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REDUCTION OF CONTROL SPILLOVER IN ACTIVE VIBRATION CONTROL OF DISTRIBUTED STRUCTURES USING MULTIOPTIMAL SCHEMES

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1. INTRODUCTION

Distributed structures are infinite-dimensional. Since it is impossible, in practice, to control or estimate the entire infinity of modes, vibration control of such a structure is limited to a finite number of modes. Both observation and control spillover problems thus arise due to the modal truncation [1]. They both degrade the system's performance, and the former can even cause the system to become unstable [2].

In reference [3], strategies for reducing observation spillover are discussed. This letter concerns an approach to the reduction of control spillover under the assumption that observation spillover is eliminated using such approaches.

One approach to vibration control, feedback wave control [4–6], involves the control of the propagation and transmission of vibrational waves using sensors and actuators. These are often collocated (but need not be), and their effect can be interpreted in terms of impedance-matched, frequency-dependent damping. The controller can be designed to optimally damp or absorb the vibrational energy over a broad frequency band, and works especially well at relatively high frequencies. The approach has been used in the vibration suppression of structures with one-dimensional members.

In this letter, wave feedback control and conventional quadratic optimal control are combined. The feedback wave controller adds damping to the high-frequency residual modes and reduces their resonant response, hence reducing the effects of control spillover. Stability and robustness issues are also discussed.

2. MODAL DECOMPOSITION AND STATE-SPACE REPRESENTATION OF THE STRUCTURE

The equation of motion of a distributed system with viscous damping can be written as [2]

$$\vartheta w(x,t) + \upsilon \dot{w}(\mathrm{x},\mathrm{t}) + m(x)\ddot{w}(x,t) = f(x,t), \tag{1}$$

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where $w(x,t)$ is the displacement, ϑ and υ stiffness and damping operators, $m(x)$ the mass density, t the time, $f(x,t)$ the applied force(s) and x a one-, two- or three-component position vector for one-, two- or three-dimensional structures respectively.

Expressing the response as a sum of modal components using the undamped modes of the structure gives

$$w(x,t) = \sum_{j=1}^{T} \phi_j(x) q_j(t), \tag{2}$$

with $\phi_j(x)$ denoting the jth mode shape and $q_j(t)$ the jth modal co-ordinate. As stated earlier, T is perhaps large. Among the T modes, assume that only N modes are to be controlled. The remaining $(T-N)$ modes are then the residual modes. Hence, the displacement can be expressed as

$$w(x,t) = w_C(x,t) + w_R(x,t), \tag{3}$$

where

$$w_C(x,t) = \sum_{j=1}^{N} \phi_j(x) q_j(t), \quad w_R(x,t) = \sum_{j=N+1}^{T} \phi_j(x) q_j(t) \tag{4}$$

give the contributions to the response of the controlled and residual modes respectively. Substituting equation (2) into equation (1) and using the orthonormality properties of the mode shapes, equation (1) becomes

$$\ddot{q}_j(t) + \sum_{k=1}^{T} c_{jk} \dot{q}_j(t) + \omega_j^2 q_j(t) = f_j(t), \quad j = 1, 2, \ldots, T, \tag{5}$$

where ω_j denotes the undamped natural frequency of the jth mode and

$$c_{jk} = \int \phi_j \upsilon \phi_k \, \mathrm{d}x, \quad f_j(t) = \int f(x,t) \phi_j(x) \, \mathrm{d}x, \qquad k, j = 1, 2, \ldots, T. \tag{6}$$

Equation (5) can be written in matrix form as

$$\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t), \tag{7}$$

where $\mathbf{C} = [c_{jk}]$ is the damping matrix, $\mathbf{K} = diag(\omega_j^2)$ the diagonal stiffness matrix, and $\mathbf{q}(t) = [q_1(t)\, q_2(t) \ldots q_T(t)]^{\mathrm{T}}$ and $\mathbf{f}(t) = [f_1(t)\, f_2(t) \ldots f_T(t)]^{\mathrm{T}}$ denote vectors of modal amplitudes and forces respectively. If the damping is proportional then $\mathbf{C}$ is also diagonal, while if the structure is undamped, $\mathbf{C} = \mathbf{0}$.

