Journal of Sound and Vibration (2002) **250**(5), 955–963
doi:10.1006/jsvi.2001.3783, available online at http://www.idealibrary.com on IDEAL®

NUMERICAL STUDY OF A NON-STANDARD FINITE-DIFFERENCE SCHEME FOR THE VAN DER POL EQUATION

R. E. Mickens

Department of Physics, Clark Atlanta University, Atlanta, GA 30314, *U.S.A.*

AND

A. B. Gumel

Department of Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2

(*Received* 21 *June* 2001)

The van der Pol equation provides an important mathematical model for dynamical systems having a single (unstable) fixed point, along with a single (stable) limit cycle. Examples of such phenomena arise in all of the natural and engineering sciences [1, 2]. The details of the general dynamics of such a system are determined by one parameter, ε. In dimensionless form, this non-linear differential equation can be written [1] as

$$\ddot{x} + x = \varepsilon(1 - x^2)\dot{x}, \quad \varepsilon > 0. \tag{1}$$

Another important use for this equation is that it provides an interesting model for testing numerical integration methods. The main purpose of this Letter to the Editor is to carry out a detailed numerical investigation of a new finite-difference scheme for equation (1). The new scheme is based on the non-standard procedures of Mickens [3] for constructing discrete models of differential equations. The results to follow generalize the previous work of Mickens [4] on the "unplugged" van der Pol oscillator equation,

$$\ddot{x} + x = -\varepsilon x^2 \dot{x}. \tag{2}$$

To proceed, consider a system modelled by two first order, ordinary differential equations (ODE)

$$\dot{x} = ax + by + f(x, y), \qquad \dot{y} = cx + dy + g(x, y), \tag{3a, b}$$

where $(\bar{x},\ \bar{y}) = (0,\ 0)$ is the unique fixed point or constant solution, and (a, b, c, d) are constant parameters. As derived by Mickens [5], the non-standard finite-difference (NSFD) scheme for equations (3) is

$$\frac{x_{k+1} - \psi x_k}{\phi} = ax_k + by_k + f(x_k, y_k), \tag{4a}$$

$$\frac{y_{k+1} - \psi y_k}{\phi} = cx_k + dy_k + g(x_{k+1}, y_k), \tag{4b}$$

0022-460X/02/100955 + 09 $35.00/0

where ϕ and ψ can be explicitly written in terms of the parameters (a, b, c, d) and the time step-size $h = \Delta t$. In these equations, the discrete time t_k is given by $t_k = hk$, where k is an integer, and x_k and y_k are, respectively, approximations to $x(t_k)$ and $y(t_k)$. If equation (1) is rewritten as

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x + \varepsilon(1 - x^2)y, \tag{5a, b}$$

then the corresponding values for (a, b, c, d) and the functions (ϕ, ψ) are given by the following expressions [5]:

$$a = 0, \qquad b = 1, \qquad c = -1, \qquad d = \varepsilon, \tag{6a}$$

$$f(x, y) = 0, \qquad g(x, y) = -\varepsilon x^2 y, \tag{6b}$$

$$\phi(\varepsilon, h) = \left[\frac{e^{\varepsilon h/2}}{\sqrt{1 - \varepsilon^2/4}}\right] \sin\left[\left(\sqrt{1 - \frac{\varepsilon^2}{4}}\right)h\right], \tag{6c}$$

$$\psi(\varepsilon, h) = -\left[\frac{\varepsilon e^{\varepsilon h/2}}{2\sqrt{1 - \varepsilon^2/4}}\right] \sin\left[\left(\sqrt{1 - \frac{\varepsilon^2}{4}}\right)h\right] + [e^{\varepsilon h/2}] \cos\left[\left(\sqrt{1 - \frac{\varepsilon^2}{4}}\right)h\right]. \tag{6d}$$

Note that $\phi(\varepsilon, h)$ and $\psi(\varepsilon, h)$ are defined for all values of ε; for example, if $\varepsilon > 2$, then

$$\phi(\varepsilon, h) = \left[\frac{e^{\varepsilon h/2}}{\sqrt{\varepsilon^2/4 - 1}}\right] \sinh\left[\left(\sqrt{\frac{\varepsilon^2}{4} - 1}\right)h\right], \tag{7a}$$

$$\psi(\varepsilon, h) = -\left[\frac{\varepsilon e^{\varepsilon h/2}}{2\sqrt{\varepsilon^2/4 - 1}}\right] \sinh\left[\left(\sqrt{\frac{\varepsilon^2}{4} - 1}\right)h\right] + [e^{\varepsilon h/2}] \cosh\left[\left(\sqrt{\frac{\varepsilon^2}{4} - 1}\right)h\right]. \tag{7b}$$

