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ON THE ROLES OF COMPLEMENTARY AND ADMISSIBLE BOUNDARY CONSTRAINTS IN FOURIER SOLUTIONS TO THE BOUNDARY VALUE PROBLEMS OF COMPLETELY COUPLED *r*TH ORDER PDES

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A heretofore unavailable double Fourier series based approach, for obtaining non-separable solution to a system of completely coupled linear *r*th order partial differential equations with constant coefficients and subjected to general (completely coupled) boundary conditions, has been presented. The method has been successfully implemented to solve a class of hitherto unsolved boundary-value problems, pertaining to free and forced vibrations of arbitrarily laminated anisotropic doubly curved thin panels of rectangular planform, with arbitrarily prescribed (both symmetric and asymmetric with respect to the panel centerlines) admissible boundary conditions and subjected to general transverse loading.

Existing solutions such as those due to Navier or Levy are based on the well-known method of separation of variables. Such solutions represent particular solutions whenever the method of separation of variables work, and when these particular solution functions fortuitously satisfy the boundary conditions. For derivation of the complementary solution, the complementary boundary constraints are introduced through boundary discontinuities of some of the particular solution functions and their partial derivatives. Such discontinuities form sets of measure zero.

Various cases of lamination, geometry and dynamic response (forced and free vibrations) of a class of thin anisotropic laminated shells (curved panels) have been shown to follow from the above. Six sets of boundary conditions are used to illustrate the present method for the derivation of complementary solutions. Navier-type solutions whenever available form special cases of the present general solution.

1. INTRODUCTION

Many boundary-value problems of mathematical physics, with domains of rectangular planform, are represented by systems of highly coupled linear partial differential equations (PDE) with constant coefficients, where the prescribed boundary conditions can also be quite general. Various subclasses of the general system, such as $r = 8, 6, 4, 2$ are often encountered in the problems of structural mechanics. For example, a subclass, represented by a system of completely coupled linear fourth order PDEs with constant coefficients, can be treated, without much loss of generality, as a representative of the above. The boundary conditions in this case may contain at the most third derivatives. The objective of the present study is to present a general method of solution to a general linear system of completely coupled PDEs of *r*th ($r =$ even) order, which is subject to general admissible

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(prescribed) boundary conditions up to the order $r - 1$, using double Fourier series, which may be continuous or discontinuous at an edge. The present study is motivated by the need to find exact (in the limit) double Fourier series solutions to the problems of arbitrarily laminated thin and thick doubly curved, cylindrical or flat panels (open shells), with arbitrary boundary conditions.

A detailed literature search (see e.g., reference [1]) reveals that a vast majority of the existing studies, on obtaining exact solutions are largely restricted to Navier- or Levy-type particular solutions alone. In these studies, solutions are usually assumed in the form of a double Fourier series, such that either all four (Navier) or two opposite (Levy-type solution) boundary conditions are satisfied *a priori*. These assumed solutions are then substituted into the governing partial differential equations, which yield a set of a system of linear albegraic equations in terms of as many unknown Fourier coefficients for each combination of m, n, where m and n denote the wave numbers (i.e., number of terms of the Fourier series). This approach has been successful only in the case of cross-ply and homogeneous orthotropic/isotropic curved panels or flat plates of rectangular planform (see references [2, 3]), with the SS3-type simply supported boundary conditions, and antisymmetric angle-ply rectangular plates with the SS2-type boundary conditions prescribed at all four edges (Navier) or two opposite edges (Levy). Such solutions are based on the well-known method of separation of variables, which does not work even for a symmetric angle-ply plate because of the presence of bending–twisting coupling rigidities, D_{16} and D_{26} (see reference [2]), let alone arbitrarily laminated plates and all shells with the exception of cross-ply curved panels. This is because the variables are, in general, not separable, and more important, boundary conditions are not satisfied *a priori*. The primary objective of the present investigation is to bridge this long-standing analytical gap.

Chaudhuri [1] recently presented a double Fourier series approach for the solution to a system of completely coupled linear second order partial differential equations (PDE) with constant coefficients, satisfying Dirichlet, Neumann and arbitrary (mixed) admissible boundary conditions. This approach has been applied by Chaudhuri and Abu-Arja [4, 5], and Chaudhuri and Kabir [6, 7] to solve the FSDT (first order shear deformation theory)-based problems of (1) doubly curved moderately thick panels of antisymmetric angle-ply, and (2) homogeneous isotropic (metallic) and general cross-ply constructions respectively. Special cases of flat isotropic and cross-ply panels were also presented by Chaudhuri and Kabir [8, 9]. The underlying mathematical principle is concerned with well-posedness or lack thereof of the Fourier-type formulation, and the existence of the resulting series solution. This kind of ill-posedness can be removed by the addition of mathematical "structures" to the formulation, which, for a system of coupled second order PDEs, is accomplished through the introduction of certain constraints [1], termed here as the complementary boundary constraints. However, it is worthwhile to note that the levels of ill-posedness in the Fourier formulations of systems of coupled fourth or higher order PDEs are vastly more complex compared to their second order counterpart. Consequently, significantly more complex mathematical "structures" through the introduction of additional constraints are needed in order to obtain Fourier-type solutions for such problems, which is the primary objective of the present investigation. Additionally, although the boundary-discontinuous Fourier series theory has been expounded earlier by Hobson [10] and Carslaw [11], and the method has been applied by other investigators, such as Green [12], Winslow [13], and Whitney [14, 15], the criteria determining as to when the boundary Fourier series are needed or not needed have never been clearly spelled out. Second and more important, the boundary-discontinuous Fourier method has never been applied to the problem of a plate/shell subjected to asymmetric (with respect to panel centerlines) boundary conditions, which along with the general lack of non-separable

Fourier solution has so far remained an enigma in the literature. A clear exposition of this important issue for completely coupled systems of rth order PDEs, subjected to completely coupled general admissible boundary conditions is the subject matter of the present investigation.

The present study will obtain, in a direct manner, the most general solutions to the boundary-value problems of a system of completely coupled rth order PDEs, with constant coefficients and subjected to completely coupled admissible boundary conditions—a general self-adjoint linear differential system. This study first presents a method for non-separable particular (double Fourier series) solutions. Care is taken of the discontinuities of the particular functions or their derivatives at the boundary, which will yield additional unknown coefficients, i.e., the appropriate boundary Fourier coefficients, through introduction of complementary and admissible boundary constraints, so that the number of equations will finally become equal to the number of unknown coefficients.