Introducing the state vector

$$\mathbf{X}(t) = [\mathbf{q}^{\mathrm{T}}(t) : \dot{\mathbf{q}}^{\mathrm{T}}(t)]^{\mathrm{T}} \tag{8}$$

allows equation (7) to be written in state-space form as

$$\dot{\mathbf{X}}(t) = \mathbf{A}\mathbf{X}(t) + \mathbf{B}\mathbf{f}(t), \tag{9}$$

where the matrices

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} & -\mathbf{C} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \tag{10}$$

and $\mathbf{0}$ and $\mathbf{I}$ denote null and identity matrices respectively.

Equation (9) can be partitioned into controlled and residual modes $\mathbf{q}_C$ and $\mathbf{q}_R$ and their states $\mathbf{X}_C$ and $\mathbf{X}_R$. For the case of proportional damping, when both $\mathbf{K}$ and $\mathbf{C}$ are diagonal, we have

$$\begin{bmatrix} \dot{\mathbf{X}}_C^{\mathrm{T}} \\ \dot{\mathbf{X}}_R^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_C & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_R \end{bmatrix} \begin{bmatrix} \mathbf{X}_C^{\mathrm{T}} \\ \mathbf{X}_R^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_C \\ \mathbf{B}_R \end{bmatrix} \mathbf{f}(t), \tag{11}$$

where

$$\mathbf{A}_C = \begin{bmatrix} \mathbf{0} & \mathbf{I}_C \\ (-\mathbf{K})_C & (-\mathbf{C})_C \end{bmatrix}, \quad \mathbf{B}_C = \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_C \end{bmatrix}, \tag{12}$$

$$\mathbf{A}_R = \begin{bmatrix} \mathbf{0} & \mathbf{I}_R \\ (-\mathbf{K})_R & (-\mathbf{C})_R \end{bmatrix}, \quad \mathbf{B}_R = \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_R \end{bmatrix}. \tag{13}$$

Thus the state-space equation can be written as

$$\dot{\mathbf{X}}_C(t) = \mathbf{A}_C X_C(t) + \mathbf{B}_C \mathbf{f}(t) \tag{14}$$

and

$$\dot{\mathbf{X}}_R(t) = A_R \mathbf{X}_R(t) + \mathbf{B}_R \mathbf{f}(t). \tag{15}$$

3. CONTROL DESIGN

3.1. OPTIMAL CONTROL WITH QUADRATIC PERFORMANCE INDICES

In optimal state-space control, control based on equation (14) is designed for N of the T modes of the structure. A quadratic performance index

$$\mathbf{J} = \mathbf{X}_C^{\mathrm{T}}(t)\mathbf{Q}\mathbf{X}_C(t) + \mathbf{f}^{\mathrm{T}}(t)\mathbf{R}\mathbf{f}(t) \tag{16}$$

can be chosen, where $\mathbf{R}$ is a real, symmetric, positive definite matrix, and $\mathbf{Q}$ a real, symmetric, non-negative definite matrix. The problem can be interpreted as driving the state variables $\mathbf{X}_C$ as close as possible to zero while placing a penalty on the control effort $\mathbf{f}$.

The optimal control force is given by

$$\mathbf{f}(t) = -\mathbf{R}^{-1}\mathbf{B}_C^{\mathrm{T}}\mathbf{G}(t)\mathbf{X}_C(t), \tag{17}$$

where $\mathbf{G}(t)$ is the solution of the Riccati equation [7, 8]. Partitioning $\mathbf{G}(t)$ into upper and lower parts $\mathbf{G}(t) = [\mathbf{G}_U \,\vdots\, \mathbf{G}_L]$ and noting equations (8), (12) and (13), the closed-loop state-space equation is then

$$\begin{bmatrix} \dot{\mathbf{q}}_C^{\mathrm{T}} \\ \ddot{\mathbf{q}}_C^{\mathrm{T}} \\ \dot{\mathbf{q}}_R^{\mathrm{T}} \\ \ddot{\mathbf{q}}_R^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_C & \vdots & \mathbf{0} & \mathbf{0} \\ (-\mathbf{K})_C & (-\mathbf{R}^{-1}\mathbf{G}_L)_{CC} & \vdots & \mathbf{0} & \mathbf{0} \\ \cdots & \cdots & \vdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \vdots & \mathbf{0} & \mathbf{I}_R \\ \mathbf{0} & (-\mathbf{R}^{-1}\mathbf{G}_L)_{RC} & \vdots & (-\mathbf{K})_R & (-\mathbf{C})_R \end{bmatrix} \begin{bmatrix} \mathbf{q}_C^{\mathrm{T}} \\ \dot{\mathbf{q}}_C^{\mathrm{T}} \\ \mathbf{q}_R^{\mathrm{T}} \\ \dot{\mathbf{q}}_R^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_C \\ \mathbf{0} \\ \mathbf{I}_R \end{bmatrix} \mathbf{f}. \tag{18}$$

It can be seen that control spillover arises from and is described by the term $(-\mathbf{R}^{-1}\mathbf{G}_L)_{RC}$.

