \mathrm{i}2\alpha^{1/2} q_n B_n^{(0)} \sin j v_n \exp\{\mathrm{i}\alpha^{1/2} q_n t\}, \tag{5.17}$$

$$Ey_j^{(1)} = -\sum_{n=1}^{N} \mathrm{i}2\lambda_n A_n^{(0)} \sin j\gamma_n \exp\{\mathrm{i}\lambda_n t\}. \tag{5.18}$$

The derivation of equation (5.18) is straightforward but that of equation (5.17) involves the use of equation (5.15) and the identity

$$\sum_{n=1}^{N} n \sin nx = \{(N+1)\sin Nx - N\sin(N+1)x\}/2(1-\cos x), \tag{5.19}$$

which may be deduced from identities given in reference [5, p. 30].

It is apparent from the right-hand sides of equations (5.17) and (5.18) that secularity will not usually occur at this order and so there is no need for a slow variable being proportional to μ. It will become apparent that secularity will always arise at the $O(\mu^2)$ stage and hence the choice of the slow variable. There is an exception to this in that it is possible, for $N \geqslant 2$ and for a particular value of α (depending upon N and ε), that one of the "radial" frequencies may coincide with one of the "transverse" frequencies. This special case is not considered here (i.e., here it is taken that α will not assume that value) but will be presented in another work whose emphasis will differ from the present one.

The required solution to equation (5.17) is

$$\begin{aligned} x_j^{(1)} = & \sum_{n=1}^{N} A_n^{(1)}(\tau) \sin j\gamma_n \exp\{\mathrm{i}\lambda_n t\} \\ & + \sum_{n=1}^{N} \frac{\mathrm{i}2\alpha^{1/2}}{(1-\alpha)q_n} B_n^{(0)} \left\{ \sin j v_n - \sin j\zeta_n \frac{\sin(N+1)v_n}{\sin(N+1)\zeta_n} \right\} \exp\{\mathrm{i}\alpha^{1/2} q_n t\}, \end{aligned} \tag{5.20}$$

where ζ_n is such that

$$\cos\zeta_n = 1 - \alpha + \alpha\cos v_n \tag{5.21}$$

and the introduction of the parameter ζ_n has been made so that $x_{N+1}^{(1)} = 0$.

Similarly, the solution to equation (5.18) is

$$\begin{aligned} y_j^{(1)} = & \sum_{n=1}^{N} B_n^{(1)}(\tau) \left\{ \sin j v_n - \frac{j}{N+1} \sin(N+1) v_n \right\} \exp\{\mathrm{i}\alpha^{1/2} q_n t\} \\ & + \sum_{n=1}^{N} \frac{\mathrm{i}2}{(1-\alpha)\lambda_n} A_n^{(0)} \left\{ \sin j\gamma_n + \varepsilon \frac{\alpha \sin N\gamma_n}{\lambda_n^2} \left[\frac{j}{N+1} - \frac{\sin j\xi_n}{\sin(N+1)\xi_n} \right] \right\} \exp\{\mathrm{i}\lambda_n t\}, \end{aligned} \tag{5.22}$$

where

$$\cos\xi_n = 1 - \frac{1}{\alpha}(1 - \cos\gamma_n), \tag{5.23}$$

ξ_n being introduced in order that $y_{N+1}^{(1)} = 0$.

Note that the first order perturbations $x_j^{(1)}$ and $y_j^{(1)}$ contain both the "radial" and the "transverse" spectra and so the radial motion and the transverse motion involve both spectra.

When the zero order and first order solutions are put into the second order equations (5.9) they become

$$Dx_j^{(2)} = \sum_{n=1}^{N} U_j \exp\{\mathrm{i}\lambda_n t\} + \sum_{n=1}^{N} U_j^* \exp\{\mathrm{i}\alpha^{1/2} q_n t\},$$

$$Ey_j^{(2)} = \sum_{n=1}^{N} V_j^* \exp\{\mathrm{i}\lambda_n t\} + \sum_{n=1}^{N} V_j \exp\{\mathrm{i}\alpha^{1/2} q_n t\}, \tag{5.24}$$

where U_j and V_j are given in Appendix B, along with the consequences of no secularity.

These consequences are that $A_n^{(0)}(\tau)$ and $B_n^{(0)}(\tau)$ must satisfy the differential equations (B.10) and (B.21) respectively. Both equations are readily solved for $A_n^{(0)}$ and $B_n^{(0)}$ in terms of τ, that is $\mu^2 t$. When these are put into the zero order solutions (5.10) and (5.11) it is found that

$$x_j^{(0)} = \sum_{n=1}^{N} a_n^{(0)} \sin j\gamma_n \exp(\mathrm{i} f_n t),$$

$$y_j^{(0)} = \sum_{n=1}^{N} b_n^{(0)} \left\{ \sin jv_n - \frac{\mathrm{j}}{N+1} \sin(N+1)v_n \right\} \exp(\mathrm{i} g_n t), \tag{5.25}$$

where the $a_n^{(0)}$ and $b_n^{(0)}$ are the undetermined constant amplitudes and

$$f_n = \lambda_n \left\{ 1 + \mu^2 \frac{1}{2\lambda_n^2} \left(\frac{3+\alpha}{1-\alpha} \right) + \hat{\varepsilon}\mu^2 \frac{12\alpha^2(1+\cos\gamma_n)}{(1-\alpha)^2 N(2N+1)\lambda_n^4} \right\}, \tag{5.26}$$

and

$$\begin{aligned} g_n = \alpha^{1/2}\lambda_n \Bigg\{ & 1 + \hat{\varepsilon}\frac{3(1+\cos\gamma_n)}{N(2N+1)\lambda_n^2} - \mu^2 \frac{1}{2\alpha\lambda_n^2}\left[\frac{1+3\alpha}{1-\alpha} + \frac{(1-\alpha)\cos\gamma_n}{2(1+\cos\gamma_n)} \right] \\ & - \hat{\varepsilon}\mu^2 \frac{3}{\alpha N(2N+1)\lambda_n^2} \Bigg[\frac{(-3+2\alpha+9\alpha^2)(1+\cos\gamma_n)}{2(1-\alpha)^2\lambda_n^2} \\ & + (1-\alpha)\left\{ \frac{1}{3}(N+1)^2 + \frac{7}{24} - \frac{5+\cos\gamma_n}{8\sin^2\gamma_n} \right\} \Bigg] \Bigg\}. \end{aligned} \tag{5.27}$$

Here, f_n and g_n are the dimensionless natural (circular) frequencies associated essentially with the radial and transverse vibrations respectively. This is a slightly simplistic labelling as both frequency spectra depend to some extent upon the other mode of vibration due to the Coriolis interaction.