Various cases of lamination, geometry and dynamic response (forced and free vibrations) of a class of thin anisotropic laminated shells (curved panels) are shown to follow from the above. Six sets of boundary conditions are used to illustrate the present method for derivation of complementary solutions. Navier-type solutions whenever available, are shown to form special cases of the present general solution.

2. STATEMENT OF THE PROBLEM

We consider the following system of completely coupled rth order ($r = 1, 2, \ldots$) partial differential equations with constant coefficients representing a linear undamped elasto-dynamic system:

$$L_{ij}u'_j = a_{ij}u'_j + b_{ijk}u'_{j,k} + c_{ijkl}u'_{j,kl} + d_{ijklm}u'_{j,klm} + e_{ijklmn}u'_{j,klmn} + \cdots = C'_i - f_i,$$

$$\text{for } i, j = 1, 2, \ldots, N; \quad k, l, m, n, \ldots = 1, 2, \tag{1}$$

where a ',' followed by subscripts k, l, m, n denotes partial differentiation with respect to general (curvilinear) spatial co-ordinates x_1, x_2. u'_i denotes the time-dependent displacement (including rotation) component, while C'_i and f_i represent the inertia term and periodic forcing function, respectively, and are written in the form

$$C'_j = m_j u_{j,\tau\tau} \quad (\text{no sum on either } j \text{ or } \tau);\ j = 1, 2, \ldots, N, \tag{2a}$$

$$f_j(x_1, x_2, \tau) = q_j(x_1, x_2)\mathrm{e}^{\mathrm{i}\omega\tau}, \quad \mathrm{i} = \sqrt{-1};\ j = 1, 2, \ldots, N, \tag{2b}$$

where τ and ω denote time and angular frequency respectively. It then follows that

$$u'_j(x_1, x_2, \tau) = u_j(x_1, x_2)\mathrm{e}^{\mathrm{i}\omega\tau}, \quad \mathrm{i} = \sqrt{-1};\ j = 1, 2, \ldots, N, \tag{3a}$$

$$C'_j = C_j\mathrm{e}^{\mathrm{i}\omega\tau}, \quad \mathrm{i} = \sqrt{-1};\ j = 1, 2, \ldots, N, \tag{3b}$$

with

$$C_j = -\,\omega^2 m_j u_j\ (\text{no sum on } j),\ j = 1, 2, \ldots, N. \tag{3c}$$

The above operation reduces the linear elasto-dynamic system, given by equation (1) to that of a boundary-value problem (BVP), to be solved in the frequency domain, in conjunction with prescribed boundary conditions given below. In the absence of the periodic forcing function (i.e., the free vibration case), the BVP is concerned with determination of the eigenvalues and eigenfunctions. The linear partial differential operator, L_{ij}, $(i, j = 1, 2, \ldots, N)$, is defined in such a way that (1) its inverse exists and is unique, (2) the adjoint operator exists, and (3) the Fredholm alternative theorem holds [16]. It follows from Sobolev's theorem and the homogeneous boundary conditions, which determine the domain of the operator L_{ij}, when applied at an edge, $x_p = \bar{x}_p$, where $\bar{x}_p$ is a constant, can, in general, take the form

$$B_{\alpha ij} u_j$$

$$= \bar{a}^{(p)}_{\alpha ij} u_j + \bar{b}^{(p)}_{\alpha ijk} u_{j,k} + \bar{c}^{(p)}_{\alpha ijkl} u_{j,kl} + \bar{d}^{(p)}_{\alpha ijklm} u_{j,klm} + \cdots = 0 \text{ at an edge } x_p = \text{constant},$$

$$\text{for } \alpha = 1, 2, \ldots r/2 \ (r = \text{even});\ i, j = 1, 2, \ldots, N;\ k, l, m, \ldots = p, t, \tag{4}$$

where p, t denote the directions normal and tangential to the edge, $x_p = $ constant. For example, when $p = 1$, $t = 2$ and *vice versa*. The above are referred to as general or mixed boundary conditions, where all the unknown dependent variables (i.e., response functions) and their derivatives are completely coupled. In equations (1) and (4), $a_{ij}, b_{ijk}, c_{ijkl}, \ldots, \bar{a}^{(p)}_{\alpha ij}, \bar{b}^{(p)}_{\alpha ijk}, \ldots,$ are constant coefficients. For $r = $ even, although the total number of boundary conditions prescribed at an edge is $r/2$, the boundary conditions can contain normal derivatives of order $r - 1$ at the most. The above boundary conditions arise from a variational principle such that equations (1) and (4) form a self-adjoint differential system.

3. METHOD FOR SELECTION OF PARTICULAR SOLUTION

Since the particular solution depends on the loading, $q_i(x_1, x_2)$ is first expanded in the form of

$$q_i(x_1, x_2) = \sum_{s=1}^{4} q_i^{(s)}(x_1, x_2) = Q^{(s)}_{imn} f^{(s)}_{mn}(x_1, x_2), \quad i = 1, \ldots, N;\ s = 1, \ldots, 4. \tag{5}$$

The most general non-separable particular solution to the problem, represented by equations (1) and (2), is then assumed in the form

$$u_j^p(x_1, x_2) = \sum_{s=1}^{4} u_j^{p(s)}(x_1, x_2) = U^{(s)}_{jmn} f^{(s)}_{mn}(x_1, x_2), \quad s = 1, \ldots, 4. \tag{6}$$

It may be noted that Einstein's summation convention has been used on subscripts m, n, and superscript, s, except when "no sum" is mentioned.

$$u_j^{(1)}(x_1, x_2) = U^{(1)}_{jmn} f^{(1)}_{mn}(x_1, x_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U^{(1)}_{jmn} \sin(\alpha_m x_1) \sin(\beta_n x_2), \tag{7a}$$

$$u_j^{(2)}(x_1, x_2) = U^{(2)}_{jmn} f^{(2)}_{mn}(x_1, x_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} U^{(2)}_{jmn} \cos(\alpha_m x_1) \cos(\beta_n x_2), \tag{7b}$$

$$u_j^{(3)}(x_1, x_2) = U_{jmn}^{(3)} f_{mn}^{(3)}(x_1, x_2) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} U_{jmn}^{(3)} \sin(\alpha_m x_1) \cos(\beta_n x_2), \tag{7c}$$

$$u_j^{(4)}(x_1, x_2) = U_{jmn}^{(4)} f_{mn}^{(4)}(x_1, x_2) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} U_{jmn}^{(4)} \cos(\alpha_m x_1) \sin(\beta_n x_2) \tag{7d}$$

for $j = 1, 2, \ldots, N$ with $\alpha_m = m\pi/a, \quad \beta_n = n\pi/b$.