Consequently, the NSFD scheme for the system form of the van der Pol equation is

$$\frac{x_{k+1} - \psi x_k}{\phi} = y_k, \tag{8a}$$

$$\frac{y_{k+1} - \psi y_k}{\phi} = -x_k + \varepsilon y_k - \varepsilon x_{k+1}^2 y_k. \tag{8b}$$

If equation (8a) is used to eliminate the y_k variable, then equation (8b) becomes

$$\frac{x_{k+1} - 2x_k + x_{k-1}}{\phi^2} + \frac{2(1 - \psi)x_k + (\psi^2 + \phi^2 - 1)x_{k-1}}{\phi^2} = \varepsilon(1 - x_k^2)\left[\frac{x_k - \psi x_{k-1}}{\phi}\right]. \tag{9}$$

The discrete form of the van der Pol equation in system form, equation (8), is semi-implicit. This means that given (x_k, y_k), the value for x_{k+1} can be calculated from equation (8a). Substitution for (x_k, y_k) and x_{k+1} into equation (8b) then allows the determination of y_{k+1}. Also of interest is the result given by equation (9) for the discrete form of the second order form of the van der Pol equation, i.e., equation (1). The following points should be indicated:

(1) The second order time derivative, in equation (1), is replaced by the usual form of a central difference expression, i.e.,

$$\ddot{x} \to \frac{x_{k+1} - 2x_k + x_{k-1}}{\phi^2}, \tag{10}$$

where from equation (6c)

$$\phi = h + O(h^2). \tag{11}$$

(2) The non-linear term $(1 - x^2)\dot{x}$ is replaced by an expression that is locally evaluated for the function $(1 - x^2)$, but with a non-standard backward Euler representation for the first derivative [5], i.e.,

$$(1 - x^2)\dot{x} \to (1 - x_k^2)\left[\frac{x_k - \psi x_{k-1}}{\phi}\right], \tag{12}$$

where from equation (6d)

$$\psi = 1 + O(h). \tag{13}$$

(3) The linear x term in the original van der Pol equation is replaced by the combination

$$x \to \frac{2(1 - \psi)x_k + (\psi^2 + \phi^2 - 1)x_{k-1}}{\phi^2}. \tag{14}$$

Non-standard finite-difference schemes generally give expressions for linear terms, occurring in the ODEs having oscillatory solutions, as linear combinations of x_k and x_{k-1}. (The background to all of the above analysis can be found in reference [5].)

The remainder of the discussion will center on using equations (8) for the calculation of numerical solutions to the van der Pol equation. This analysis is compared with results obtained by using a direct forward-Euler scheme, i.e.,

$$\frac{x_{k+1} - x_k}{h} = y_k, \tag{15a}$$

$$\frac{y_{k+1} - y_k}{h} = -x_k + \varepsilon(1 - x_k^2)y_k. \tag{15b}$$

All comparisons between the two schemes were made using the same values of (ε, h) and the initial values (x_0, y_0).

An extremely important feature of problems involving solutions with oscillations is that the time step-size must be selected to be small in comparison to the "characteristic time" of the oscillation. This characteristic time may depend on the parameters appearing in the original equations and/or the initial conditions [4]. Consequently, to begin the numerical study of equations (8) and (15), an estimate of the characteristic time must be made. Using

TABLE 1

Properties of the numerical solutions for $\varepsilon = 0{\cdot}5$, $x_0 = 0{\cdot}5$, $y_0 = 0$ as h varies. SLC means stable limit cycle

h	Non-standard method	Euler's method
0·01	SLC	SLC
0·1	SLC	SLC
0·2	SLC	SLC
0·5	SLC	SLC
0·51	SLC	Overflow
0·55	Overflow	Overflow

dimensional analysis, see section 1.3.3 of reference [1], the following result can be derived for the van der Pol equation:

$$\varepsilon = \frac{(\text{period of free oscillations})}{(\text{damping time})} = \frac{2\pi}{T_d}. \tag{16}$$