It should be noted that if $N = 1$, i.e., the single-mass case, then equations (5.26) and (5.27) reduce to equation (2.17) as then $n = 1$, $\gamma_1 = \pi/2$ and $\lambda_1 = 2^{1/2}$.

The frequencies f_n and g_n depend upon the parameters μ, $\hat{\varepsilon}$ and α (and also N and n) and this dependence will now be addressed. The comments to be made will of course only be valid for small values of the parameters μ^2, $\hat{\varepsilon}$ and $\hat{\varepsilon}\mu^2$.

It is evident from equation (5.26) that *all* the "radial" natural frequencies (i.e., for any N and all n) increase from their static values λ_n under the influence of an imposed rotation.

Again this is at variance with the purely radial vibrations [1] where, for $\hat{\varepsilon} = 0$, the frequencies decrease with increasing rotation. Furthermore, if there is coupling between the masses and the turntable (i.e., $\hat{\varepsilon} > 0$) then this increase is compounded. These observations are valid for all possible values of α, and all the frequencies will increase with increasing α provided there is rotation present. In the absence of rotation the frequencies are independent of α, that is of the static tension.

For the transverse frequencies g_n, the situation is somewhat more complicated. The following observations may be made:

(i) if $\mu = 0$, $\hat{\varepsilon} > 0$, i.e., no imposed rotation but with coupling present then all the frequencies are increased due to the presence of the coupling for any α.

(ii) If $\hat{\varepsilon} = 0$, $\mu > 0$, i.e., with imposed rotation but no coupling then the frequencies will decrease only if

$$\frac{1 + 3\alpha}{1 - \alpha} + \frac{(1 - \alpha)}{2} \frac{\cos \gamma_n}{1 + \cos \gamma_n} > 0$$

that is for

$$\cos \gamma_n > \frac{-2(1 + 3\alpha)}{(1 + \alpha)(3 + \alpha)}$$

or, in alternative form, for

$$\alpha > 1 - 8\left\{3 + \left(\frac{9 + \cos \gamma_n}{1 + \cos \gamma_n}\right)^{1/2}\right\}^{-1}. \tag{5.28}$$

For $N = 1$ equation (5.28) is valid for all possible α, and so the frequency decreases with increasing rotation, as indicated in section 2. Similarly, equation (5.28) is valid for both frequencies when $N = 2$ for all α, but for $N = 3$ it is valid for $n = 1$ and 2 but for $n = 3$ it is valid only if $\alpha > 0{\cdot}0386$. For $0 < \alpha < 0{\cdot}0386$ this frequency will increase. Similarly, for higher values of N the lower frequencies will decrease due to rotation for all α whilst the higher ones may increase unless a restriction is put upon α, this restriction becoming more severe with increasing $n/(N + 1)$. It can be seen from equation (5.28) that the frequencies, for which this restriction is required (to ensure a decrease), are those for which

$$n > \frac{(N + 1)}{\pi} \cos^{-1}\left(-\frac{2}{3}\right) \simeq 0{\cdot}7323\,(N + 1).$$

(iii) To assess the combined effect of both rotation and coupling upon the "transverse" frequencies the sign of the coefficient of $\mu^2\hat{\varepsilon}$ needs to be considered. As this is not entirely trivial the analysis is given in Appendix C, the outcome being that all the frequencies other than the lowest will decrease due to this contribution. For the lowest frequency there will be an increase for some range of values of α and a decrease for other values. For instance if $N \geqslant 3$ there will be an increase for $0 < \alpha < \alpha_0$ and a decrease for $\alpha_0 < \alpha < 1$, where $\alpha_0 \simeq 0{\cdot}43$ (this value varying little with N). The appropriate ranges of α for $N = 1$ and $N = 2$ are given in Appendix C.

6. THE CASE $N \to \infty$

When the limit is achieved the problem becomes one of a continuous elastic string. The length of each segment when at rest is $L/(N + 1)$ and its natural length is $(1 - \alpha)L/(N + 1)$

and hence the extension of each segment is $\alpha L/(N+1)$. This means that the tension T at rest in each segment is given by $T = \alpha kL/(N+1)$. Also the total mass $M = Nm = \rho L$, where ρ may be considered to be the mass density (per unit length) at rest. It follows that

$$\left(\frac{k}{m}\right)^{1/2} = \left\{\frac{N(N+1)}{\alpha}\right\}^{1/2}\frac{c}{L}, \tag{6.1}$$

where $c = (T/\rho)^{1/2}$, the notation being suggested by that for the wave speed for a continuous elastic string. Now

$$\lambda_n = 2\sin\frac{n\pi}{2(N+1)} \sim \frac{n\pi}{N+1}$$

as $N \to \infty$ and n finite, and so

$$\left(\frac{k}{m}\right)^{1/2}\lambda_n \to \frac{n\pi c}{L}\alpha^{-1/2} \quad \text{as} \quad N \to \infty. \tag{6.2}$$

Further

$$\mu^2 = m\Omega^2/k = \frac{\alpha}{N(N+1)}\left(\frac{\Omega L}{c}\right)^2. \tag{6.3}$$

Note that for the expansion procedures presented in this work to be valid (for any N) then $\Omega L/c \ll 1$, i.e., the maximum linear speed of the system (on the circumference) must be much smaller than the wave speed of any disturbance.