3.2. COMBINED LINEAR QUADRATIC OPTIMAL CONTROL AND OPTIMAL DAMPING FEEDBACK WAVE CONTROL

3.2.1. *Optimal damping feedback wave control*

Vibrations can be described as waves travelling in solids. In the feedback wave control considered here, a collocated sensor and actuator are assumed to be applied to a region of a structure to control the propagation of waves there. It is difficult to go much further without specifying the details of the structure at this control position, since the characteristics of wave motion depend on these local details. In this letter, it is assumed that the control is applied to a part of the structure that vibrates as a thin beam in bending.

Assuming the wave control force is a point force, it can be written in the frequency domain as

$$F_w = -H_w(\omega)w, \tag{19}$$

where $H_w(\omega)$ is the frequency response of the controller.

This point control force acts as a discontinuity in the one-dimensional waveguide. Upon meeting this discontinuity, an incident wave is partially transmitted and partially reflected. The transmission and reflection coefficients can be obtained by considering the continuity and equilibrium conditions at the excitation point [9]. For waves in a beam, and neglecting any incident near-field flexural waves, the transmission and reflection coefficients are

$$t = 1 + i\mu, \quad r = i\mu, \tag{20}$$

where

$$\mu = \frac{H(\omega)}{\omega^{3/2} - (1+i)H(\omega)}, \quad H(\omega) = \frac{H_w(\omega)}{4\sqrt[4]{m^3EI}}. \tag{21}$$

In this letter, the controller is designed to absorb incident vibrational energy by adding optimal damping to the structure. Furthermore, in this example the controller $H_w(\omega)$ is constrained to be a pure imaginary function in the frequency domain, i.e.,

$$H_w(\omega) = \mathrm{i}\omega g. \tag{22}$$

The optimal control gain g can be found by assuming that a wave is incident on one side of the control location and then by designing the control gain so as to maximize the absorbed incoming energy, in other words to minimize $|r|^2 + |t|^2$. In this case, the control gain is found to be

$$g = 4\sqrt[4]{m^3 EI}\sqrt{\omega/2}. \tag{23}$$

It follows that the optimal controller is

$$H_w(\omega) = \mathrm{i}\omega 4\sqrt[4]{m^3 EI}\sqrt{\omega/2}. \tag{24}$$

The ideal controller is frequency-dependent and non-causal [10]. For a practical implementation, a causal controller must be found which is an approximation to the ideal. One such approximation is found by setting the controller to be constant and tuning it to be optimal at a certain frequency ω_d. The controller then becomes velocity feedback, and the control gain becomes

$$g = 4\sqrt[4]{m^3 EI}\sqrt{\omega_d/2}. \tag{25}$$

Apart from the optimal, tuned, damping wave feedback described above, other control strategies may also be adopted. For example, a strategy may aim at either maximizing the energy absorbed by the controller or isolating the vibrations (i.e., stopping energy transmission from one part of the structure to another) [5, 6, 11].

3.2.2. *Combined linear quadratic optimal control and feedback wave control*

To alleviate control spillover, it is proposed that conventional linear quadratic optimal control, designed on the basis of the lowest modes of the structure, is combined with feedback wave control designed to add damping to the higher modes.

In the time domain, the tuned causal wave feedback control force (assumed to be applied at point x_w) described by equations (19), (22) and (25) is

$$f_w(x,t) = -g\dot{w}(x,t)\delta(x - x_w). \tag{26}$$

From equation (2), one has

$$f_w(x,t) = -g\sum_{i=1}^{T}\phi_i(x)\dot{q}_i(t)\delta(x - x_w). \tag{27}$$

Transforming the force into modal space by using equation (6) gives the modal forces

$$f_{wj}(x,t) = \int_0^L f_w(x,t)\phi_j(x)\,\mathrm{d}x = -g\sum_{i=1}^{T}\phi_j(x_w)\phi_i(x_w)\dot{q}_i(t), \tag{28}$$

where $j = 1, 2, \ldots, T$.

