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# NON-LINEAR VIBRATION AND THERMAL BUCKLING OF AN ORTHOTROPIC ANNULAR PLATE WITH A CENTRIC RIGID MASS

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A computational analysis of the non-linear vibration and thermal post-buckling of a heated orthotropic annular plate with a central rigid mass is examined for the cases of immovably hinged as well as clamped constraint conditions of the outer edge. First, based on von Karman's plate theory and Hamilton's principles, the governing equations, in terms of the displacements of the middle plane, of the problem are derived. Then, upon assuming that harmonic responses of the system exist, the non-linear partial differential equations are converted into the corresponding non-linear ordinary differential equations through elimination of the time variable by using the Kantorovich time-averaging method. Finally, by applying a shooting method, the fundamental responses of the non-linear vibration and thermal post-buckling of the plate are numerically obtained. For some prescribed values of the parameters, such as the material rigidity ratio, temperature rise and so on, the curves of the fundamental frequency versus specified amplitude and the thermal post-buckled equilibrium paths of the plate are numerically presented.

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## 1. INTRODUCTION

In recent years, the studies of dynamic response of plates exhibiting anisotropic characteristics have received greater attention due to increasing use of fiber-reinforced materials in aerospace, ocean engineering, electronic equipment, etc. [1–11]. Different researchers have used different analytical or numerical methods; Laura *et al.* [1–3], Gupta *et al.* [4, 5], Gunaratnam [6] and others analyzed linear free vibration and buckling of polar orthotropic circular and annular plates. On considering the geometric non-linearity of the plates, large-amplitude axisymmetric vibrations of this kind of plates were presented in many literatures. Dumir *et al.* [7, 8] studied the non-linear vibrations of orthotropic circular plates by the orthogonal point collocation method, in which the elastic foundations and

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a rigid central mass were considered. By using Kantorovich time-averaging method, C. L. Huang gave a computational analysis of the non-linear oscillations of orthotropic annular plates with variable thickness [9] and a circular isotropic plate with a concentric rigid mass [12]. In reference [10], S. Huang presented an investigation for the non-linear vibration of hinged orthotropic circular plates with a concentric rigid mass by using finite element method. However, to the authors' knowledge, little information has been found in the literature regarding non-linear vibrations and thermal post-buckling of orthotropic circular and annular plates subjected to in-plane loads, especially to an in-plane temperature rise. In general, in-plane pressures, induced by in-plane temperature increase in a restrained plate will weaken the flexural rigidity, then the fundamental frequency of the plate will be decreased [11, 13]. Especially, when the temperature rise exceeds a certain value, or the critical temperature, the plate will be in a buckled state. So, quantitative studies of thermal vibration and buckling of plates are of great importance for the design of those structures working in the environments with severe temperature change.

The present work is concerned with the axisymmetric non-linear vibration and thermal post-buckling of a polar orthotropic annular plate with the edge immovably constrained and subjected to a statically axisymmetric temperature rise. First, on the basis of von Karman's plate theory and Hamilton's principle [14], the governing equations, in terms of the in-plane displacements, of the problem are derived. Then, upon assuming that harmonic vibrations of the system exist, the non-linear partial differential equations are converted into two ordinary differential equations through elimination of the time variable by Kantorovich's time-averaging method [9, 11–13]. Finally, by applying the shooting method [15], the fundamental responses of non-linear vibration and thermal post-buckling of the plate with different values of the geometric and material parameters are given.

## 2. MATHEMATICAL FORMULATION

Consider a thin polar orthotropic annular plate, with inner radius $a$, outer radius $b$, and constant thickness $h$. A concentric rigid mass $M_c$ is attached rigidly along the inner edge as shown in Figure 1. A cylindrical co-ordinate system $(r, \theta, z)$ is located in the middle plane of the plate. Immovably hinged as well as clamped outer boundary conditions of the plate will be considered respectively. Assume that a steady and non-resource temperature rise field $T = T(r)$ is imposed on the plate in its natural state. Let us examine the free transverse vibrations and buckling of the heated plate–rigid mass system with large axisymmetric deflections. On the basis of geometric non-linear theory of thin plates in von Karman's sense, one obtains the strain–displacement relations

$$\varepsilon_r = \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 - z\frac{\partial^2 w}{\partial r^2}, \qquad \varepsilon_\theta = \frac{u}{r} - \frac{z}{r}\frac{\partial w}{\partial r}, \tag{1}$$

where $u(r, t)$ and $w(r, t)$ denote the radial and transverse displacements of the middle plane, $t$ is the time variable, $\varepsilon_r$ and $\varepsilon_\theta$ are the radial and tangential strains respectively. Suppose that the material of the plate is linearly thermal and elastic, the constitutive equations are given as follows:

$$\sigma_r = \frac{E_r}{1 - v_{\theta r}v_{r\theta}}\left[\varepsilon_r + v_{\theta r}\varepsilon_\theta - (\alpha_r + v_{\theta r}\alpha_\theta)T\right], \tag{2}$$

$$\sigma_\theta = \frac{E_\theta}{1 - v_{\theta r}v_{r\theta}}\left[\varepsilon_\theta + v_{r\theta}\varepsilon_r - (\alpha_\theta + v_{r\theta}\alpha_r)T\right], \tag{3}$$