From equation (5.26) the dimensional "radial" circular frequencies may be written as, for $N \to \infty$

$$F_n = \left(\frac{k}{m}\right)^{1/2} f_n = \frac{n\pi c}{L}\alpha^{-1/2}\left\{1 + \left(\frac{\Omega L}{\pi c}\right)^2(\tilde{f}_n + \hat{\varepsilon}\hat{f}_n)\right\},$$

where

$$\tilde{f}_n = \frac{1}{n^2}\frac{\alpha(3+\alpha)}{2(1-\alpha)} \quad \text{and} \quad \hat{f}_n = \frac{1}{n^4}\frac{12\alpha^3}{\pi^2(1-\alpha)^2}. \tag{6.4}$$

Similarly, from equation (5.27) the dimensional "transverse" circular frequencies are, for $N \to \infty$,

$$G_n = \left(\frac{k}{m}\right)^{1/2} g_n = \frac{n\pi c}{L}\left\{1 + \hat{\varepsilon}\frac{3}{n^2\pi^2} + \left(\frac{\Omega L}{\pi c}\right)^2(\tilde{g}_n + \hat{\varepsilon}\hat{g}_n)\right\},$$

where

$$\tilde{g}_n = -\frac{1}{n^2}\frac{(5+10\alpha+\alpha^2)}{8(1-\alpha)}, \quad \hat{g}_n = -\frac{1}{n^2}\left\{\frac{1}{2}(1-\alpha) + \frac{1}{n^2}\frac{3(-15+17\alpha+27\alpha^2+3\alpha^3)}{8\pi^2(1-\alpha)^2}\right\}. \tag{6.5}$$

In conclusion, note that the leading term in equation (6.5), that is $n\pi c/L$, is the familiar frequency spectrum for a continuous elastic string undergoing transverse vibrations. The

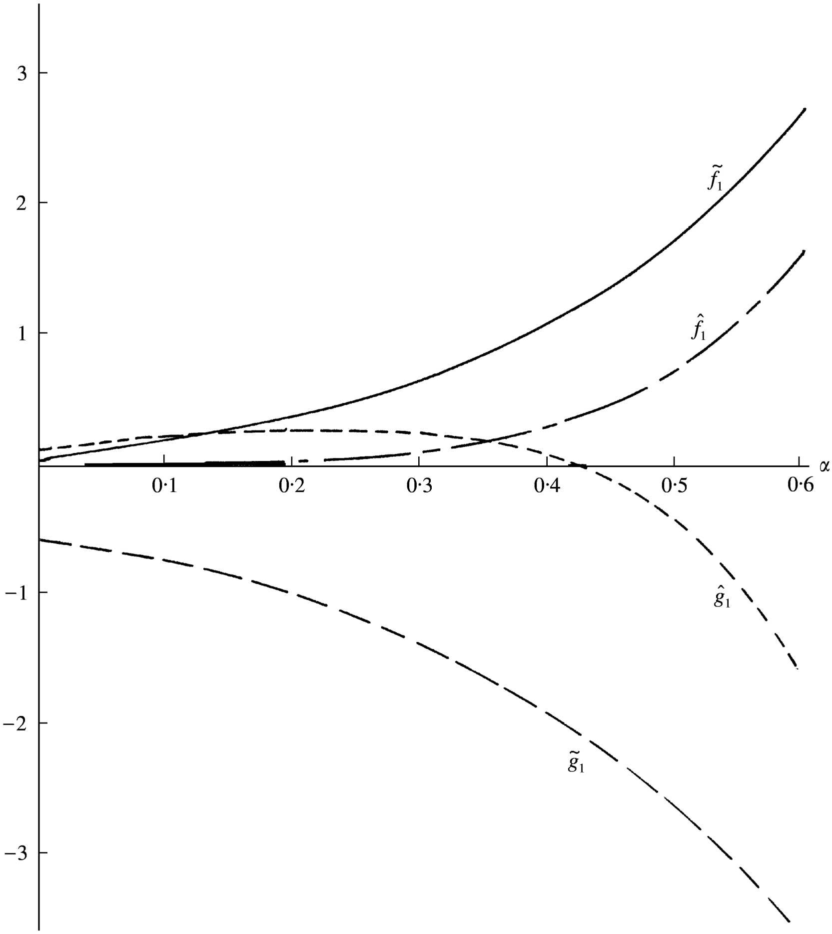

Figure 5. The dependence upon α of the lowest frequencies of the continuous elastic string.

influence of rotation and coupling upon the frequencies is much the same as for the finite $N(\geqslant 3)$ case though simpler to visualize from equations (6.4) and (6.5). As $\tilde{f}_n$ and $\hat{f}_n$ are both positive for all n and α, the "radial" frequencies must all increase with rotation and be further increased if coupling is present. Similarly, for the "transverse" frequencies there is increase if there is coupling but no imposed rotation (due to the second term in the expression for G_n), but when there is imposed rotation then as $\tilde{g}_n$ and $\hat{g}_n$ are negative for all $n \geqslant 2$ and α, these frequencies will decrease from their static values.

The exception to this general rule arises for the smallest transverse frequency, $n = 1$. For this case $\tilde{g}_1$ is always negative but $\hat{g}_1$ is only negative if $\alpha > 0{\cdot}4308$, below this value it is positive. This indicates that for $\alpha < 0{\cdot}4308$ the coupling will increase the frequency but rotation will reduce it. As is apparent from equations (6.4) and (6.5) the influence of rotation and coupling will be confined largely to the lower frequencies (due to the dependence upon inverse powers of n of the coefficients). This means that the lowest frequencies F_1 and G_1 will be pre-eminent in displaying the effect that rotation and coupling have upon the vibrations. The coefficients $\tilde{f}_1, \hat{f}_1, \tilde{g}_1$ and $\hat{g}_1$ are shown as functions of α in Figure 5, illustrating the above comments.