Term by term partial differentiation of the assumed particular solution functions given by equations (6, 7) and substitution into the governing partial differential equations (1), and finally, equating the coefficients of $f_{mn}^{(s)}(x_1, x_2)$ for $s = 1, \ldots, 4$, and $i, j = 1, 2, \ldots, N$, will yield $N(4mn + 2m + 2n + 1)$ equations for arbitrary m, n in terms of as many unknown Fourier coefficients, $F_{jmn}^{(s)}$, $s = 1, \ldots, 4$, and $j = 1, 2, \ldots, N$, which can easily be evaluated. The assumed solution, given by equations (6, 7), would represent the complete solution, provided it satisfies the prescribed boundary conditions, as it happens in the case of the available Navier solutions. However, in most cases of practical interest, the assumed particular solution would fail to satisfy one or more of the prescribed boundary conditions, which when satisfied would provide the complementary solution to the problem under consideration.

4. METHOD OF DERIVATION OF COMPLEMENTARY SOLUTION

4.1. BOUNDARY CONDITIONS AND THE TOTAL NUMBER OF EQUATIONS

The first task here is to identify the total number of equations available from satisfying the prescribed boundary conditions. In order to accomplish this, the same procedure as applied to the case of the governing partial differential equations will be followed. Substitution of the assumed particular solution functions given by equations (6, 7) and their partial derivatives obtained by term-wise differentiation (i.e., ignoring the boundary discontinuities at this point) into the general or mixed type of prescribed boundary conditions given by equation (4), yields

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(\alpha_m x_1) \sin(\beta_n x_2) \{ \bar{a}_{\alpha ij}^{(p)} U_{jmn}^{(1)} - \bar{b}_{\alpha ij1}^{(p)} \alpha_m U_{jmn}^{(4)} - \bar{b}_{\alpha ij2}^{(p)} \beta_n U_{jmn}^{(3)} - \bar{c}_{\alpha ij11}^{(p)} \alpha_m^2 U_{jmn}^{(1)}$$

$$- \bar{c}_{\alpha ij22}^{(p)} \beta_n^2 U_{jmn}^{(1)} + (\bar{c}_{\alpha ij12}^{(p)} + \bar{c}_{\alpha ij21}^{(p)}) \alpha_m \beta_n U_{jmn}^{(2)} + \bar{d}_{\alpha ij111}^{(p)} \alpha_m^3 U_{jmn}^{(4)} + \bar{d}_{\alpha ij222}^{(p)} \beta_n^3 U_{jmn}^{(3)}$$

$$+ (\bar{d}_{\alpha ij112}^{(p)} + \bar{d}_{\alpha ij121}^{(p)} + \bar{d}_{\alpha ij211}^{(p)}) \alpha_m^2 \beta_n U_{jmn}^{(3)} + (\bar{d}_{\alpha ij122}^{(p)} + \bar{d}_{\alpha ij212}^{(p)} + \bar{d}_{\alpha ij221}^{(p)}) \alpha_m \beta_n^2 U_{jmn}^{(4)} + \cdots \}$$

$$+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos(\alpha_m x_1) \cos(\beta_n x_2) \{ \bar{a}_{\alpha ij}^{(p)} U_{jmn}^{(2)} + \bar{b}_{\alpha ij1}^{(p)} \alpha_m U_{jmn}^{(4)} + \bar{b}_{\alpha ij2}^{(p)} \beta_n U_{jmn}^{(3)} - \bar{c}_{\alpha ij11}^{(p)} \alpha_m^2 U_{jmn}^{(2)}$$

$$- \bar{c}_{\alpha ij22}^{(p)} \beta_n^2 U_{jmn}^{(2)} + (\bar{c}_{\alpha ij12}^{(p)} + \bar{c}_{\alpha ij21}^{(p)}) \alpha_m \beta_n U_{jmn}^{(1)} + \bar{d}_{\alpha ij111}^{(p)} \alpha_m^3 U_{jmn}^{(3)} + \bar{d}_{\alpha ij222}^{(p)} \beta_n^3 U_{jmn}^{(4)}$$

$$- (\bar{d}_{\alpha ij112}^{(p)} + \bar{d}_{\alpha ij121}^{(p)} + \bar{d}_{\alpha ij211}^{(p)}) \alpha_m^2 \beta_n U_{jmn}^{(4)} - (\bar{d}_{\alpha ij122}^{(p)} + \bar{d}_{\alpha ij212}^{(p)} + \bar{d}_{\alpha ij221}^{(p)}) \alpha_m \beta_n^2 U_{jmn}^{(3)} + \cdots \}$$

$$+ \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \sin(\alpha_m x_1) \cos(\beta_n x_2) \{ \bar{a}_{\alpha ij}^{(p)} U_{jmn}^{(3)} - \bar{b}_{\alpha ij1}^{(p)} \alpha_m U_{jmn}^{(2)} + \bar{b}_{\alpha ij2}^{(p)} \beta_n U_{jmn}^{(1)} - \bar{c}_{\alpha ij11}^{(p)} \alpha_m^2 U_{jmn}^{(3)}$$

$$-\bar{c}^{(p)}_{\alpha ij22}\beta_n^2 U^{(3)}_{jmn}-\bar{c}^{(p)}_{\alpha ij12}\alpha_m\beta_n U^{(4)}_{jmn}+\bar{c}^{(p)}_{\alpha ij21}\alpha_m\beta_n U^{(4)}_{jmn}+\bar{d}^{(p)}_{\alpha ij111}\alpha_m^3 U^{(2)}_{jmn}-\bar{d}^{(p)}_{\alpha ij222}\beta_n^3 U^{(1)}_{jmn}$$