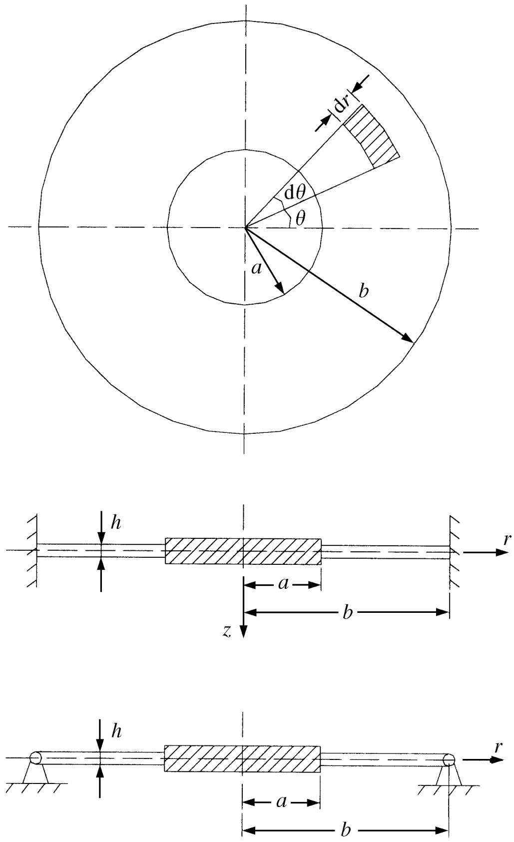


Figure 1. Geometry, boundary conditions and co-ordinates of the plates.

where $\sigma_r$ and $\sigma_\theta$ are the stresses, $E_r$ and $E_\theta$ the elastic moduli, $v_{\theta r}$ and $v_{r\theta}$ the Poisson ratios, $\alpha_r$ and $\alpha_\theta$ the thermal expansion coefficients, in $r$ and $\theta$ directions respectively. Here, it is assumed that these material constants of the plate do not change along with the temperature rise.

Substituting equation (1) into equations (2) and (3) and integrating the stresses through the thickness of the plate, we get the membrane forces and the bending moments of it as follows:

$$N_r = \int_{-h/2}^{h/2} \sigma_r \,\mathrm{d}z = C\left[\frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{v_{\theta r}}{r} u - \alpha_r(1 + \beta v_{\theta r})T\right], \tag{4}$$

$$N_\theta = \int_{-h/2}^{h/2} \sigma_\theta \,\mathrm{d}z = C\left[v_{\theta r}\left(\frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2\right) + \frac{k}{r} u - \alpha_r(k\beta + v_{\theta r})T\right], \tag{5}$$

$$M_r = \int_{-h/2}^{h/2} \sigma_r z \,\mathrm{d}z = -D\left(\frac{\partial^2 w}{\partial r^2} + \frac{v_{\theta r}}{r}\frac{\partial w}{\partial r}\right), \tag{6}$$

$$M_\theta = \int_{-h/2}^{h/2} \sigma_\theta z \,\mathrm{d}z = -D\left(v_{\theta r}\frac{\partial^2 w}{\partial r^2} + \frac{k}{r}\frac{\partial w}{\partial r}\right). \tag{7}$$

Here $k = E_\theta/E_r = \nu_{\theta r}/\nu_{r\theta}$, $\beta = \alpha_\theta/\alpha_r$, $D = E_r h^3/[12(1-\nu_{\theta r}\nu_{r\theta})]$, and $C = 12D/h^2$ are defined, where $k$ is regidity ratio, $\beta$ is the ratio of linear expansion coefficients, and $D$ is the flexural rigidity of the plate.

By neglecting the in-plane as well as the rotary inertia and applying Hamilton's principle [14], it can be shown [10, 11] that the non-dimensional equations of motion and the corresponding boundary conditions for the large-amplitude vibrations of the heated plate–rigid mass system have the forms of

$$\frac{\partial^2 U}{\partial x^2} + \frac{1}{x}\frac{\partial U}{\partial x} - \frac{k}{x^2}U + \frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x^2} + \frac{1-\nu_\theta}{2x}\left(\frac{\partial W}{\partial x}\right)^2 = \frac{\lambda}{12\delta^2}\left[(1-\mu)\frac{\Theta}{x} + \frac{\mathrm{d}\Theta}{\mathrm{d}x}\right], \tag{8}$$

$$\frac{\partial^4 W}{\partial x^4} + \frac{2}{x}\frac{\partial^3 W}{\partial x^3} - \frac{k}{x^2}\frac{\partial^2 W}{\partial x^2} + \frac{k}{x^3}\frac{\partial W}{\partial x} + \lambda\Theta\left(\frac{\partial^2 W}{\partial x^2} + \frac{1}{x}\frac{\partial W}{\partial x}\right) + \lambda\frac{\mathrm{d}\Theta}{\mathrm{d}x}\frac{\partial W}{\partial x} + \frac{\partial^2 W}{\partial \tau^2}$$
$$= 12\delta^2\frac{1}{x}\frac{\partial}{\partial x}\left[x\left(\frac{\partial U}{\partial x} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2 + \frac{\nu_\theta}{x}U\right)\frac{\partial W}{\partial x}\right], \tag{9}$$