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APPENDIX A

Here, a sufficient condition for all springs to be in tension in the equilibrium configuration is found.

In this configuration, the dimensionless distance between consecutive masses is $R_{j-1} - R_j$.
Then as

$$(R_{j+1} - R_j) - (R_j - R_{j-1}) = -\mu^2 R_j < 0$$

the intermass distance decreases with increasing j. This means that the spring of smallest size is the final one, and its size is $L(1 - R_N)$. If the spring is in tension then so are all the other springs. The condition for tension is

$$L(1 - R_N) > \frac{(1-\alpha)L}{N+1} \tag{A.1}$$

that is

$$\frac{\sin N\beta}{\sin(N+1)\beta} < \frac{N+\alpha}{N+1}. \tag{A.2}$$

A condition upon μ so that equation (A.2) is true is required.

As $(N+1)\beta < \pi/2$ then both $\sin N\beta$ and $\sin(N+1)\beta$ are positive. Hence equation (A.2) may be written as

$$\frac{\sin(N+1)\beta}{\sin N\beta} = \cos\beta + \frac{\sin\beta\cos N\beta}{\sin N\beta} > \frac{N+1}{N+\alpha},$$

i.e.

$$\frac{\cos N\beta \sin\beta}{\sin N\beta} > \frac{1}{2}\mu^2 + \frac{1-\alpha}{N+\alpha}, \tag{A.3}$$

Inequality (A.3) is an alternative to inequality (A.2) and has been neither strengthened nor weakened.

Now as $N\beta < \pi/2$ it is evident that

$$\cos N\beta > 1 - \frac{1}{2}N^2\beta^2, \quad \sin\beta > \beta\left(1 - \frac{1}{6}\beta^2\right) \text{ and } \sin N\beta < N\beta$$

and so

$$\frac{\cos N\beta \sin\beta}{\sin N\beta} > \frac{1}{N}\left\{1 - \frac{1}{6}(3N^2+1)\beta^2\right\}.$$

It follows that if

$$\frac{1}{2}\mu^2 + \frac{1-\alpha}{N+\alpha} < \frac{1}{N}\left\{1 - \frac{1}{6}(3N^2+1)\beta^2\right\}$$

then inequality (A.3) and hence inequality (A.2) are valid. Now as

$$1 - \frac{1}{2}\mu^2 = \cos\beta > 1 - \frac{1}{2}\beta^2 \quad \text{then} \quad \beta^2 > \mu^2$$

and so if

$$\frac{1}{2}\beta^2 + \frac{1-\alpha}{N+\alpha} < \frac{1}{N}\left\{1 - \frac{1}{6}(3N^2+1)\beta^2\right\}$$

then inequality (A.2) is valid. That is

$$\beta^2 < \frac{6\alpha(N+1)}{(3N^2+3N+1)(N+\alpha)}. \tag{A.4}$$

As $\mu = 2\sin\beta/2$ and $\sin\beta/2$ is monotonic in $0 < \beta < \pi/2$, inequality (A.2) is valid if

$$\mu < 2\sin\left\{\frac{3\alpha(N+1)}{2(3N^2+3N+1)(N+\alpha)}\right\}^{1/2}. \tag{A.5}$$

In view of the successive approximations made in order to express the restriction explicitly in μ, then inequality (A.5) can only be a sufficient condition, i.e., if the rotation rate is such that inequality (A.5) is satisfied then the springs are certainly all in tension, though there will no doubt be larger values of μ for which the springs remain in tension.

For the case $N = 1$ there is the "exact" upper bound $\mu^{(e)}$ on μ given by equation (2.7), namely

$$\mu^{(e)} < \left(\frac{2\alpha}{1+\alpha}\right)^{1/2},$$

whereas the upper bound given by inequality (A.5) is

$$\mu < 2\sin\left\{\frac{3\alpha}{7(1+\alpha)}\right\}^{1/2}.$$

It is not difficult to show that $\mu \leqslant \mu^{(e)}$ for all α in (0, 1).

From inequality (A.5), for $N \to \infty$ the restriction takes the form $\mu < (2\alpha)^{1/2}/N$.

APPENDIX B

The expressions for U_j and V_j in equations (5.24) are readily found to be

$$U_j = -\,\mathrm{i}2\lambda_n \frac{\mathrm{d}}{\mathrm{d}\tau} A_n^{(0)} \sin j\gamma_n - \frac{3+\alpha}{1-\alpha} A_n^{(0)} \sin j\gamma_n + \varepsilon\frac{4\alpha}{1-\alpha} A_n^{(0)} \frac{\sin N\gamma_n}{\lambda_{n^2}} \frac{\sin j\xi_n}{\sin(N+1)\xi_n} \tag{B.1}$$

and

$$V_j = -\,\mathrm{i}2\alpha^{1/2} q_n \frac{\mathrm{d}B_n^{(0)}}{\mathrm{d}\tau}\left\{\sin j v_n - \frac{j}{N+1}\sin(N+1)v_n\right\}$$
$$+ B_n^{(0)}\left\{\frac{4\alpha}{1-\alpha}\left[\sin j v_n - \sin j\xi_n \frac{\sin(N+1)v_n}{\sin(N+1)\xi_n}\right] + \alpha\left[\sin j v_n - \frac{j}{N+1}\sin(N+1)v_n\right]\right.$$
$$+ (1+\alpha)\left[\left(1 - \frac{1}{6}N(N+2)q_n^2 + \frac{1}{2}q_n^2 j^2\right)\sin j v_n - j\cos j v_n \sin v_n\right]$$
$$\left. + \frac{1}{6}\frac{j}{N+1}\, q_n^2 \sin(N+1)v_n[\alpha\{(N+1)^2 - j^2\} - (1-\alpha)N(2N+1)]\right\} \qquad \text{(B.2)}$$

and U_j^* and V_j^* are not required for this work as they do not give rise to secular terms. In the derivation of equation (B.1), identity (5.19) has again been used and also terms of $O(\varepsilon^2)$ have been omitted.