Equation (28) can be partitioned as

$$\mathbf{f}_w = -[\mathbf{0}_{wC} \;\; \mathbf{G}_{wC} \;\; \mathbf{0}_{wR} \;\; \mathbf{G}_{wR}]\begin{bmatrix}\mathbf{q}_C^{\mathrm{T}} \\ \dot{\mathbf{q}}_C^{\mathrm{T}} \\ \mathbf{q}_R^{\mathrm{T}} \\ \dot{\mathbf{q}}_R^{\mathrm{T}}\end{bmatrix}, \tag{29}$$

where $\mathbf{f}_w(t) = [f_{w1}(t)\, f_{w2}(t) \cdots f_{wT}(t)]^{\mathrm{T}}$ is a generalized wave control force vector, and $\mathbf{0}_{wC}$ and $\mathbf{0}_{wR}$ are null matrices of the same dimensions as matrices $\mathbf{G}_{wC}$ and $\mathbf{G}_{wR}$, respectively, and

$$(\mathbf{G}_{wC})_{ij} = g\phi_i(x_w)\phi_j(x_w), \quad i = 1, 2, \ldots, r;\ j = 1, 2, \ldots, N, \tag{30}$$

$$(\mathbf{G}_{wR})_{ij} = g\phi_i(x_w)\phi_{N+j}(x_w), \quad i = 1, 2, \ldots, r;\ j = 1, 2, \ldots, T - N. \tag{31}$$

The closed-loop state-space equation then becomes

$$\begin{bmatrix} \dot{\mathbf{q}}_C^{\mathrm{T}} \\ \ddot{\mathbf{q}}_C^{\mathrm{T}} \\ \dot{\mathbf{q}}_R^{\mathrm{T}} \\ \ddot{\mathbf{q}}_R^{\mathrm{T}} \end{bmatrix} = \left[\begin{array}{cc:cc} \mathbf{0} & \mathbf{I}_C & \mathbf{0} & \mathbf{0} \\ (-\mathbf{K})_C & (-\mathbf{R}^{-1}\mathbf{G}_L)_{CC} - \mathbf{G}_{wC} & \mathbf{0} & -\mathbf{G}_{wR} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_R \\ \mathbf{0} & (-\mathbf{R}^{-1}\mathbf{G}_L)_{RC} - \mathbf{G}_{wC} & (-\mathbf{K})_R & (-\mathbf{C})_R - \mathbf{G}_{wR} \end{array}\right] \begin{bmatrix} \mathbf{q}_C^{\mathrm{T}} \\ \dot{\mathbf{q}}_C^{\mathrm{T}} \\ \mathbf{q}_R^{\mathrm{T}} \\ \dot{\mathbf{q}}_R^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_C \\ \mathbf{0} \\ \mathbf{I}_R \end{bmatrix} \mathbf{f}. \tag{32}$$

The optimal wave feedback controller is thus seen to add damping to all the modes but also couples the modes of the structure.

It is now appropriate to make some comments regarding stability and robustness. Bounded input bounded output stability requires that no eigenvalues of a system have positive real parts for the system to be asymptotically stable. For an asymptotically stable system, the relative stability is measured by the dominant eigenvalues: the larger their (negative) real parts, the more stable the system.

From equation (5), it can be seen that with damping, the original uncontrolled system is an asymptotically stable system. Since the stability of the state-space equations ensures the existence of the optimal controller that guarantees the stability of the closed-loop system [8], equation (18), the closed-loop state-space equation after the implementation of the optimal quadratic controller, is also stable. If $\mathbf{R}$ is chosen to be a diagonal matrix, minimum multivariable gain and phase margins are guaranteed [12].