Note that the period of free oscillations for equation (1) is 2π. Consequently, equation (16) can be used to determine the characteristic damping time, T_d in terms of the parameter ε; it is given by

$$T_d = \frac{2\pi}{\varepsilon}. \tag{17}$$

An alternative way of viewing this situation is to understand that when ε is small, the characteristic time is determined by the period of the free oscillations, while for large values of ε, i.e., strong damping, the characteristic time is given by equation (17). For any value of $\varepsilon \geqslant 0$, a useful estimate of the characteristic time is given by the formula

$$T(\varepsilon) = \frac{2\pi}{1+\varepsilon}. \tag{18}$$

It has the useful feature of nicely extrapolating between the low and high values of ε. The time step-size, $\Delta t = h$, used to calculate numerical solutions should be small compared to $T(\varepsilon)$, i.e.,

$$h \ll T. \tag{19}$$

A typical choice is to use

$$h = \frac{T}{20} = \left(\frac{1}{10}\right)\left(\frac{\pi}{1+\varepsilon}\right). \tag{20}$$

The net result of this discussion is that for a given value of the parameter ε, there should exist a maximum time step-size for which realistic oscillatory behavior is observed in the calculated numerical solutions. Larger values of the step-size will lead to meaningless numerical values (generally overflow is expected).

The results presented in Table 1 clearly illustrate the above discussion. The parameter ε was fixed at $\varepsilon = 0{\cdot}5$ and the initial condition was taken as $x_0 = 0{\cdot}5$ and $y_0 = 0$. The NSFD and Euler schemes, respectively, given by equations (8) and (15) were then used to obtain numerical solutions for the van der Pol equation. For values of ε in the range $0 < \varepsilon \leqslant 0{\cdot}5$, both methods gave numerical results consistent with the existence of a unique limit cycle. However, the NSFD scheme could be extended to a higher value of the step-size, in comparison with the Euler method, before overflow took place. Figure 1 shows the calculated phase-space trajectories using the NSFD scheme of equation (8). Similar results were obtained using the Euler method for values of h for which its numerical solutions were bounded. Computations were also done for other values of ε with various values for h. The same general behavior was observed.