In equations (5.24) the terms on the right-hand side involving U_j and V_j will give rise to secularity unless avoiding action is taken. To see what action needs to be taken consider the following:

$$\sum_{j=1}^{N} \sin j\gamma_p D x_j^{(2)} = \sum_{j=1}^{N} \sin j\gamma_p \left\{\frac{\partial^2}{\partial t^2} x_j^{(2)} - x_{j+1}^{(2)} + 2x_j^{(2)} - x_{j-1}^{(2)}\right\}$$
$$= \frac{\partial^2}{\partial t^2}\sum_{j=1}^{N} \sin j\gamma_p x_j^{(2)} - \sum_{j=1}^{N} \sin j\gamma_p x_{j+1}^{(2)} + 2\sum_{j=1}^{N} \sin j\gamma_p x_j^{(2)} - \sum_{j=1}^{N} \sin j\gamma_p x_{j-1}^{(2)}$$
$$= \frac{\partial^2}{\partial t^2}\sum_{j=1}^{N} \sin j\gamma_p x_j^{(2)} - \sum_{j=2}^{N+1} \sin (j-1)\gamma_p x_j^{(2)} + 2\sum_{j=1}^{N} \sin j\gamma_p x_j^{(2)}$$
$$- \sum_{j=0}^{N-1} \sin (j-1)\gamma_p x_j^{(2)}.$$

As $x_{N+1}^{(2)}$, $x_0^{(2)}$ and $\sin(N+1)\gamma_p$ all vanish then it follows that

$$\sum_{j=1}^{N} \sin j\gamma_p D x_j^{(2)} = \left(\frac{\partial^2}{\partial t^2} + \lambda_p^2\right)\sum_{j=1}^{N} \sin j\gamma_p x_j^{(2)}. \qquad \text{(B.3)}$$

Similarly, it can be shown that

$$\sum_{j=1}^{N} \sin j v_p E y_j^{(2)} = \left(\frac{\partial^2}{\partial t^2} + \alpha q_p^2\right)\sum_{j=1}^{N} \sin j v_p y_j^{(2)}, \qquad \text{(B.4)}$$

where use has been made in equation (5.19).

It follows from equation (5.24) that

$$\left(\frac{\partial^2}{\partial t^2} + \lambda_p^2\right)\sum_{j=1}^{N} \sin j\gamma_p x_j^{(2)} = \sum_{n=1}^{N}\left(\sum_{j=1}^{N} \sin j\gamma_p U_j\right)\exp(\mathrm{i}\lambda_n t)$$
$$+ \sum_{n=1}^{N}\left(\sum_{j=1}^{N} \sin j\gamma_p U_j^*\right)\exp(\mathrm{i}\alpha^{1/2} q_n t).$$

Hence secular terms will be generated by the term, in the first summation on the right-hand side of the above, for which $n = p$. However, this may be eliminated provided that

$$\sum_{j=1}^{N} \sin j\gamma_n U_j = 0 \quad (\text{for all } n). \tag{B.5}$$

Similarly, from the second of equations (5.24) secular terms will arise unless

$$\sum_{j=1}^{N} \sin jv_n V_j = 0 \quad (\text{for all } n). \tag{B.6}$$

Conditions (B.5) and (B.6) are then those required to ensure that secularity does not occur. The consequences of the imposition of these conditions is now pursued.

From the definition of U_j, equation (B.1) and condition (B.5) it is evident that

$$\begin{aligned} &\mathrm{i}2\lambda_n \frac{\mathrm{d}A_n^{(0)}}{\mathrm{d}t} \sum_{j=1}^{N} \sin^2 j\gamma_n \\ &\quad + A_n^{(0)} \left\{ \frac{3+\alpha}{1-\alpha} \sum_{j=1}^{N} \sin^2 j\gamma_n - \varepsilon \frac{4\alpha}{1-\alpha} \frac{\sin N\gamma_n}{\lambda_n^2 \sin(N+1)\xi_n} \sum_{j=1}^{N} \sin j\gamma_n \sin j\xi_n \right\} = 0. \end{aligned} \tag{B.7}$$

The summations in this equation may be evaluated by use of some of the standard forms given in reference [5], or variants thereof. For instance from formula 1·342 (2) in reference [5] it is possible to show that

$$\sum_{n=1}^{N} \sin nx \sin ny = \frac{\sin Nx \sin(N+1)y - \sin(N+1)x \sin Ny}{2(\cos y - \cos x)} \tag{B.8}$$

from which may be deduced (cf., formula 1·351(1))

$$\sum_{n=1}^{N} \sin^2 nx = \frac{1}{2}\left\{ N - \frac{\cos(N+1)x \sin Nx}{\sin x} \right\}. \tag{B.9}$$

Hence,

$$\sum_{j=1}^{N} \sin^2 j\gamma_n = \frac{1}{2}(N+1), \qquad \sum_{j=1}^{N} \sin j\gamma_n \sin j\xi_n = -\frac{\alpha}{1-\alpha} \frac{\sin N\gamma_n \sin(N+1)\xi_n}{\lambda_n^2},$$

where equation (5.23) is used. Equation (B.7) may now be written as

$$\frac{\mathrm{d}A_n^{(0)}}{\mathrm{d}\tau} = \frac{\mathrm{i}}{2\lambda n} \left\{ \frac{3+\alpha}{1-\alpha} + \varepsilon \frac{8\alpha^2 \sin^2 \gamma_n}{(1-\alpha)^2 (N+1) \lambda_n^4} \right\} A_n^{(0)}. \tag{B.10}$$