Comparing equations (18) and (32), it can be seen that the feedback wave controller adds damping to each mode. As a result, the poles/eigenvalues of the vibration modes are moved further to the left in the s-plane. This, of course, improves the stability of the system. To check the improvement of the robustness, denoting the characteristics matrices and the poles/eigenvalues of the ith mode related to equations (32) and (18) as $\mathbf{A}_{wave}$, $\mathbf{A}_{LQ}$, λ_{wave-i} and λ_{LQ-i}, respectively, we have

$$|\lambda_{wave-i}| > |\lambda_{LQ-i}|. \tag{33}$$

The sensitivity $\mathbf{S}$ is closely related to the return difference matrix $\mathbf{R}_D$. The exact form of the return difference matrix is dependent on the location of the loop broken point. However, the determinants of the return difference matrices are the same [13]:

$$|\mathbf{R}_D| = \frac{|s\mathbf{I} - \mathbf{A}_{wave}|}{|s\mathbf{I} - \mathbf{A}_{LQ}|}. \tag{34}$$

We then have

$$|\mathbf{S}| = \frac{1}{|\mathbf{R}_D|} = \frac{|s\mathbf{I} - \mathbf{A}_{LQ}|}{|s\mathbf{I} - \mathbf{A}_{wave}|} = \frac{|\prod_{i=1}^{2T} \lambda_{LQ-i}|}{|\prod_{i=1}^{2T} \lambda_{wave-i}|}. \tag{35}$$

From equations (33) and (35), it is seen that $|\mathbf{S}| < 1$, which indicates an improvement in the robustness of the system. Hence, compared to the traditional linear quadratic optimal control, the multi-optimal control scheme not only reduces the control spillover, but also improves the stability and the robustness of the system. These statements reflect the obvious fact that adding damping through the collocated wave control improves stability and robustness.

4. NUMERICAL EXAMPLE

The bending vibration control of a uniform simply supported beam is considered as an example. It is assumed that the first 5 modes are excited and only the first 3 modes are modelled accurately and included in the quadratic optimal control. Thus $T = 5$ and $N = 3$. The natural frequencies ω_i and mass-normalized mode shapes $\phi_i(x)$ are given by

$$\omega_i = \sqrt{\frac{EI}{\rho A}}\left(\frac{\mathrm{i}\pi}{L}\right)^2, \quad \phi_i(x) = \sqrt{\frac{2}{\rho AL}}\sin\left(\frac{\mathrm{i}\pi}{L}x\right), \quad i = 1, 2, \ldots, T, \tag{36}$$

where EI is the bending stiffness, ρA the mass per unit length and L the length of the beam. For simplicity, all these quantities equal unity in these examples. The position x is measured from one end of the beam. A damping ratio of 0·01 is assumed for the modes of the uncontrolled system.

Linear quadratic optimal control is designed for the first three modes of vibration, while the wave velocity feedback controller is tuned to be optimal at the fourth natural frequency. The locations of the disturbance input, the linear quadratic optimal controller and the wave controller are $0{\cdot}43L$, $0{\cdot}70L$ and $0{\cdot}10L$, respectively, and the response is observed at $0{\cdot}36L$ (see Figure 1).

The fraction of the incoming vibrational energy absorbed after the implementation of the ideal optimal damping wave controller and the approximated tuned optimal damping wave controller are shown in Figure 2. In the ideal case, the optimal damping wave feedback controller absorbs 41·4% of the total incoming energy. The approximated tuned controller absorbs the same amount of energy as the ideal case at the frequency to which the control is tuned. At other frequencies, less energy is absorbed, but the performance is reasonable, except at low frequencies, where the quadratic control is effective.

Figure 3 shows the response without control, with linear quadratic optimal control and with combined linear, quadratic and damping, wave optimal control. Without control, clear, sharp resonances can be observed. With the implementation of the linear quadratic optimal control, the resonances of the first three modes are significantly less sharp. However, spillover is observed in the uncontrolled modes. After the implementation of the combined optimal controllers, the spillover in the residual modes is eliminated and

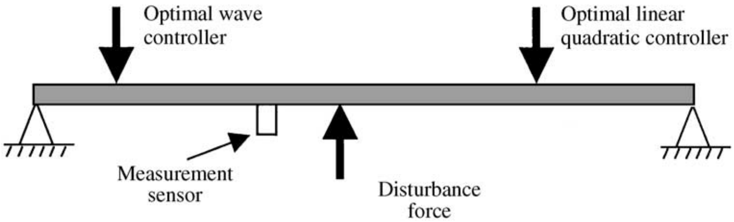

Figure 1. The simply supported beam.