$$-(\bar{d}^{(p)}_{\alpha ij112}+\bar{d}^{(p)}_{\alpha ij121}+\bar{d}^{(p)}_{\alpha ij211})\alpha_m^2\beta_n U^{(1)}_{jmn}+(\bar{d}^{(p)}_{\alpha ij122}+\bar{d}^{(p)}_{\alpha ij212}+\bar{d}^{(p)}_{\alpha ij221})\alpha_m\beta_n^2 U^{(2)}_{jmn}+\cdots\}$$

$$+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\cos(\alpha_m x_1)\sin(\beta_n x_2)\{\bar{a}^{(p)}_{\alpha ij}U^{(4)}_{jmn}+\bar{b}^{(p)}_{\alpha ij1}\alpha_m U^{(1)}_{jmn}-\bar{b}^{(p)}_{\alpha ij2}\beta_n U^{(2)}_{jmn}-\bar{c}^{(p)}_{\alpha ij11}\alpha_m^2 U^{(4)}_{jmn}$$

$$-\bar{c}^{(p)}_{\alpha ij22}\beta_n^2 U^{(4)}_{jmn}-(\bar{c}^{(p)}_{\alpha ij12}+\bar{c}^{(p)}_{\alpha ij21})\alpha_m\beta_n U^{(3)}_{jmn}-\bar{d}^{(p)}_{\alpha ij111}\alpha_m^3 U^{(1)}_{jmn}+\bar{d}^{(p)}_{\alpha ij222}\beta_n^3 U^{(2)}_{jmn}$$

$$+(\bar{d}^{(p)}_{\alpha ij112}+\bar{d}^{(p)}_{\alpha ij121}+\bar{d}^{(p)}_{\alpha ij211})\alpha_m^2\beta_n U^{(2)}_{jmn}-(\bar{d}^{(p)}_{\alpha ij122}+\bar{d}^{(p)}_{\alpha ij212}+\bar{d}^{(p)}_{\alpha ij221})\alpha_m\beta_n^2 U^{(1)}_{jmn}+\cdots\}$$

$$=0. \tag{8}$$

Applying the boundary conditions at the edges $x_1 = 0$, a, to equation (8), and equating the coefficients of $\cos(\beta_n x_2)$ and $\sin(\beta_n x_2)$ to zero will yield $4n + 2$ equations for each $\alpha = 1, 2, \ldots, r/2$; $i = 1, 2, \ldots, N$; i.e., $rN(2n + 1)$ equations for a system of rth order PDEs. Similar operations at the edges $x_2 = 0$, b will yield $rN(2m + 1)$ equations. Therefore, the total additional equations arising out of satisfying the general or mixed boundary conditions at the edges will number $2rN(m + n + 1)$.

4.2. COMPLEMENTARY BOUNDARY CONSTRAINTS AND THE ASSOCIATED BOUNDARY DISCONTINUITIES

For derivation of the complementary solution, the complementary boundary constraints play as important a role as the (prescribed) admissible boundary conditions. The complementary boundary constraints enter into the picture through boundary discontinuities of some of the particular solution functions, assumed in the form of equations (6, 7) and their partial derivatives. In order for this method to furnish a complete solution to the type of boundary-value problems given by equations (1–4), $2rN(m + n + 1)$, additional unknown coefficients must be furnished by the complementary boundary constraints. For a system of fourth order completely coupled PDEs, this number reduces to $8N(m + n + 1)$ additional unknown coefficients. The admissible boundary constraints, which are equalities, are conjugates of the associated complementary boundary constraints, which are inequalities. They are selected at an edge in a direction normal to that edge in order to guarantee the self-adjointness of the corresponding one-dimensional differential system.

The prescription of the associated complementary boundary constraints at an edge implies that in order for equation (4) to yield the required number of equations at each of the four edges, not all of the $r(r + 1)/2$ quantities for each $i = 1, 2, \ldots, N$ — $u_i^{(s)}$, $u_{i,p}^{(s)}$, $u_{i,t}^{(s)}$, $u_{i,pp}^{(s)}$, $u_{i,tt}^{(s)}$, $u_{i,pt}^{(s)}, \ldots, s = 1, \ldots, 4$—can be permitted to vanish at an edge, $x_p = \bar{x}_p$, because that will reduce equation (4) to identities. Assignment of associated complementary boundary constraints at an edge results in "ordinary" discontinuities of the solution functions and/or their partial derivatives at that edge. In contrast, prescription of the admissible boundary constraints dictates that vanishing of some of them must be permitted, which includes but is not limited to the prescribed geometric boundary conditions, and which must not be regarded as violations of the physics of the problem. Assignment of admissible boundary constraints at an edge insures continuity of the solution functions and/or their partial derivatives at that edge. As a first step, admissible boundary constraints are assumed to be absent, and only complementary boundary constraints are present, which implies that vanishing of both the assumed solution functions and their partial derivatives of up to $(r-1)$th order at an edge can be regarded as violations of the physics of the problem, resulting in "ordinary" discontinuities at that edge. In what follows, the procedure is illustrated for the case of a system of completely coupled second ($r=2$) and fourth order ($r=4$) PDEs, then extended to a system of completely coupled sixth order ($r=6$) PDEs and is finally generalized to the case of rth order.

Winslow [13], following Hobson's [10] lead, discussed the mathematical conditions of differentiation, of functions and their partial derivatives represented by ordinary Fourier series, in the presence of ordinary discontinuities and has concluded that unless additional conditions, imposed by term-wise differentiation are fulfilled, the hypothetical representation by Fourier series may not have sufficient generality to satisfy all the required conditions and furnish a solution. A series obtained by differentiating a convergent Fourier series (here a double Fourier series), $u_i^{(s)}(x_1, x_2)$, $i = 1, \dots, N$, $s = 1, \dots, 4$, given by equations (6 , 7) is, in general, not convergent; nor is the series so obtained necessarily the Fourier series corresponding to the particular partial derivative of $u_i^{(s)}(x_1, x_2)$, $i = 1, \dots, N$, $s = 1, \dots, 4$. In general, $u_i^{(s)}(x_1, x_2)$ is a bounded function, which is piecewise continuous (i.e., continuous except for a finite number of ordinary discontinuities, $d = d^{(N_p)}$, in the direction of x_p at $x = x_{pd}$); in the problem under investigation, $d = d^{(N_p)} = 2$ at the most. The partial derivatives, $u_{i,p}^{(s)}(x_1, x_2)$, $p = 1, 2$, are assumed to be Lebesgue integrable in the domain $(0, a) \times (0, b)$ and also, if these have lines of infinite discontinuity (e.g., Dirac Delta function), such lines form reducible sets. This is consistent with there being a set of lines of zero measure at which $u_{i,p}^{(s)}(x_1, x_2)$, $p = 1, 2$, has no definite value. Further details relating to the discontinuities in the particular solution functions and their first partial derivatives are available in Appendix A of Chaudhuri [1]. Discontinuities of higher derivatives can be similarly dealt with, and will not be discussed here in the interest of brevity. Let $F^{(s)}(x_1, x_2)$, $s = 1, \dots, 4$, denote a function, $u_i^{(s)}(x_1, x_2)$ or any of its derivatives. As an example, $F^{(1)}(x_1, x_2)$, which is defined to be an odd function with respect to both x_1 and x_2 will be considered:

$$F^{(1)}(-x_1, x_2) = -F^{(1)}(x_1, x_2), \quad F^{(1)}(x_1, -x_2) = -F^{(1)}(x_1, x_2). \tag{9}$$

The half-range double Fourier series expansion for the function and its two first partial derivatives are

$$F^{(1)}(x_1, x_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{(1)} \sin(\alpha_m x_1) \sin(\beta_n x_2), \tag{10a}$$

$$F_{,1}^{(1)}(x_1, x_2) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} B_{mn}^{(1)} \cos(\alpha_m x_1) \sin(\beta_n x_2), \tag{10b}$$

$$F_{,2}^{(1)}(x_1, x_2) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} C_{mn}^{(1)} \sin(\alpha_m x_1) \cos(\beta_n x_2), \tag{10c}$$

wherein

$$A_{mn}^{(1)} = \frac{4}{ab}\int_0^a\int_0^b F^{(1)}(x_1, x_2)\sin(\alpha_m x_1)\sin(\beta_n x_2)\mathrm{d}x_1\,\mathrm{d}x_2 \quad \text{for } m, n = 1, 2, \ldots, \infty, \tag{11a}$$

$$B_{mn}^{(1)} = \alpha_m A_{mn}^{(1)} + \frac{4}{ab}\int_0^b \{F^{(1)}(a-0, x_2)(-1)^m - F^{(1)}(0+0, x_2)\}\sin(\beta_n x_2)\,\mathrm{d}x$$

$$\text{for } m, n = 1, 2, \ldots, \infty, \tag{11b}$$

$$B_{0n}^{(1)} = \frac{2}{ab}\int_0^b \{F^{(1)}(a-0, x_2) - F^{(1)}(0+0, x_2)\}\sin(\beta_n x_2)\,\mathrm{d}x_2 \quad \text{for } n = 1, 2, \ldots, \infty, \tag{11c}$$

$$C_{mn}^{(1)} = \beta_n A_{mn}^{(1)} + \frac{4}{ab}\int_0^a \{F^{(1)}(x_1, b-0)(-1)^n + F^{(1)}(x_1, 0+0)\}\sin(\alpha_m x_1)\,\mathrm{d}x_1$$

$$\text{for } m, n = 1, 2, \ldots, \infty, \tag{11d}$$

$$C_{m0}^{(1)} = \frac{2}{ab}\int_0^b \{F^{(1)}(x_1, b-0) - F^{(1)}(x_1, 0+0)\}\sin(\alpha_m x_1)\,\mathrm{d}x_1 \quad \text{for } m = 1, 2, \ldots, \infty. \tag{11e}$$

By identifying $F^{(1)}(x_1, x_2)$ as $u_i^{(1)}(x_1, x_2)$, it can be easily seen that in the presence of complementary boundary constraints (boundary discontinuities) at $x_p = 0$, $\bar{x}_p$ ($\bar{x}_p = a$ or b depending on whether $p = 1$ or 2), the two first partial derivatives expressed in the form of equations 10(b, c) and 11(b–e) are given by equations (A1b) and (A1c) in Appendix A with boundary Fourier coefficients, $\bar{a}_{in}$, $\bar{b}_{in}$, and $\bar{c}_{im}$, $\bar{d}_{im}$, being given by equations (B1a, b) and (B1e, f), respectively, in Appendix B.

If complementary boundary constraint of the function $F^{(1)}(x_1, x_2)$ is assigned only at $x_p = 0$ ($p = 1$ or 2), then the Fourier coefficients of its first partial derivatives are given as

$$B_{mn}^{(1)} = \alpha_m A_{mn}^{(1)} + \frac{4}{ab}\int_0^b \{-F^{(1)}(0+0, x_2)\}\sin(\beta_n x_2)\,\mathrm{d}x_2 \quad \text{for } m, n = 1, 2, \ldots, \infty, \tag{12a}$$

$$B_{0n}^{(1)} = \frac{2}{ab}\int_0^b \{-F^{(1)}(0+0, x_2)\}\sin(\beta_n x_2)\,\mathrm{d}x_2 \quad \text{for } n = 1, 2, \ldots, \infty, \tag{12b}$$

$$C_{mn}^{(1)} = \beta_n A_{mn}^{(1)} + \frac{4}{ab}\int_0^a \{-F^{(1)}(x_1, 0+0)\}\sin(\alpha_m x_1)\,\mathrm{d}x_1 \quad \text{for } m, n = 1, 2, \ldots, \infty, \tag{12c}$$

$$C_{m0}^{(1)} = \frac{2}{ab}\int_0^b \{-F^{(1)}(x_1, 0+0)\}\sin(\alpha_m x_1)\,\mathrm{d}x_1 \quad \text{for } m = 1, 2, \ldots, \infty. \tag{12d}$$

By identifying $F^{(1)}(x_1, x_2)$ as $u_i^{(1)}(x_1, x_2)$, it can easily be seen that in the presence of complementary boundary constraints (boundary discontinuities) at only $x_p = 0$ ($p = 1$ or 2), the two first partial derivatives expressed in the form of equations 10(b, c) and 11(b–e) are given by modified equations (A1b) and (A1c) in Appendix A with boundary Fourier coefficients, $\bar{a}_{in} = -\bar{b}_{in}$ and $\bar{c}_{im} = -\bar{d}_{im}$, being given by equation (B1a, b) and (B1e, f), respectively, in Appendix B. In this case, the first terms vanish from the integrands in the r.h.s. of equations (B1a, b) and (B1e, f).