Similarly, from equation (B.6) and the definition of V_j it follows that

$$\begin{aligned} &-\mathrm{i}2\alpha^{1/2} q_n \frac{\mathrm{d}B_n^{(0)}}{\mathrm{d}\tau} \left\{ \sum_{j=1}^{N} \sin^2 jv_n - \frac{\sin(N+1)v_n}{N+1} \sum_{j=1}^{N} j \sin jv_n \right\} \\ &\quad + B_n^{(0)} \left\{ \frac{4\alpha}{1-\alpha} \left[\sum_{j=1}^{N} \sin^2 jv_n - \frac{\sin(N+1)v_n}{\sin(N+1)\zeta_n} \sum_{j=1}^{N} \sin jv_n \sin j\zeta_n \right] \right. \end{aligned}$$

$$+\alpha\left[\sum_{j=1}^{N}\sin^2 jv_n-\frac{\sin(N+1)v_n}{N+1}\sum_{j=1}^{N} j\sin jv_n\right]$$
$$+(1+\alpha)\left[\left(1-\frac{1}{6}N(N+2)q_n^2\right)\sum_{j=1}^{N}\sin^2 jv_n+\frac{1}{2}q_n^2\sum_{j=1}^{N} j^2\sin^2 jv_n\right.$$
$$\left.-\sin v_n\sum_{j=1}^{N} j\sin jv_n\cos jv_n\right]-\frac{1}{6}q_n^2\,\frac{\sin(N+1)v_n}{N+1}$$
$$\times\left[(\alpha(N+1)^2-(1-\alpha)N(2N+1))\sum_{j=1}^{N} j\sin jv_n-\alpha\sum_{j=1}^{N} j^3\sin jv_n\right]\Bigg\}=0. \tag{B.11}$$

This equation is of the form

$$\frac{\mathrm{d}B_n^{(0)}}{\mathrm{d}\tau}=\frac{-\mathrm{i}}{2\lambda_n}\alpha^{-1/2}(Q_n^{(0)}+\varepsilon Q_n^{(1)})B_n^{(0)} \tag{B.12}$$

when written to display the dependence upon ε, for small ε (i.e., again neglecting all terms of $O(\varepsilon^2)$ and smaller). In order to find $Q_n^{(0)}$ and $Q_n^{(1)}$ the various sums in equation (B.11) need to be evaluated up to and including terms of $O(\varepsilon)$. It is known from equations (5.16) that

$$v_n=\gamma_n+\varepsilon\frac{\sin\gamma_n}{\lambda_n^2(N+1)},\quad q_n^2=\lambda_n^2+\varepsilon\frac{2\sin^2\gamma_n}{\lambda_n^2(N+1)} \tag{B.13}$$

and so

$$\sin(N+1)v_n=\varepsilon\frac{(-1)^n\sin\gamma_n}{\lambda_n^2},\quad \cos(N+1)v_n=(-1)^n,$$
$$\sin Nv_n=(-1)^{n+1}\sin\gamma_n\left(1-\varepsilon\frac{N\cos\gamma_n}{(N+1)\lambda_n^2}\right),$$
$$\cos Nv_n=(-1)^n\left(\cos\gamma_n+\varepsilon\frac{N\sin^2\gamma_n}{(N+1)\lambda_n^2}\right),$$
$$\sin v_n=\sin\gamma_n\left(1+\varepsilon\frac{\cos\gamma_n}{(N+1)\lambda_n^2}\right),\quad \cos v_n=\cos\gamma_n-\varepsilon\frac{\sin^2\gamma_n}{(N+1)\lambda_n^2}. \tag{B.14}$$

It now follows from equations (B.9) and (B.14) that

$$\sum_{j=1}^{N}\sin^2 jv_n=\frac{1}{2}(N+1)\left\{1-\varepsilon\frac{\cos\gamma_n}{(N+1)\lambda_n^2}\right\} \tag{B.15}$$

and also from equation (5.19)

$$\frac{\sin(N+1)v_n}{N+1}\sum_{j=1}^{N} j\sin jv_n=-\varepsilon\frac{\sin^2\gamma_n}{\lambda_n^4} \tag{B.16}$$

and from equations (B.8), (B.14) and (5.21) that

$$\frac{\sin(N+1)v_n}{\sin(N+1)\zeta_n}\sum_{j=1}^{N}\sin jv_n\sin j\zeta_n=-\varepsilon\frac{\sin^2\gamma_n}{(1-\alpha)\lambda_n^4}. \tag{B.17}$$

If both sides of equation (5.19) are differentiated an expression for $\sum_{n=1}^{N} n^3 \sin nx$ may be derived, and from it that

$$\frac{1}{6} q_n^2 \frac{\sin(N+1)v_n}{N+1} \sum_{j=1}^{N} j^3 \sin jv_n = -\varepsilon \frac{\sin^2\gamma_n}{\lambda_n^2}\left\{\frac{1}{6}(N+1)^2 - \frac{1}{\lambda_n^2}\right\}. \tag{B.18}$$

Finally, the first and second derivatives of equation (B.9) lead to, respectively,

$$\sum_{j=1}^{N} j \sin jv_n \cos jv_n = -\frac{1}{4}(N+1)\frac{\cos\gamma_n}{\sin\gamma_n} - \varepsilon \frac{1}{2\lambda_n^2 \sin\gamma_n}\{(N+1)\sin^2\gamma_n - 1\} \tag{B.19}$$

and

$$\sum_{j=1}^{N} j^2 \sin^2 jv_n = \frac{1}{12} N(N+1)(2N-H) + \frac{1}{4}(N+1)^2 - \frac{1}{4}\frac{(N+1)}{\sin^2\gamma_n}$$
$$- \varepsilon \frac{\cos\gamma_n}{4\lambda_n^2}\left\{2(N+1)^2 - \frac{3}{\sin^2\gamma_n}\right\}. \tag{B.20}$$