$$U = 0, \frac{\partial W}{\partial x} = 0, \frac{\partial^3 W}{\partial x^3} + \frac{1}{x}\frac{\partial^2 W}{\partial x^2} + \frac{c\gamma}{2}\frac{\partial^2 W}{\partial \tau^2} = 0 \quad x = c, \tag{10}$$

$$U = 0,\ W = 0, \frac{\partial W}{\partial x} + K\frac{\partial^2 W}{\partial x^2} = 0 \quad \text{at } x = 1. \tag{11}$$

The non-dimensional quantities in the above equations are defined as

$$(c, x, U, W) = (a, r, u, w)/b, \qquad \tau = (t/b^2)(D/\rho h)^{1/2}, \quad \lambda = 12(1-\nu_\theta\beta)\delta^2\alpha_r T_0,$$
$$\delta = b/h, \quad \gamma = M_c/(\rho\pi h a^2), \quad \mu = (\nu_\theta + \beta k)/(1+\beta\nu_\theta), \quad \nu_\theta = \nu_{\theta r}, \tag{12}$$

where $\rho$ is the density of the plate, $K$ is a parameter related to the outer edge constraints and $K = 0$, $1/\nu_\theta$ represent the clamped and simply supported boundary conditions respectively. $T_0$ is a uniform temperature rise distribution, by which the temperature field of the plate can be expressed as $T(r) = T_0\Theta(x)$. For a specific temperature rise $T(r)$, $\Theta(x)$ is a given function.

If one lets the inertia terms equal zero and denotes $U(x,\tau) = U(x)$ and $W(x,\tau) = W(x)$ in equations (8)–(11), then the governing equations of thermal post-buckling of the heated plate can be obtained.

## 3. APPROXIMATE ANALYSIS

It is difficult to get any exact solutions of the dynamic equations (8)–(11) because of the inclusion of the coupled as well as the non-linear terms of the derivatives of the displacements. In the analysis and solution of this kind of equations two approximate methods are commonly used. One of them is known as "assumed-space-mode" solution, generally, which is achieved by implementing some assumed spatial shape functions and by using a variational method to eliminate the spatial variables and reduce the partial differential equations to ordinary ones only including time variable [7, 8, 10]. Another method is to find an "assumed-time-mode" solution. Upon assuming that a harmonic response for the non-linear vibrations exists, the time variable is eliminated by using a Kantorovich averaging method [9, 12, 13] and a non-linear boundary value problem is obtained. In the present investigation, the latter method is employed.

Assuming that the vibration is prior to the buckling of the plate, i.e., $\lambda < \lambda_{cr}$, where $\lambda_{cr}$ is the critical temperature parameter, and that harmonic response of equations (8)–(11) exists, the displacements $U(x, z)$ and $W(x, z)$ can be expressed as

$$U(x,\tau) = \xi_0(x) + \xi(x)\cos^2\omega\tau, \quad W(x,\tau) = \eta(x)\cos\omega\tau, \tag{13}$$

where $\omega$ is the non-dimensional frequency of the system, $\xi(x)$ and $\eta(x)$ are the shape functions corresponding to displacement $U$ and $W$ respectively. $\xi_0(x)$ is the solution of the static plane thermal stress problem

$$\xi_0'' + \frac{1}{x}\xi_0' - \frac{k}{x^2}\xi_0 = \frac{\lambda}{12\delta^2}\left[(1-\mu)\frac{\Theta}{x} + \Theta'\right]. \tag{14}$$

$$\xi_0(c) = 0, \quad \xi_0(1) = 0, \tag{15}$$

where the prime as the superscript denotes the ordinary differentiation with respect to $x$.

Substituting harmonic responses (13) into equation (8) and by using equations (14) and (15), one obtains the homogenous ordinary differential equation

$$\xi'' + \frac{1}{x}\xi' - \frac{k}{x^2}\xi + \eta''\eta' + \frac{1-\nu_\theta}{2x}(\eta')^2 = 0. \tag{16}$$

Equation (13) cannot satisfy equation (9) identically for all values of $\tau$ and, moreover, a residual may exist. So, a Kantorovich time-averaging method is applied to equation (9), yielding the homogenous ordinary differential equation

$$L(\eta,\lambda) = C_1\varphi(\xi,\eta) + C_2\psi(\xi_0,\eta), \tag{17}$$

where $C_1 = 9\delta^2$, $C_2 = 12\delta^2$ and

$$L(\eta,\lambda) = \eta'''' + \frac{2}{x}\eta''' - \frac{k}{x^2}\eta'' + \frac{k}{x^3}\eta' - \omega^2\eta + \lambda\Theta\left(\eta'' + \frac{1}{x}\eta'\right) + \lambda\eta'\Theta', \tag{18}$$

$$\varphi(\xi,\eta) = \frac{(1+\nu_\theta)}{x}\xi'\eta' + \xi'\eta'' + \xi''\eta' + \frac{\nu_\theta}{x}\xi\eta'' + \frac{3}{2}(\eta')^2\eta'' + \frac{1}{2x}(\eta')^3, \tag{19}$$

$$\psi(\xi_0,\eta) = \frac{(1+\nu_\theta)}{x}\xi_0'\eta' + \xi_0'\eta'' + \xi_0''\eta' + \frac{\nu_\theta}{x}\xi_0\eta''. \tag{20}$$