If complementary boundary constraint of the function $F^{(1)}(x_1, x_2)$ is assigned only at $x_p = \bar{x}_p$ ($\bar{x}_p = a$ or b depending on whether $p = 1$ or 2), then the Fourier coefficients of its first partial derivatives are given as

$$B_{mn}^{(1)} = \alpha_m A_{mn}^{(1)} + \frac{4}{ab}\int_0^b \{F^{(1)}(a-0, x_2)(-1)^m\} \sin(\beta_n x_2)\,\mathrm{d}x_2 \quad \text{for } m, n = 1, 2, \ldots, \infty, \tag{13a}$$

$$B_{0n}^{(1)} = \frac{2}{ab}\int_0^b \{F^{(1)}(a-0, x_2)\} \sin(\beta_n x_2)\,\mathrm{d}x_2 \quad \text{for } n = 1, 2, \ldots, \infty, \tag{13b}$$

$$C_{mn}^{(1)} = \beta_n A_{mn}^{(1)} + \frac{4}{ab}\int_0^b \{F^{(1)}(x_1, b-0)(-1)^n\} \sin(\alpha_m x_1)\,\mathrm{d}x_1 \quad \text{for } m, n = 1, 2, \ldots, \infty, \tag{13c}$$

$$C_{m0}^{(1)} = \frac{2}{ab}\int_0^b \{F^{(1)}(x_1, b-0)\} \sin(\alpha_m x_1)\,\mathrm{d}x_1 \quad \text{for } m = 1, 2, \ldots, \infty. \tag{13d}$$

By identifying $F^{(1)}(x_1, x_2)$ as $u_i^{(1)}(x_1, x_2)$, it can easily be seen in the presence of complementary boundary constraint (boundary discontinuities) at only $x_p = \bar{x}_p (p = 1$ or 2), the two first partial derivatives expressed in the form of equations 10(b, c) and 11(b–e) are given by modified equations (A1b) and (A1c) in Appendix A with boundary Fourier coefficients, $\bar{a}_{in} = \bar{b}_{in}$ and $\bar{c}_{im} = \bar{d}_{im}$, being given by equations (B1a, b) and (B1e, f), respectively, in Appendix B. In this case, the second terms vanish from the integrands in the r.h.s. of equations (B1a, b) and (B1e, f).

Extension of the above to higher partial derivatives of $u_i^{(1)}(x_1, x_2)$ or all partial derivatives of $u_i^{(s)}(x_1, x_2)$, $s = 2, 3, 4$ is straightforward, and hence will not be pursued here any further.

For a system of completely coupled fourth order PDEs subjected to complementary boundary constraints assigned at both ends in each direction, the additional coefficients, $\bar{a}_{in}, \bar{b}_{in}, \ldots, \bar{\bar{h}}_{im}, \ldots, a'_{in}, b'_{in}, \ldots, h''_{in}$, defined by equations (B1, B2) of Appendix B, number $16N(m + n + 1)$, while the number of equations arising out of satisfying the prescribed geometric and natural boundary conditions, as noted earlier, equal $8N(m + n + 1)$. This mismatch will necessitate revision of the previous hypothesis through the introduction of additional constraints, i.e., the admissible boundary constraints. In the case of two special cases, where complementary boundary constraint inequalities are enforced at either $x_p = 0$ or $\bar{x}_p$, the corresponding opposite ends are assigned the related boundary constraints, and no such mismatch will occur. It is worthwhile to offer a formal definition of the admissible boundary constraints. Admissible boundary constraints (and the associated complementary boundary constraints) refer to the vanishing (and non-vanishing) of certain one-dimensional beam functions, such as $\sin(\alpha_m x_1)$ and their normal derivatives. For prescribed geometric boundary conditions, the admissible boundary constraints are identical to the corresponding boundary conditions, while for prescribed natural boundary conditions admissible boundary constraints only imply vanishing of the corresponding second or third derivatives, which correspond to the moment or shear for a beam (in the case of a fourth order problem), and not that of a physical quantity, such as moment or shear force of a plate or shell of rectangular planform.

Generalization of the above to a system of rth (r = even) order completely coupled PDEs would result in $4rN(m + n + 1)$ boundary Fourier coefficients, while the number of equations arising out of satisfying the boundary conditions, as noted earlier, equal $2rN(m+n+1)$. Since the assumed particular solution functions are comprised of $\sin(\alpha_m x_1)$ or $\cos(\alpha_m x_1)$ and $\sin(\beta_n x_2)$ or $\cos(\beta_n x_2)$, vanishing of $u_i^{(s)}(x_1, x_2)$ at an edge $x_p = \bar{x}_p$, will automatically lead to vanishing of the corresponding tangential derivatives, $u_{i,t}^{(s)}(x_1, x_2)$, $u_{i,tt}^{(s)}(x_1, x_2), \dots$ at that edge and *vice versa*. Vanishing of the corresponding normal derivatives, $u_{i,p}^{(s)}(x_1, x_2)$, $u_{i,pp}^{(s)}(x_1, x_2), \dots$, is, however, independent of that of the function itself. The following discussion illustrates the number of admissible boundary constraints and the associated complementary boundary constraints, which must be satisfied in order to produce a solution to the class of boundary-value problems involving completely coupled second and fourth order PDEs, which is extended to the case of sixth order PDEs, and is finally generalized to the case of rth order PDEs.

4.2.1. *A system of completely coupled second order PDEs*

In the case of a boundary-value problem involving a system of completely coupled second order PDEs, the following two mutually independent combinations of complementary boundary constraints are possible, one of which must be introduced in order for the total number of unknowns to become equal to the total number of equations. Table 1 summarizes these combinations in the presence of complementary boundary constraints (boundary discontinuities) at both ends, $x_p = 0$, $\bar{x}_p$ ($\bar{x}_p = a$ or b depending on whether $p = 1$ or 2).

Combination (1): $u_i^{(s)}(x_1, x_2)$ and the associated tangential derivative are not permitted to vanish at an edge $x_p =$ constant, which will result in non-vanishing of the single-barred boundary Fourier coefficients. Vanishing of $u_{i,p}^{(s)}(x_1, x_2)$ at that edge, that corresponds to vanishing of the single-primed coefficients, constitutes satisfaction of an admissible boundary constraint, and will not constitute a violation of the physics of the problem.