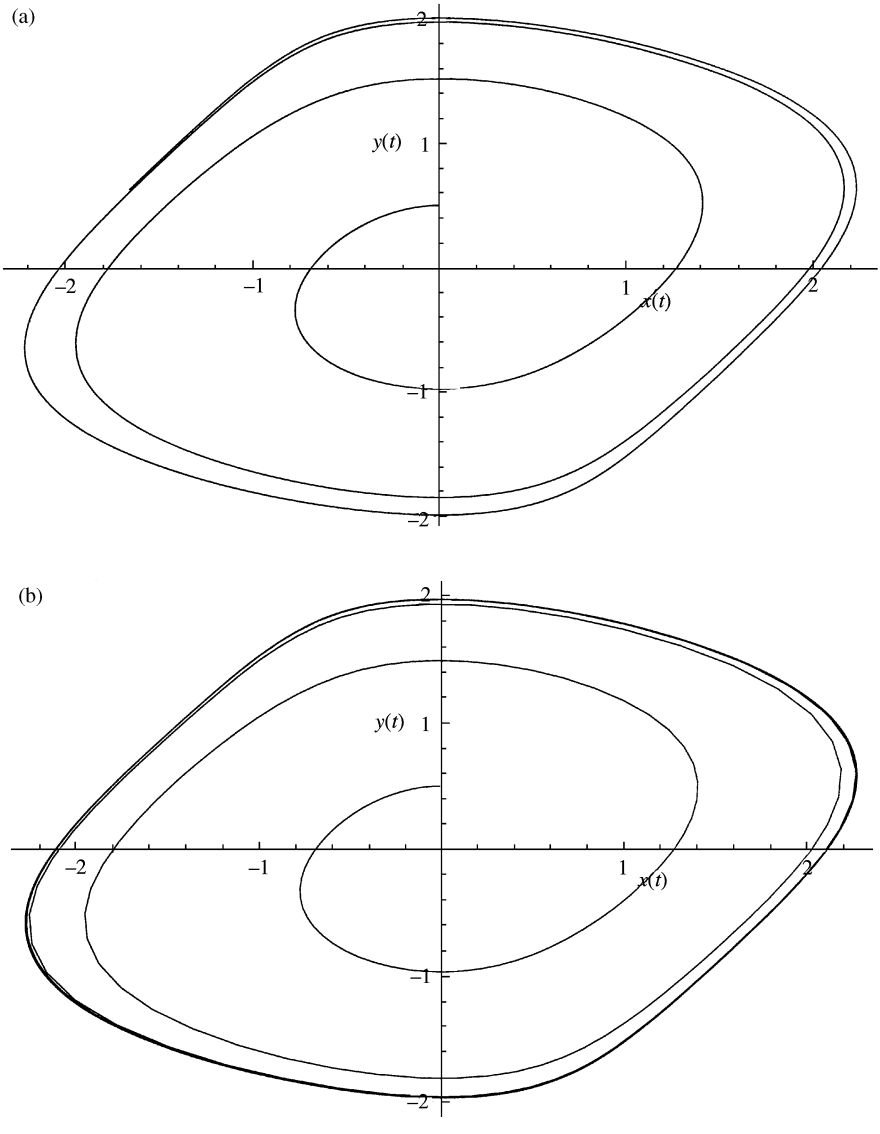

Figure 1. Phase-space trajectory using NSFD method with $\varepsilon = 0{\cdot}5$, $x_0 = 0{\cdot}5$, $y_0 = 0$, and $h =$ (a) 0·01, (b) 0·1, (c) 0·2, (d) 0·5, (e) 0·51, (f) 0·54.

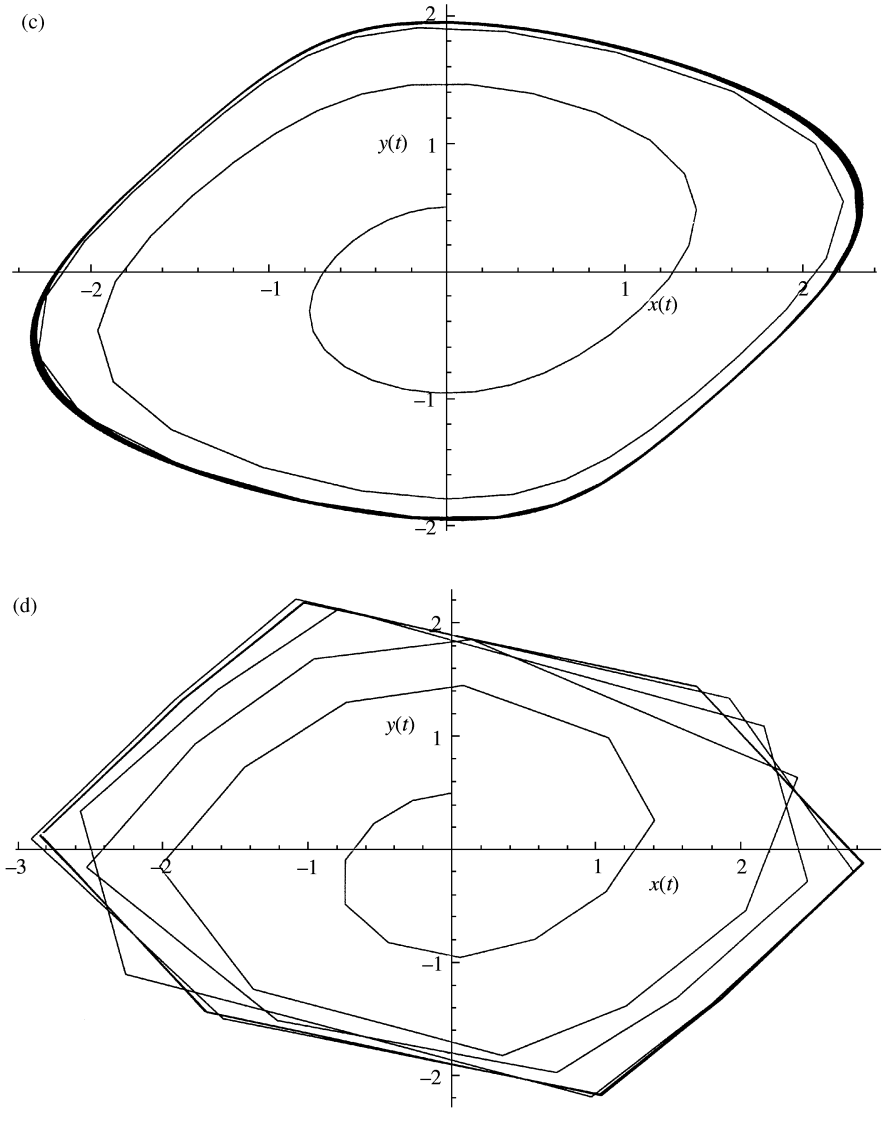

Figure 1. Continued.

In more detail, the following procedure was carried out.