Substituting equation (13) into equations (10) and (11) gives the boundary conditions about shape functions $\xi(x)$ and $\eta(x)$ as

$$\xi(x) = 0,\ \eta'(x) = 0,\ \eta'''(x) + \frac{1}{x}\eta''(x) - \frac{\gamma c\omega^2}{2}\eta(x) = 0 \quad \text{at } x = c, \tag{21}$$

$$\xi(x) = 0,\ \eta(x) = 0,\ \eta'(x) + K\eta''(x) = 0 \quad \text{at } x = 1. \tag{22}$$

In addition to the boundary conditions, a normal relationship is proposed for the system, i.e.,

$$\eta(c) = A/\delta, \tag{23}$$

where $A = \delta\eta(c) = w(a,0)/h$ is the non-dimensional transverse amplitude of the inner edge of the plate. Equations (16)–(22) with normalization condition (23) constitute a non-linear boundary value problem including dynamic parameter $\omega$ and temperature, or load parameter, $\lambda$.

Letting $\omega = 0$, and $C_1 = C_2 = 12\delta^2$ in equations (16)–(23), the governing equations of thermal post-buckling of the heated plate can be obtained. The corresponding fundamental post-buckling responses are $(U, W) = (\xi_0 + \xi, \eta)$.

## 4. SHOOTING METHOD OF THE BOUNDARY-VALUE PROBLEM

It is difficult to obtain analytical solutions of the non-linear boundary-value problem of (16)–(23). Here, a shooting method, or trial and error method [11, 12, 15] is employed to get a numerical solution of the problem. For convenience, equations (16)–(23) are written in a standard form, a system of first order non-linear ordinary differential equations, as follows:

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x} = \mathbf{H}(x, \mathbf{Y}; \lambda) \quad (c < x < 1), \tag{24}$$

$$\mathbf{B}_1\mathbf{Y}(c) = \{0, A/\delta, 0, 0\}^{\mathrm{T}}, \quad \mathbf{B}_2\mathbf{Y}(1) = \{0, 0, 0\}^{\mathrm{T}}. \tag{25a, b}$$

with

$$\mathbf{Y} = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}^{\mathrm{T}} = \{\xi, \xi', \eta, \eta', \eta'', \eta''', \omega^2\}^{\mathrm{T}}, \tag{26}$$

$$\mathbf{H} = \{y_2, \phi_1, y_4, y_5, y_6, \phi_2, 0\}^{\mathrm{T}}, \tag{27}$$

$$\phi_1 = -y_2/x + ky_1/x^2 - y_4y_5 - (1 - \nu_\theta)y_4^2/(2x), \tag{28}$$

$$\phi_2 = -\frac{2}{x}y_6 + \frac{k}{x^2}y_5 - \frac{k}{x^3}y_4 + y_7y_3 - \lambda\Theta\left(y_5 + \frac{1}{x}y_4\right) - \lambda\Theta' y_4 + C_1\varphi + C_2\psi, \tag{29}$$

$$\varphi = (1 + \nu_\theta)y_2y_4/x + y_2y_5 + \phi_1y_4 + \nu_\theta y_1y_5/x + 3y_4^2y_5/2 + y_4^3/(2x), \tag{30}$$

$$\psi = (1 + \nu_\theta)\xi_0' y_4/x + \xi_0' y_5 + \xi_0'' y_4 + \nu_\theta\xi_0 y_5/x, \tag{31}$$

$$\mathbf{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{c\gamma A}{2\delta} & 0 & \dfrac{1}{c} & 1 & 0 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & K & 0 & 0 \end{bmatrix}. \tag{32}$$

Let us consider the initial problem corresponding to boundary-value problem (24)–(25)

$$\frac{\mathrm{d}\mathbf{Z}}{\mathrm{d}x} = \mathbf{H}(x, \mathbf{Z}; \lambda) \quad \text{for } x > c, \tag{33}$$

$$\mathbf{Z}(c) = \mathbf{I}(A, \mathbf{V}) = \{0, v_1, A/\delta, 0, v_2, (-v_2/c + Av_3\gamma c/2\delta), v_3\}^{\mathrm{T}}, \tag{34}$$

where $\mathbf{Z} = \{z_1, z_2, z_3, z_4, z_5, z_6, z_7\}^{\mathrm{T}}$, $\mathbf{V} = \{v_1, v_2, v_3\}^{\mathrm{T}}$ is an unknown vector related to the missing initial values of $\mathbf{Y}$ at $x = c$. A solution of initial problem (33)–(34) can be symbolically expressed as

$$\mathbf{Z}(x; A, \mathbf{V}, \lambda) = \mathbf{I}(A, \mathbf{V}) + \int_c^x \mathbf{H}(\zeta, \mathbf{Z}; \lambda)\,\mathrm{d}\zeta. \tag{35}$$

