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LETTERS TO THE EDITOR

GENERALIZED HARMONIC BALANCE/NUMERICAL METHOD FOR DETERMINING ANALYTICAL APPROXIMATIONS TO THE PERIODIC SOLUTIONS OF THE $x^{4/3}$ POTENTIAL

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In a previous publication, Mickens [1] considered the dynamics of a system modelled by the equation of motion

$$\ddot{x} + x^{1/(2n+1)} = 0, \tag{1}$$

where n is a positive integer. Using phase-space techniques [2], it was shown that all the solutions of equation (1) were periodic. If $n = 1$, then the method of (simple) harmonic balance provided the following approximation to the periodic solutions:

$$x(t) \simeq x_0 \cos\left[\left(\frac{4}{3x_0^2}\right)^{1/6} t\right], \tag{2}$$

where the initial conditions were taken to be

$$x(0) = x_0, \qquad \dot{x}(0) = 0. \tag{3}$$

The main purpose of this Letter to the Editor is to determine an improved approximation by using a functional form suggested from generalized harmonic balance [2], namely,

$$x(t) \simeq \frac{A\cos(\omega t)}{1 + B\cos(2\omega t)}, \tag{4}$$

where A, B and ω are to be determined as functions of the initial conditions expressed in equation (3). The ultimate procedure used to calculate A, B, and ω was based on the numerical integration of equation (3) for one particular set of values for the initial conditions. The reasons for this modified technique will be given.

To apply the method of generalized harmonic balance, first start by rewriting equatio (1), for $n = 1$, as

$$(\ddot{x})^3 + x = 0. \tag{5}$$

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Second, calculate the second derivative of the expression for $x(t)$ given by equation (4); doing this gives

$$\ddot{x}(t) \simeq \omega^2 A \left[\frac{C_1 \cos\theta + C_2 \cos 3\theta + C_3 \cos 5\theta}{(1 + B\cos 2\theta)^3} \right], \tag{6}$$

where $\theta = \omega t$ and (C_1, C_2, C_3) are given functions of only B. Thus,

$$(\ddot{x})^3 \simeq (\omega^6 A^3) \left[\frac{\sum_{k=0}^{7} D_k \cos(2k+1)\theta}{(1 + B\cos 2\theta)^9} \right], \tag{7}$$

where each D_k is known in terms of B. Now setting equation (7) "equal" to equation (4) gives

$$(\omega^6 A^3) \left[\frac{\sum_{k=0}^{7} D_k \cos(2k+1)\theta}{(1 + B\cos 2\theta)^9} \right] \simeq -\frac{A\cos\theta}{1 + B\cos 2\theta}, \tag{8}$$

which can be rationalized to an expression taking the form

$$(\omega^6 A^2) \sum_{k=0}^{7} D_k \cos(2k+1)\theta \simeq -(1 + B\cos 2\theta)^8 \cos\theta \tag{9}$$

or

$$(\omega^6 A^2) \sum_{k=0}^{7} D_k \cos(2k+1)\theta \simeq -\sum_{m=0}^{8} F_m \cos(2m+1)\theta, \tag{10}$$

where the F_m are known functions of the parameter B. This last relation can be rewritten as

$$(\omega^6 A^2 D_0 + F_0)\cos\theta + (\omega^6 A^2 D_1 + F_1)\cos 3\theta + HOH \simeq 0. \tag{11}$$

On applying harmonic balancing for the $\cos\theta$ and $\cos 3\theta$ terms, the following two equations are obtained:

$$\omega^6 A^2 D_0 + F_0 = 0, \qquad \omega^6 A^2 D_1 + F_1 = 0. \tag{12a, b}$$

From the initial conditions, see equation (3), and the assumed solution, given by equation (4), A can be calculated as a function of x_0 and B; it is given by

$$A = x_0(1 + B). \tag{13}$$

Eliminating $\omega^6 A^2$ in equations (12) gives

$$D_1 F_0 - D_0 F_1 = 0, \tag{14}$$

which is a polynomial expression for the parameter B. In principle, the smallest magnitude root of equation (14) provides the required value of B. The estimate for the angular frequency can now be determined from either one of equations (12); for example, using equation (12a), gives

$$\omega(x_0) = \left[\frac{F_0(B)}{(1 + B) F_1(B)} \right]^{1/6} \left(\frac{1}{x_0^{1/3}} \right). \tag{15}$$

In equation (15), the explicit dependence of F_0 and F_1 on B is indicated. Since B is just a pure number, ω in this equation is only a function of x_0. Note that for the simple harmonic balance procedure, $\omega(x_0)$ is

$$\omega(x_0) = \left(\frac{4}{3} \right)^{1/6} \left(\frac{1}{x_0^{1/3}} \right). \tag{16}$$

For $x_0 = 1$, the period, $T = 2\pi/\omega$, was determined by numerically integrating equation (1), for $n = 1$. A comparison of the numerically derived value and that obtained from equation (16) gives

$$\omega_{num} \simeq 1{\cdot}054, \qquad \omega_{shb} \simeq 1{\cdot}049 \tag{17}$$

for a fractional error of less than one-half percent. Thus, the simple harmonic balance estimate for ω is very good.

In practice, the actual derivation of equation (14) is quite difficult. However, an alternate procedure can be used to estimate B. This method uses the numerical solution for a particular value of x_0. Let $x_0 = 1$ and consider the value of $t = \bar{t}$ such that $\bar{t} = T/8$. Then,

$$\cos(\omega\bar{t}) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \qquad \cos(2\omega\bar{t}) = \cos\left(\frac{\pi}{2}\right) = 0 \tag{18}$$

and, from equation (4)

$$x(\bar{t}) = \frac{A}{\sqrt{2}}. \tag{19}$$

Substitution of equation (13) into equation (19), and solving for B gives

$$B = \sqrt{2}x(\bar{t}) - 1. \tag{20}$$

From the numerical solution, $x(\bar{t})$ can be easily determined; its value is $x(\bar{t}) = 0{\cdot}7305$. Hence, from equation (20), the following value is obtained for B:

$$B = 0{\cdot}033. \tag{21}$$

Putting all of this together gives the following analytical approximation to equation (1), for $n = 1$, for the initial conditions of equation (3):

$$x(t) \simeq \frac{(1{\cdot}033)\,x_0 \cos\omega t}{1 + (0{\cdot}033)\cos 2\omega t}, \tag{22}$$

where

$$\omega(x_0) \simeq \frac{1{\cdot}054}{x_0^{1/3}}. \tag{23}$$

It should be indicated that the result of equation (22) contains approximations to *all* of the higher harmonic components for the exact periodic solutions of equation (1) with $N = 1$. (For the details of this, see Mickens [3].)

In summary, a higher order harmonic balance method, combined with a numerical solution for one particular value of the initial conditions, has been used to construct an analytical approximation to a system modelled by an $x^{4/3}$ potential, i.e., $n = 1$ in equation (1). This procedure can be generalized for any positive integer n.

Finally, there seem to be no major difficulties in applying this combined analytic/numerical method to other conservative oscillator problems.

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