When these forms are put into equation (B.11) the form (B.12) is recovered in explicit form as

$$\frac{\mathrm{d}B_n^{(0)}}{\mathrm{d}\tau} = \frac{-\mathrm{i}\alpha^{-1/2}}{2\lambda_n}\left\{\frac{1+3\alpha}{1-\alpha} + \frac{1-\alpha}{2}\frac{\cos\gamma_n}{1+\cos\gamma_n}\right.$$
$$\left. + \varepsilon\left[\frac{(-3+2\alpha+9\alpha^2)}{2(1-\alpha)^2}\frac{(1+\cos\gamma_n)}{\lambda_n^2} + (1-\alpha)\left\{\frac{1}{3}(N+1)^2 + \frac{7}{24} - \frac{5+\cos\gamma_n}{8\sin^2\gamma_n}\right\}\right]\right\} B_n^{(0)}, \tag{B.21}$$

which then provides $Q_n^{(0)}$ and $Q_n^{(1)}$.

APPENDIX C

Here, the combined effect of rotation and coupling is discussed. This effect is determined by the coefficient of $\hat{\varepsilon}\mu^2$ in equation (5.27) which may be written as

$$-\frac{3(1+\cos\gamma_n)}{\alpha(1-\alpha)^2 N(2N+1)\lambda_n^4} S_{Nn}, \tag{C.1}$$

where

$$S_{Nn}(\hat{\alpha}) = s_{Nn}\hat{\alpha}^3 + \frac{9}{2}\hat{\alpha}^2 - 10\hat{\alpha} + 4$$

with

$$\hat{\alpha} = 1 - \alpha,$$

and

$$s_{Nn} = 2\frac{(1-\cos\gamma_n)}{1+\cos\gamma_n}\left\{\frac{1}{3}(N+1)^2 + \frac{7}{24} - \frac{5+\cos\gamma_n}{8\sin^2\gamma_n}\right\}.$$

If $S_{Nn} > 0$ then the nth frequency will be decreased by the combination, and increased if $S_{Nn} < 0$. The determination of the sign of S_{Nn} is complicated by the presence of three parameters N, n and $\hat{\alpha}$ (and hence α). This may be reduced to consideration of just two parameters N and n by use of the following stratagem.

It will be shown that if, and only if, $s_{Nn} > 2{\cdot}3240$ then $S_{Nn} > 0$ for all possible $\hat{\alpha}$, and also that if $s_{Nn} < 2{\cdot}3240$ then there will be a range of $\hat{\alpha}$ in (0, 1) for which $S_{Nn} < 0$. To show this note that $S_{Nn}(0) = 4$ and $S_{Nn}(1) = s_{Nn} - 1{\cdot}5$, and so s_{Nn} must exceed 1·5 for $S_{Nn} > 0$ for all permissible $\hat{\alpha}$. However, this is not sufficient as it is possible that, in $0 < \hat{\alpha} < 1$, S_{Nn} could have a minimum value which is negative. However, as

$$\frac{\mathrm{d}}{\mathrm{d}\hat{\alpha}} S_{Nn} = 3s_{Nn}\hat{\alpha}^2 + 9\hat{\alpha} - 10$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}\hat{\alpha}^2} S_{Nn} = 6s_{Nn}\hat{\alpha} + 9 > 0 \text{ for } \hat{\alpha} > 0,\ s_{Nn} > 1{\cdot}5$$

then S_{Nn} has a minimum value at

$$\hat{\alpha} = \frac{1}{6s_{Nn}}\{-9 + (81 + 120s_{Nn})^{1/2}\}.$$

This minimum value is

$$\frac{1}{108s_{Nn}^2}\{27(16s_{Nn}^2 + 60s_{Nn} + 27) - (81 + 120s_{Nn})^{3/2}\}.$$

which is positive if and only if $s_{Nn} > 2{\cdot}3240$.

Hence at $s_{Nn} = 2{\cdot}3240$, S_{Nn} has a minimum value of zero with $\hat{\alpha} = 0{\cdot}7150$ ($\alpha = 0{\cdot}2850$), that is, within the permissible values. This means that $S_{Nn} > 0$ for all permissible $\hat{\alpha}$ if and only if $s_{Nn} > 2{\cdot}340$, and if $s_{Nn} < 2{\cdot}340$ then S_{Nn} has a negative minimum value within $0 < \hat{\alpha} < 1$ and so for some permissible $\hat{\alpha}$, S_{Nn} will be negative.

It remains to consider the sign of $(s_{Nn} - 2{\cdot}340)$. To this end consider the lower triangular matrix s_{mn} for $n \leqslant m$ and $m = 1, 2 \cdots N$. It will be shown that (i) $s_{m1} < 2{\cdot}3240$, (ii) $s_{mn} = \ > 2{\cdot}3240$ for $n = 2, 3 \cdots m$. In order to show (i) it is noted that by direct calculation $s_{11} = 2$, $s_{21} = 1{\cdot}5833$, $s_{31} = 1{\cdot}4406$ and as $m \to \infty$, $s_{m1} \to \frac{1}{6}\pi^2 - \frac{3}{8} = 1{\cdot}2700$. s_{m1} has the appearance of a monotonically decreasing sequence and to confirm that this is in fact so consider the function

$$\Psi(u) = 2\frac{(1 - \cos u)}{(1 + \cos u)}\left(\frac{\pi^2}{3u^2} + \frac{7}{24} - \frac{5 + \cos u}{8\sin^2 u}\right),$$

which coincides with s_{m1} at $u = \pi/(m + 1)$, and it should be noted that $0 < u \leqslant \pi/2$ for any $m \geqslant 1$. As

$$\frac{\mathrm{d}\Psi}{\mathrm{d}u} = \frac{\sin u}{(1 + \cos u)^2}\left\{\frac{4}{3}\frac{\pi^2}{u^2}\left(1 - \frac{\sin u}{u}\right) + \frac{17}{12} - \frac{1}{2}\frac{5 + \cos u}{1 + \cos u}\right\},$$

$$\geqslant \frac{\sin u}{(1 + \cos u)^2}\left\{\frac{2}{9}\pi^2\left(1 - \frac{\pi^2}{80}\right) + \frac{17}{12} - \frac{5}{2}\right\} > 0 \quad \text{for } 0 < u \leqslant \pi/2.$$

Then, Ψ is a monotonically increasing function of u, and so s_{m1} monotonically decreases as m increases, and as $s_{11} < 2{\cdot}3240$ then $s_{m1} < 2{\cdot}3240$ for all $m \geqslant 1$.