For a prescribed value of $A$, the components of $\mathbf{V}$ are searched for such that solution (35) also satisfies boundary condition (25b), i.e.,

$$\mathbf{B}_2\mathbf{Z}(1; A, \mathbf{V}, \lambda) = \{0, 0, 0\}^{\mathrm{T}}. \tag{36}$$

Clearly, if $\mathbf{V} = \mathbf{V}^*$ is a root of equation (36), the solution for the boundary-value problem (24)–(25) is then obtained as

$$\mathbf{Y}(x) = \mathbf{Z}(x; A, \mathbf{V}^*, \lambda). \tag{37}$$

Therefore, a harmonic response of equations (8)–(11) is obtained in the form of equation (13).

In order to find the solution of the thermal post-buckling of the plate, a similar procedure as just mentioned above can be established through letting $C_1 = C_2$, $\omega = 0$ and $y_7 = \lambda$ in equations (24) and (25).

## 5. NUMERICAL RESULTS AND DISCUSSIONS

In this paper, only the case of a uniform temperature rise is considered. Then we have $\Theta(x) \equiv 1$. It is easy to find that the solution of equations (14) and (15) is

$$\xi_0(x) = \begin{cases} \dfrac{B}{1-k}\left(\dfrac{\left(c - c^{-\sqrt{k}}\right)x^{\sqrt{k}} - \left(c - c^{\sqrt{k}}\right)x^{-\sqrt{k}}}{c^{-\sqrt{k}} - c^{\sqrt{k}}} + x\right) & \text{for } k \neq 1 \\ 0, & \text{for } k = 1. \end{cases} \tag{38}$$

where $B = (1-\mu)\lambda/(12\delta^2)$. By employing a fourth order Runge–Kutta method with variable steps to integrate equation (35) and, at the same time, by using a Newton–Raphson method to find the root $V^*$ of algebraic equation (36), numerical solutions of equations (24) and (25) have been obtained. An $A$-dependent family of solutions for equations (24) and (25) are arrived at by analytic continuation [9, 15], if parameter $A$ is repeatedly increased by a given small step.

Throughout the following numerical computation, let geometric parameters $\delta = 30$, and Poisson ratio $v_\theta = 0{\cdot}3$. A relative error limit, $\varepsilon = 10^{-5}$, was taken to warrant that both the numerical integration of equation (35) and the successive correction of equation (36) were carried out until the error norm became less than $\varepsilon$. For an unheated ($\lambda = 0$) circular plate ($c = 0{\cdot}0001$) without rigid mass ($\gamma = 0$), a comparison of the values of linear fundamental frequency in this paper with those obtained by Ritz's method in reference [5] is presented below. For prescribed values of the rigidity ratio $k = 0{\cdot}75$, $1{\cdot}0$, $10{\cdot}0$, the corresponding fundamental frequencies in this paper are $\omega = 4{\cdot}5421$, $4{\cdot}9351$, $11{\cdot}286$ and those in reference [5] are $\omega = 4{\cdot}5418$, $4{\cdot}9351$, $11{\cdot}286$, which shows an excellent agreement with the published results.

For some prescribed values of the rigidity parameter $k$ and the parameter of thermal expansion coefficient $\beta$, the characteristic curves of the linear fundamental frequency $\omega$ versus the temperature parameter $\lambda$ of the clamped plate with $c = 0{\cdot}1$, $\lambda = 2{\cdot}0$ are plotted in Figure 2. It is found that the frequency increases monotonously with the increment of values of parameter $k$ and decreases with that of temperature rise parameter $\lambda$. The linear frequency becomes zero when temperature parameter $\lambda$ reaches its critical value $\lambda_{cr}$, over which the plate will be in a thermally buckled state. Figure 3 also shows similar

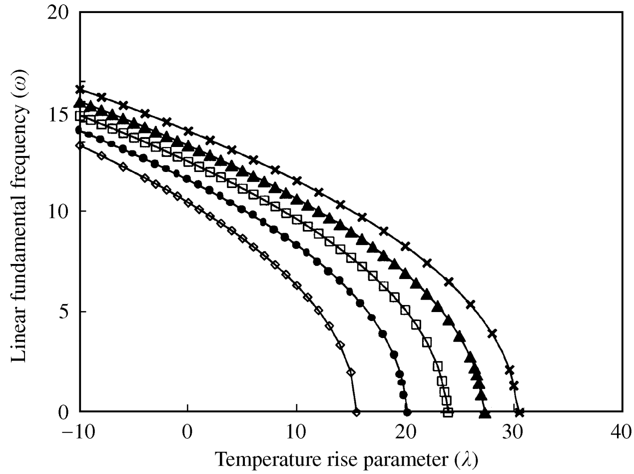


Figure 2. Temperature rise parameter $\lambda$ versus the linear frequency $\omega$ for the clamped plate with $c = 0{\cdot}1$, $\gamma = 2{\cdot}0$: —◇—, $k = 1{\cdot}0$, $\beta = 10$; —●—, $k = 2{\cdot}0$, $\beta = 0{\cdot}5$; —□—, $k = 3{\cdot}0$, $\beta = 0{\cdot}3$; —▲—, $k = 4{\cdot}0$, $\beta = 0{\cdot}25$; —×—, $k = 5{\cdot}0$; $\beta = 0{\cdot}2$.