(1) A value of ε was selected, along with a set of initial conditions (x_0, y_0).

(2) With these parameters, (ε, x_0, y_0), a series of numerical solutions were obtained with h increasing in value until overflow took place. The minimum value for h to give overflow was called the critical step-size, h_c.

(3) Writing $h_c = AT(\varepsilon)$, the value of A was calculated. This relationship assumes that the critical step-size is proportional to the estimate of the characteristic time given in equation (18). Table 2 provides a summary of h_c and A values for $0 < \varepsilon < 5$.

Two results can be obtained from studying Table 2. First, the critical step-size value decreases as ε increases. Second, the value of the proportionality "constant" A in the relation $h_c = AT(\varepsilon)$ has a weak dependence on ε, but has the approximate value of one-tenth

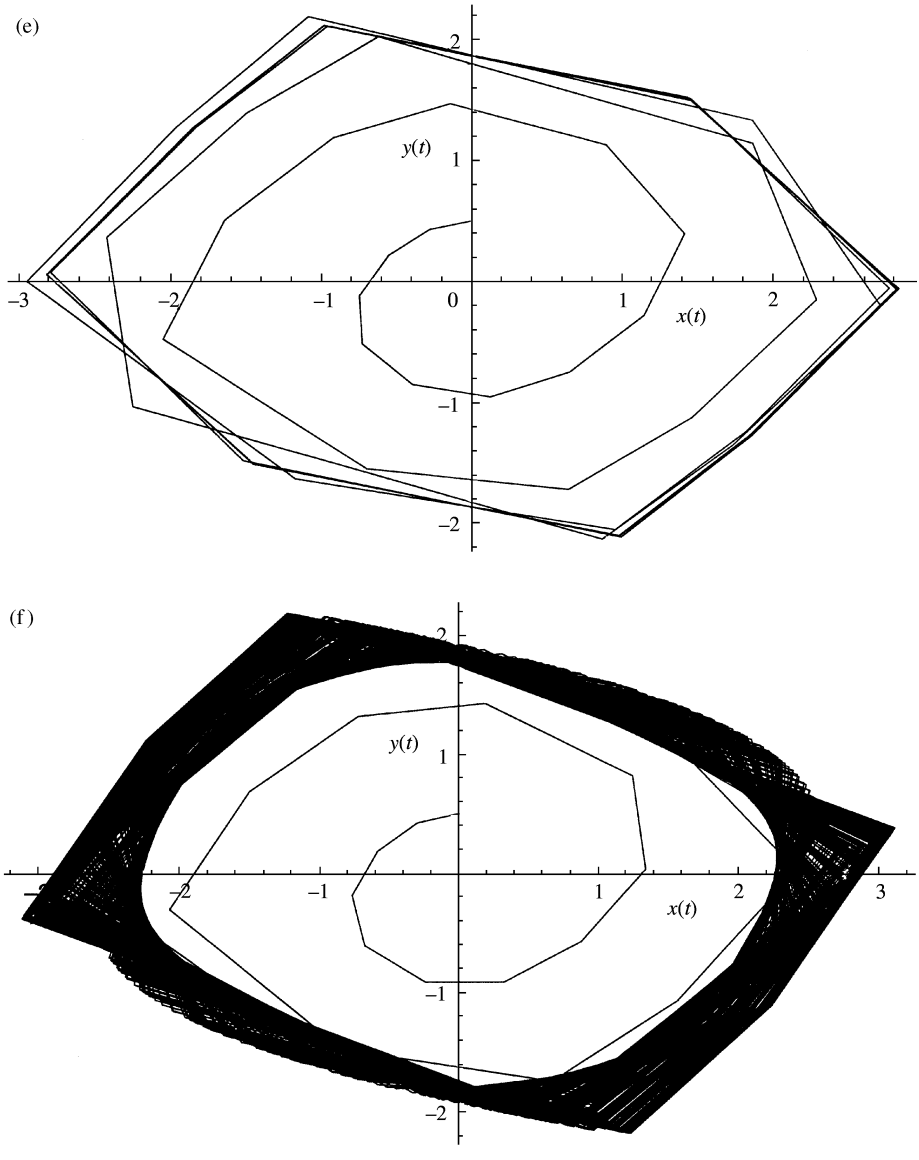

Figure 1. Continued.

for $0 < \varepsilon \leqslant 5$. Given the heuristic nature of the derivation of equation (18), this is quite a good result. Note that if $h_c \approx T/10$, then the time step-size selection, as given by equation (20), is about one-half the value of h_c. For example, let $\varepsilon = 0{\cdot}5$; then $h = 0{\cdot}209$ from equation (20) and $h_c = 0{\cdot}544$. However, a close examination of Figure 1 shows that a step-size value of 0·2 is not small enough if an accurate trajectory in phase space is needed. Even for $h = 0{\cdot}1$, non-smoothness is seen in the initial transient behaviour of the approach to the limit cycle.