The same procedure may be used to show that (ii) is true for $n = 2$, $m \geqslant 3$, starting with the direct calculations $s_{22} = 15{\cdot}25$ and $s_{m2} \to \frac{2}{3}\pi^2 - \frac{3}{8} = 6{\cdot}2047$, and repeating the monotonicity argument with the result that $s_{m2} > 6{\cdot}2047$ for $m \geqslant 3$. Indeed this method can be used to show that (ii) is true for those n and m for which $n < \frac{1}{2}(m+1)$. As this is not all-inclusive a different approach will be made. In this the elements of a row in the matrix s_{mn} for $n = 2, 3, \ldots, m$ are considered. Above it has been shown that the first term in the row s_{m2} satisfies (ii) and it will now be shown that s_{mn} for fixed m is an increasing sequence in n, leading to the conclusion that s_{mn} satisfies (ii) for $n \geqslant 2$.

The cases of m being even or odd are slightly different in that if m is odd there is a "middle" term $n = \frac{1}{2}(m+1)$ separating those terms for which $n < \frac{1}{2}(m+1)$ from those for which $n > \frac{1}{2}(m+1)$. For even m there is no such term. The two cases are treated separately but by similar methods.

(a) If m is odd then there is the middle term $n = \frac{1}{2}(m+1)$. For this term $s_{m(m+1)/2} = \frac{2}{3}m(m+2) > 2{\cdot}340$ for $m = 3, 5, \ldots$, and so satisfies (ii). Now for those terms for which $n < \frac{1}{2}(m+1)$, then $n\pi/(m+1) < \pi/2$ and so

$$s_{mn} - s_{m(n-1)} = \frac{\left[\cos\dfrac{(n-1)\pi}{m+1} - \cos\dfrac{n\pi}{m+1}\right]}{\left[1+\cos\dfrac{n\pi}{m+1}\right]^2\left[1+\cos\dfrac{(n+1)\pi}{m+1}\right]^2},$$

$$\times\left\{\left(\frac{4}{3}(m+1)^2 + \frac{11}{12}\right)\left(1+\cos\frac{n\pi}{m+1}\right)\left(1+\cos\frac{(n-1)\pi}{m+1}\right)\right.$$

$$\left. - \left(2 + \cos\frac{n\pi}{m+1} + \cos\frac{(n-1)\pi}{m+1}\right)\right\} > 0 \quad \text{for } m \geqslant 7.$$

Hence the sequence is increasing in n for $m \geqslant 7$. (Note that for $m = 3$ it has already been shown that s_{32} satisfies (ii), as does s_{33} by direct calculation. Also for $m = 5$, both s_{52} and the middle term s_{53} are known to satisfy (ii) and so the general process starts with $m = 7$). To consider those terms for which $n > \frac{1}{2}(m+1)$ set $n = m + 1 - p$, so that $p = 1, 2, \ldots, \frac{1}{2}(m-1)$ and hence $p\pi/(m+1) < \pi/2$. The previous argument may be adapted to justify that $s_{mn} > s_{m(n-1)}$ for $n > \frac{1}{2}(m+1)$ also, including $s_{m(m+3)/2} > s_{m(m+1)/2}$. As the latter is known to exceed $2{\cdot}3240$ then it follows that s_{mn} satisfies (ii) for all odd m and $n \geqslant 2$.

(b) If m is even then the technique used in (a) may be employed with a slight modification as there is now no middle term. Those terms for which $n \leqslant m/2$ and those for which $n \geqslant (m/2) + 1$ form two groups, both of which can be shown by the methods of (a) to increase with n. Similarly, it can be shown that $s_{m(m+1)/2} > s_{mm/2}$ and hence for even m s_{mn} increases with n from s_{m2} which satisfies (ii).

It has therefore been shown that $S_{Nn} > 0$ for $n = 2, 3 \cdots N$, for all possible α, but that for some α in (0, 1), $S_{N1} < 0$. It is straightforward to show from equation (C.1) that

$$S_{11} < 0 \quad \text{for} \quad 0{\cdot}1400 < \alpha < 0{\cdot}3769,$$

$$> 0 \quad \text{for} \quad \alpha \text{ elsewhere in } (0, 1)$$

and

$$S_{21} < 0 \quad \text{for} \quad 0{\cdot}0241 < \alpha < 0{\cdot}4127,$$
$$> 0 \quad \text{for} \quad \alpha \text{ elsewhere in } (0, 1)$$

and

$$S_{31} < 0 \quad \text{for} \quad 0 < \alpha < 0{\cdot}4215,$$
$$> 0 \quad \text{for} \quad 0{\cdot}4215 < \alpha < 1$$

and

$$S_{N1} < 0 \quad \text{for} \quad 0 < \alpha < 0{\cdot}4308,$$
$$> 0 \quad \text{for} \quad 0{\cdot}4308 < \alpha < 1,$$

as $N \to \infty$.

From equation (C.1) it follows that if $S_{N1} < 0$ then the frequency will increase, and decrease if $S_{N1} > 0$.