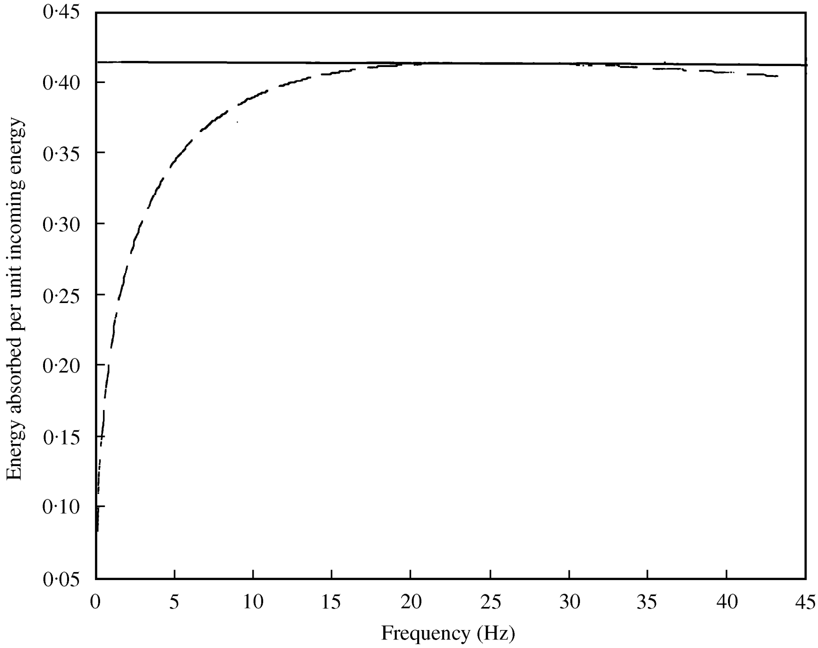

Figure 2. Energy absorbed by the ideal (——) and tuned (·········) controllers.

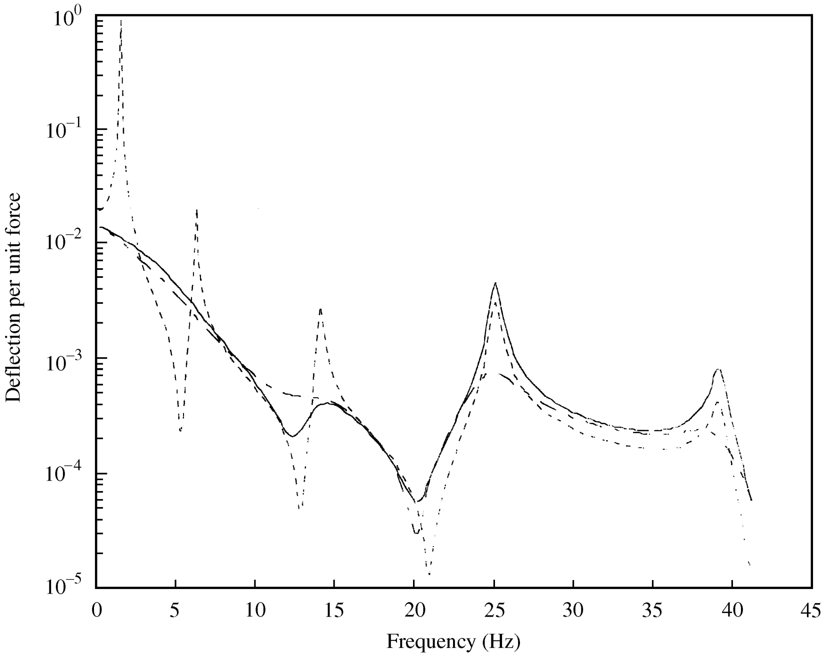

Figure 3. Frequency responses of the beam: (·········) before control; (——) after linear quadratic optimal control; (-·-·-·-) after combined linear quadratic optimal control and optimal damping wave control.

the resonances at the higher frequencies are much less sharp. This is because the wave controller absorbs the vibrational energy at those frequencies. The performance at the lower frequencies is affected little by the presence of the wave control.

5. CONCLUDING REMARKS

This letter describes a combined optimal approach to the reduction of control spillover associated with the active vibration control of a distributed structure. An optimal damping feedback wave control strategy is combined with linear quadratic optimal control. The feedback wave controller is seen to add damping to the modes of the structure, especially the higher frequency residual modes. As a result, it reduces the resonant response of these modes, and reduces the effects of control spillover. Furthermore, the additional single-input, single-output collocated wave feedback controller is robust and easy to implement.

A simple numerical example was considered. However, the approach is applicable to more complex structures: wave control as described here can be applied to some member or members which vibrate as beams in bending, while the global response of the structure can be described in terms of the global modes of vibration.

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