Combination (2): $u_{i,p}^{(s)}(x_1, x_2)$ is not permitted to vanish at an edge $x_p =$ constant, which will result in non-vanishing of the single-primed coefficients. Vanishing of $u_i^{(s)}(x_1, x_2)$ and the associated tangential derivative at that edge, that corresponds to vanishing of the single-barred coefficients, constitutes satisfaction of an admissible boundary constraint.

Special cases of these complementary boundary constraints being assigned to only one of the two opposite ends are summarized in Table 2. It may be noted here that these special cases permit us to prescribe arbitrary boundary conditions at each of the four edges independent of one another, and thus constitutes the general procedure for solving the most general form of boundary-value problems. This is in contrast to the former, where the assignment of same complementary boundary constraints (see reference [1]) on two opposite ends proves to be restrictive.

TABLE 1

Symmetrically placed complementary and admissible boundary constraints for a second order PDE at edges $x_p = 0$, $\bar{x}_p$

Case	Complementary boundary constraint	Admissible boundary constraint	Comments
1.	$u_i \neq 0$; $u_{i,t} \neq 0$	$u_{i,p} = 0$	Single-primed coefficients vanish; Single-barred coefficients non-zero
2.	$u_{i,p} \neq 0$	$u_i = u_{i,t} = 0$	Single-barred coefficients vanish; Single-primed coefficients non-zero

TABLE 2

Unsymmetrically placed complementary and admissible boundary constraints for a second order PDE at edges $x_p = 0$, $\bar{x}_p$

Combination	Complementary boundary constraint		Admissible boundary constraint		Comments	
	$x_p = 0$	$x_p = \bar{x}_p$	$x_p = 0$	$x_p = \bar{x}_p$	$p = 1, t = 2,$	$p = 2, t = 1$
1.	$u_i \neq 0$; $u_{i,t} \neq 0$		$u_{i,p} = 0$		$a'_{in} = -b'_{in}$; $g'_{in} = -h'_{in}$;	$c'_{im} = -d'_{im}$; $e'_{im} = -f'_{im}$;
		$u_{i,p} \neq 0$		$u_i = u_{i,t} = 0$	$\bar{a}_{in} = \bar{b}_{in}$; $\bar{e}_{in} = \bar{f}_{in}$;	$\bar{c}_{im} = \bar{d}_{im}$; $\bar{g}_{im} = \bar{h}_{im}$;
2.	$u_{i,p} \neq 0$		$u_i = u_{i,t} = 0$		$\bar{a}_{in} = -\bar{b}_{in}$; $\bar{e}_{in} = -\bar{f}_{in}$;	$\bar{c}_{im} = -\bar{d}_{im}$; $\bar{g}_{im} = -\bar{h}_{im}$;
		$u_i \neq 0$; $u_{i,t} \neq 0$		$u_{i,p} = 0$	$a'_{in} = b'_{in}$; $g'_{in} = h'_{in}$;	$c'_{im} = d'_{im}$; $e'_{im} = f'_{im}$.

4.2.2. *A system of completely coupled fourth order PDEs*

In the case of a boundary-value problem involving a system of completely coupled fourth order PDEs, four mutually independent cases of complementary boundary constraints are possible. Table 3 summarizes these combinations in the presence of complementary boundary constraints (boundary discontinuities) at both ends, $x_p = 0$, $\bar{x}_p(\bar{x}_p = a$ or b depending on whether $p = 1$ or 2). These four cases, in turn, produce the following four mutually independent combinations of complementary boundary constraints, one of which must be introduced in order for the total number of unknowns to become equal to the total number of equations (see also Table 4).

Combination (1): $u_i^{(s)}(x_1, x_2)$, $u_{i,p}^{(s)}(x_1, x_2)$ are not permitted to vanish at an edge $x_p =$ constant, which will result in non-vanishing of the single-barred and single-primed coefficients. Vanishing of $u_{i,pp}^{(s)}(x_1, x_2)$, $u_{i,ppp}^{(s)}(x_1, x_2)$, at that edge, that corresponds to vanishing of the double-barred and double-primed coefficients, constitutes satisfaction of admissible boundary constraints.

Combination (2): $u_i^{(s)}(x_1, x_2)$, $u_{i,pp}^{(s)}(x_1, x_2)$ are not permitted to vanish at an edge $x_p =$ constant, which will result in non-vanishing of the single- and double-barred coefficients. Vanishing of $u_{i,p}^{(s)}(x_1, x_2)$, $u_{i,ppp}^{(s)}(x_1, x_2)$ at that edge, that corresponds to vanishing of the single- and double-primed coefficients, constitutes satisfaction of admissible boundary constraints.

Combination (3): $u_{i,p}^{(s)}(x_1, x_2)$, $u_{i,ppp}^{(s)}(x_1, x_2)$ are not permitted to vanish at an edge $x_p =$ constant, which will result in non-vanishing of the single- and double-primed coefficients. Vanishing of $u_i^{(s)}(x_1, x_2)$, $u_{i,pp}^{(s)}(x_1, x_2)$ at that edge, that corresponds to vanishing of the single- and double-barred coefficients, constitutes satisfaction of admissible boundary constraints.

Combination (4): $u_{i,pp}^{(s)}(x_1, x_2)$, $u_{i,ppp}^{(s)}(x_1, x_2)$ are not permitted to vanish at an edge $x_p =$ constant, which will result in non-vanishing of the double-barred and double-primed coefficients. Vanishing of $u_i^{(s)}(x_1, x_2)$, $u_{i,p}^{(s)}(x_1, x_2)$ at that edge, that corresponds to vanishing of the single-barred and single-primed coefficients, constitutes satisfaction of admissible boundary constraints.