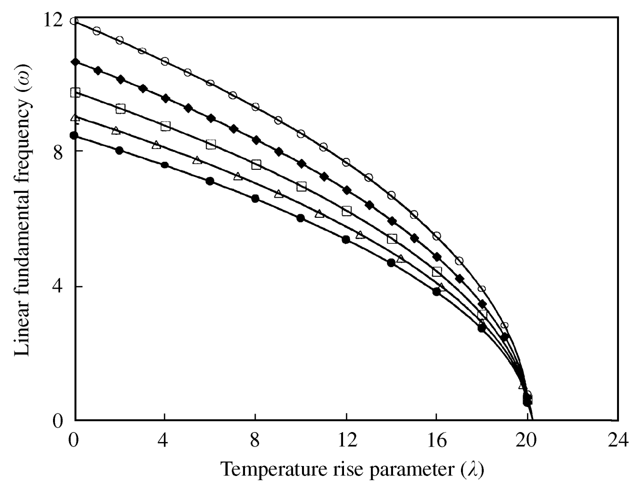


Figure 3. Temperature rise parameter $\lambda$ versus linear fundamental frequency $\omega$ for the clamped plate with $c = 0{\cdot}1$, $k = 2{\cdot}0$, $\beta = 0{\cdot}5$: —○—, $\gamma = 0$; —◆—, $\gamma = 5$; —□—, $\gamma = 10$; —△—, $\gamma = 15$; —●—, $\gamma = 20$.

characteristic relations between $\omega$ and $\lambda$ for the clamped plate with different values of rigid mass parameter $\gamma$, from which it can be seen that the frequency decreases with the increase of value of $\gamma$. As expected, the critical temperature parameter $\lambda_{cr}$ is independent of parameter $\gamma$.

The non-linear responses of fundamental frequency $\omega$ versus the non-dimensional amplitude $A$ for the heated clamped as well as the simply supported annular plates with different magnitudes of $\lambda$ are shown, in Figures 4 and 5 respectively. It can also be found that the fundamental frequencies decrease when the temperature increases. The effect of temperature parameter $\lambda$ on the frequency $\omega$ is more significant when the amplitude $A$ tends to be infinitesimal. Nevertheless, it decreases with the increment of the values of the amplitude $A$. Figure 6 illustrates the relationship of $A \sim \omega$ of a simply supported plate

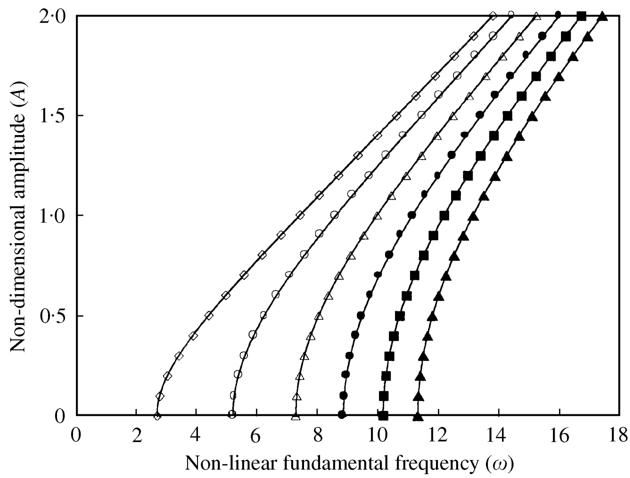


Figure 4. Non-linear fundamental frequency responses of the clamped plate with $c = 0{\cdot}1$, $k = 2{\cdot}0$, $\beta = 0{\cdot}5$ and $\gamma = 2{\cdot}0$ for some given values of $\lambda$: —◇—, $\lambda = 19$; —○—, $\lambda = 16$; —△—, $\lambda = 12$; —●—, $\lambda = 8$; —■—, $\lambda = 4$; —▲—, $\lambda = 0$.