Further consideration of Figure 1 for large values of h, i.e., $0{\cdot}30 \lesssim h \lesssim 0{\cdot}54$, shows that while the corresponding phase spaces are not numerically correct, the limit cycle structure is maintained until overflow occurs at $h_c \approx 0{\cdot}544$. This feature of the NSFD scheme is important, since it implies that an accurate solution, $x(t)$ versus t, and an accurate phase-space trajectory can be obtained by a decrease in step-size.

TABLE 2

Values of critical step-sizes (h_c) and A

ε	$h_c = AT$	A
0·5	0·544	0·130
1·5	0·269	0·107
2·0	0·218	0·104
5·0	0·103	0·097

The following is a summary of the results and issues related to the work of this Letter.

(1) An NSFD scheme was constructed for the van der Pol oscillator ODE. This particular discrete model was obtained by requiring the linear part of the difference scheme to have exactly the same stability properties as the corresponding linear part of the van der Pol equation. (See references [3, 5] for the details.)

(2) The NSFD scheme is semi-explicit, i.e., first x_{k+1} is obtained from (x_k, y_k) and then y_{k+1} is obtained from (x_k, y_k, x_{k+1}).

(3) Dimensional analysis allowed the determination of an estimate for the characteristic time of the van der Pol ODE in terms of the parameter ε. The oscillatory nature of the actual solution requires that the time step-size to be used in calculating a numerical solution, be small compared to the characteristic time $T(\varepsilon)$, i.e., $h \ll T(\varepsilon)$.

(4) Numerical studies showed that there exists a critical step-size, h_c, such that if $h \geqslant h_c$, overflow occurs, while for $0 < h < h_c$, the numerical solution had the general feature of a limit cycle. The numerical accuracy improved with a decrease in the step-size.

(5) Comparison of the NSFD and forward-Euler schemes showed them to give roughly the same numerical results. However, the NSFD scheme was more robust, i.e., provided the proper limit cycle behavior for a larger range of ε values.

It should also be indicated that several interesting features emerged from a close examination of the numerical solutions.

(1) For the same values of ε and h, the Euler method gave oscillations having larger periods than the NSFD scheme.

(2) For a fixed value of ε, the NSFD scheme gave numerical solutions for which the period decreased as the step-size h increased.

(3) The opposite of (2) was found for the Euler method, i.e., for a given ε value, the period of the oscillators increased when h was increased.

Such behaviors for the numerical solutions clearly call for a mathematical explanation.

Finally, there are two issues to be investigated in future work. First, obtain a better estimate for the characteristic time, $T(\varepsilon)$, as a function of ε. Second, determine a more accurate time step-size limitation relation than the one given by equation (20). It is also of interest to extend this work to coupled systems of ODEs, each having possible limit cycle behavior [6].

ACKNOWLEDGMENTS

The work of A. B. Gumel was supported in part by the Natural Sciences and Engineering Research Council of Canada, while that of R. E. Mickens was supported by grants from DOE and the NIH/MBRS-SCORE Program at Clark Atlanta University.

REFERENCES

1. R. E. MICKENS 1996 *Oscillations in Planar Dynamic Systems*. River Edge, NJ: World Scientific.
2. L. EDELSTEIN-KESHET 1988 *Mathematical Models in Biology*. New York: McGraw-Hill,
3. R. E. MICKENS 1994 *Nonstandard Finite Difference Models of Differential Equations*. River Edge, NJ: World Scientific.
4. R. E. MICKENS 2001 *Journal of Sound and Vibration* **245**, 757–761. Analytical and numerical study of a nonstandard finite difference scheme for the unplugged van der Pol equation.
5. R. E. MICKENS 1994 *Nonstandard Finite Difference Models of Differential Equations*. River Edge, NJ: World Scientific, sections 6.3, 6.5.2 and 6.5.5.
6. P. YU and A. B. GUMEL 2001 *Journal of Sound and Vibration* **244**, 795–820. Bifurcation and stability analyses for a coupled Brusselator model.