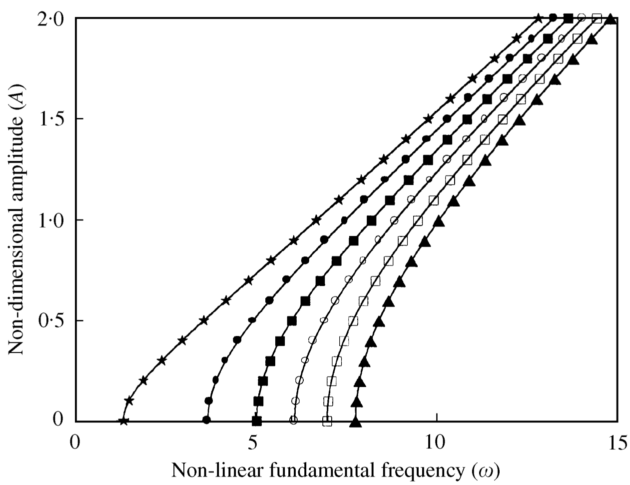


Figure 5. Non-linear fundamental frequency responses of the simply supported plate with $c = 0{\cdot}1$, $k = 2{\cdot}0$, $\beta = 0{\cdot}5$ and $\gamma = 2{\cdot}0$ for some prescribed values of $\lambda$: —★—, $\lambda = 6$; —●—, $\lambda = 4$; —■—, $\lambda = 2$; —○—, $\lambda = 0$; —□—, $\lambda = -2$; —▲—, $\lambda = -4$.

without temperature rise for some prescribed values of $k$. For some prescribed pairs of values of $k$ and $\beta$, Figure 7 shows the non-linear characteristic relationships between amplitude parameter $A$ and frequency parameter $\omega$ of a clamped annular plate with $\lambda = 5{\cdot}0$. It is obvious that non-linear frequency increases with the increments of rigidity parameter $k$. Also, from the curves in Figures 4–7 one can find that the characteristics of the amplitude–frequency responses of the plate–rigid mass system are similar to that of a hard-spring Duffing's system.

Let $C_1 = C_2 = 12\delta^2$, $\omega = 0$ and $y_7 = \lambda$ in equations (24) and (25); through a similar procedure the thermal post-buckling responses of the plate have been obtained. For some prescribed pairs of values of $(k, \beta)$, the secondary equilibrium paths in terms of

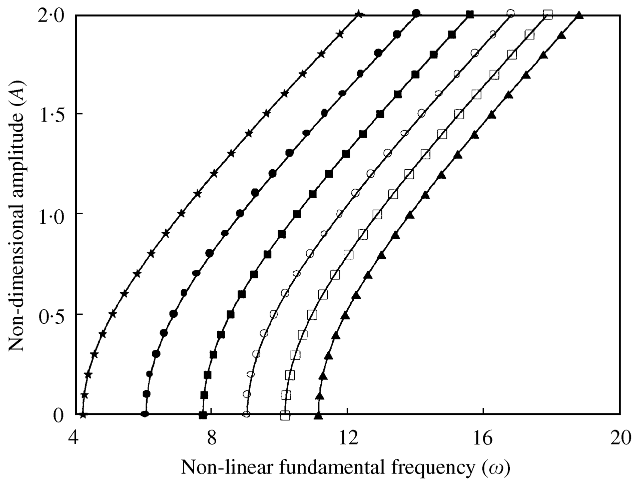


Figure 6. Non-linear fundamental frequency responses of simply supported plate with $c = 0{\cdot}1$, $\lambda = 0{\cdot}0$, and $\gamma = 2{\cdot}0$ for some prescribed values of $k$: —▲—, $k = 10$; —□—, $k = 8$; —○—, $k = 6$; —■—, $k = 4$; —●—, $k = 2$; —★—, $k = 0{\cdot}5$.

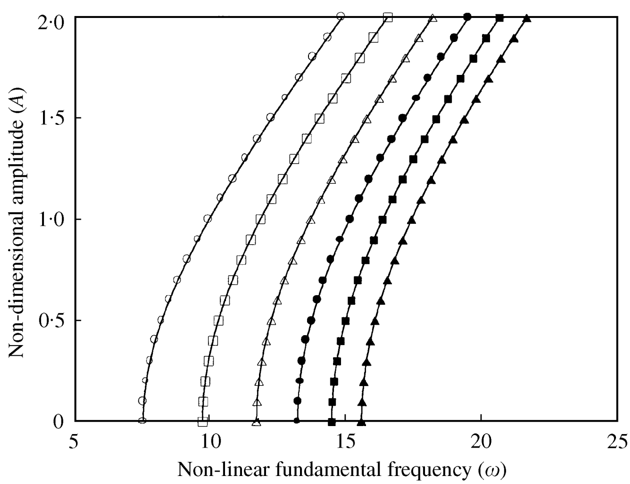


Figure 7. Non-linear fundamental frequency responses of the clamped plate with $c = 0{\cdot}1$, $\lambda = 5{\cdot}0$ and $\gamma = 2{\cdot}0$ for some prescribed values of $k$ and $\beta$: —○—, $k = 0{\cdot}5$, $\beta = 2{\cdot}0$; —□—, $k = 2{\cdot}0$, $\beta = 0{\cdot}5$; —△—, $k = 4{\cdot}0$, $\beta = 0{\cdot}25$; —●—, $k = 6{\cdot}0$, $\beta = 0{\cdot}167$; —■—, $k = 8{\cdot}0$, $\beta = 0{\cdot}125$; —▲—, $k = 10{\cdot}0$, $\beta = 0{\cdot}1$.

non-dimensional buckled deflection $A$ and temperature rise parameter $\lambda$ of a clamped annular plate are plotted in Figure 8. The same results of the simply supported plate are also shown in Figure 9. Apparently, it can be found that each of these curves bifurcates at the point $(A, \lambda) = (0, \lambda_{cr})$.

## 6. CONCLUSIONS

Based on von Karman's plate theory and Hamilton's principle the governing equations, in terms of the displacements of the middle plane, of non-linear vibration and thermal

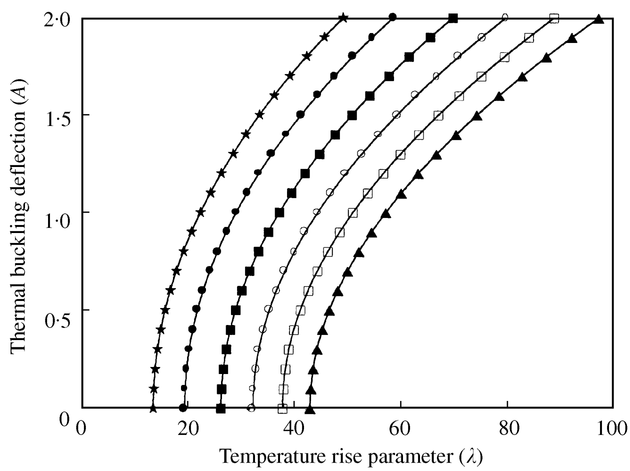


Figure 8. The thermal buckled equilibrium paths of the clamped plate with $c = 0{\cdot}1$ for some prescribed values of $k$ and $\beta$: —★—, $k = 0{\cdot}5$, $\beta = 2{\cdot}0$; —●—, $k = 2{\cdot}0$, $\beta = 0{\cdot}5$; —■—, $k = 4{\cdot}0$, $\beta = 0{\cdot}25$; —○—, $k = 6{\cdot}0$, $\beta = 0{\cdot}167$; —□—, $k = 8{\cdot}0$, $\beta = 0{\cdot}125$; —▲—, $k = 10{\cdot}0$, $\beta = 0{\cdot}1$.

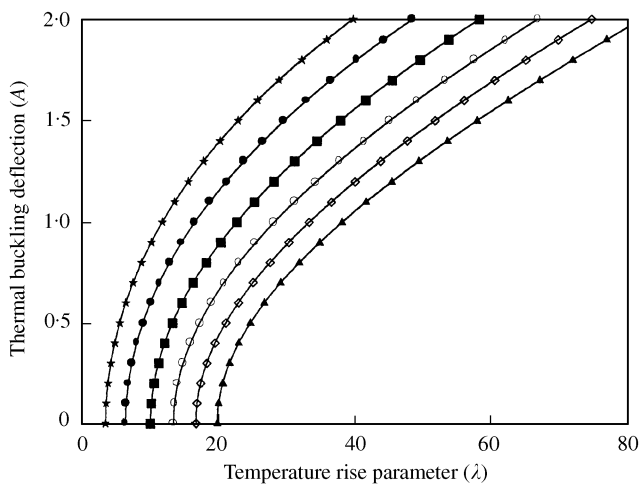


Figure 9. The thermal buckled equilibrium paths of the simply supported plate with $c = 0{\cdot}1$ for some prescribed values of $k$ and $\beta$: —★—, $k = 0{\cdot}5$, $\beta = 2{\cdot}0$; —●—, $k = 2{\cdot}0$, $\beta = 0{\cdot}5$; —■—, $k = 4{\cdot}0$, $\beta = 0{\cdot}25$; —◇—, $k = 6{\cdot}0$, $\beta = 0{\cdot}167$; —○—, $k = 8{\cdot}0$, $\beta = 0{\cdot}125$; —▲—, $k = 10{\cdot}0$, $\beta = 0{\cdot}1$.

post-buckling of a heated orthotropic annular plate with rigidly attached central rigid mass are developed. By using Kantorovich's time-averaging method the obtained non-linear partial differential equations are transformed into a boundary value problem of non-linear ordinary differential equations. The non-linear responses of harmonic vibrations of the plate–rigid mass system and also the thermal buckled configurations of the plate are numerically obtained by a shooting method. The effects of temperature rise, material rigidity, and central rigid mass on the fundamental frequencies of the clamped and simply supported plate are investigated and the corresponding numerical results are shown in the characteristic curves respectively. The characteristics exhibited by the responses of the plate are similar to those of a hard-spring Duffing system. For both kinds of the outer edge constraints, the fundamental frequency is found to decrease with the increasing values of temperature rise. This is due to the fact that the compressive membrane stresses developed by the temperature rise will reduce the transverse rigidity of the plate. So, from this standpoint one can realize a control over the vibration frequencies of elastic elements just by adjusting the temperature change imposed on them. The thermal post-buckling analysis of the heated orthotropic annular plate has been carried out also by means of the same numerical procedure. For the plate with some prescribed values of the geometrical and material parameters, secondary equilibrium paths in terms of the non-dimensional central deflections and the temperature rise are plotted.

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