Special cases of these complementary boundary constraints and the four possible combinations, when assigned at only one of the two opposite ends are summarized in Tables 5 and 6 respectively. As has been mentioned earlier, these "special" cases permit us

TABLE 3

Symmetrically placed complementary and admissible boundary constraints for a fourth order PDE at edges $x_p = 0$, $\bar{x}_p$

Case	Complementary boundary constraint	Admissible boundary constraint	Comments
1.	$u_i \neq 0$; $u_{i,t} \neq 0$; $u_{i,tt} \neq 0$; $u_{i,ttt} \neq 0$	$u_{i,ppp} = 0$	Double-primed coefficients vanish; Single-barred coefficients non-zero
2.	$u_{i,p} \neq 0$; $u_{i,pt} \neq 0$; $u_{i,ptt} \neq 0$	$u_{i,pp}=u_{i,ppt} = 0$	Double-barred coefficients vanish; Single-primed coefficients non-zero
3.	$u_{i,pp} \neq 0$; $u_{i,ppt} \neq 0$	$u_{i,p} = u_{i,pt} = u_{i,ptt} = 0$	Single-primed coefficients vanish; Double-barred coefficients non-zero
4.	$u_{i,ppp} \neq 0$	$u_i = u_{i,t} = u_{i,tt} = u_{i,ttt} = 0$	Single-barred coefficients vanish; Double-primed coefficients non-zero

TABLE 4

Symmetrically placed complementary and admissible boundary constraints for a fourth order PDE at edges $x_p = 0$, $\bar{x}_p$

Combination	Admissible complementary boundary constraint	Admissible boundary constraint	Comments
1.	$u_i \neq 0$; $u_{i,t} \neq 0$; $u_{i,tt} \neq 0$; $u_{i,ttt} \neq 0$; $u_{i,p} \neq 0$; $u_{i,pt} \neq 0$; $u_{i,ptt} \neq 0$	$u_{i,ppp} = 0$; $u_{i,pp} = u_{i,ppt} = 0$	Double-primed and double-barred coefficients vanish; Single-primed and single-barred coefficients non-zero
2.	$u_i \neq 0$; $u_{i,t} \neq 0$; $u_{i,tt} \neq 0$; $u_{i,ttt} \neq 0$; $u_{i,pp} \neq 0$; $u_{i,ppt} \neq 0$	$u_{i,ppp} = 0$; $u_{i,p} = u_{i,pt} = u_{i,ptt} = 0$	Double-primed and single-primed coefficients vanish; Double-barred and single-barred coefficients non-zero
3.	$u_{i,p} \neq 0$; $u_{i,pt} \neq 0$; $u_{i,ptt} \neq 0$; $u_{i,ppp} \neq 0$	$u_{i,pp} = u_{i,ppt} = 0$; $u_i = u_{i,t} = u_{i,tt} = u_{i,ttt} = 0$	Double-barred and single-barred coefficients vanish; Double-primed and single-primed coefficients non-zero
4.	$u_{i,pp} \neq 0$; $u_{i,ppt} \neq 0$; $u_{i,ppp} \neq 0$	$u_{i,p} = u_{i,pt} = u_{i,ptt} = 0$; $u_i = u_{i,t} = u_{i,tt} = u_{i,ttt} = 0$	Single-primed and single-barred coefficients vanish; Double-primed and double-barred coefficients non-zero

to prescribe arbitrary boundary conditions at each of the four edges independent of one another, and thus constitutes the general procedure for solving the most general form of boundary-value problems. This is in contrast to the more "general" case, where assignment of same complementary boundary constraints on two opposite ends proves to be restrictive.

4.2.3. *A system of completely coupled sixth order PDEs*

In the case of a boundary-value problem involving a system of completely coupled sixth order PDEs, six mutually independent cases of complementary boundary constraints are

TABLE 5

Unsymmetrically placed complementary and admissible boundary constraints for a fourth order PDE at edges $x_p = 0,\ \bar{x}_p$

Case	Complementary boundary constraint		Admissible boundary constraint		Comments	
	$x_p = 0$	$x_p = \bar{x}_p$	$x_p = 0$	$x_p = \bar{x}_p$	$p = 1,\ t = 2$	$p = 2,\ t = 1$
1.	$u_i \neq 0; \dots;$ $u_{i,ttt} \neq 0$		$u_{i,ppp} = 0$		$a''_{in} = -\,b''_{in};$ $g''_{in} = -\,h''_{in};$	$c''_{im} = -\,d''_{im};$ $e''_{im} = -f''_{im};$
		$u_i \neq 0; \dots;$ $u_{i,ttt} \neq 0$		$u_{i,ppp} = 0$	$a''_{in} = b''_{in};$ $g''_{in} = h''_{in};$	$c''_{im} = d''_{im};$ $e''_{im} = f''_{im};$
2.	$u_{i,p} \neq 0; \dots;$ $u_{i,ptt} \neq 0$		$u_{i,pp} =$ $u_{i,ppt} = 0$		$\bar{\bar{a}}_{in} = -\,\bar{\bar{b}}_{in};$ $\bar{\bar{e}}_{in} = -\,\bar{\bar{f}}_{in};$	$\bar{c}_{im} = -\,\bar{\bar{d}}_{im};$ $\bar{\bar{g}}_{im} = -\,\bar{\bar{h}}_{im};$
		$u_{i,p} \neq 0; \dots;$ $u_{i,ptt} \neq 0$		$u_{i,pp} =$ $u_{i,ppt} = 0$	$\bar{\bar{a}}_{in} = \bar{\bar{b}}_{in};$ $\bar{\bar{e}}_{in} = \bar{\bar{f}}_{in};$	$\bar{\bar{c}}_{im} = \bar{\bar{d}}_{im};$ $\bar{\bar{g}}_{im} = \bar{\bar{h}}_{im};$
3.	$u_{i,pp} \neq 0;$ $u_{i,ppt} \neq 0;$		$u_{i,p} = u_{i,pt}$ $= u_{i,ptt} = 0$		$a'_{in} = -\,b'_{in};$ $g'_{in} = -\,h'_{in};$	$c'_{im} = -\,d'_{im};$ $e'_{im} = -f'_{im};$
		$u_{i,pp} \neq 0;$ $u_{i,ppt} \neq 0;$		$u_{i,p} = u_{i,pt}$ $= u_{i,ptt} = 0;$	$a'_{in} = b'_{in};$ $g'_{in} = h'_{in};$	$c'_{im} = d'_{im};$ $e'_{im} = f'_{im};$
4.	$u_{i,ppp} \neq 0;$		$u_i = u_{i,t}$ $= u_{i,tt} =$ $u_{i,ttt} = 0;$		$\bar{a}_{in} = -\,\bar{b}_{in};$ $\bar{e}_{in} = -\bar{f}_{in};$	$\bar{c}_{im} = -\,\bar{d}_{im};$ $\bar{g}_{im} = -\,\bar{h}_{im};$
		$u_{i,ppp} \neq 0;$		$u_i = u_{i,t}$ $= u_{i,tt} =$ $u_{i,ttt} = 0;$	$\bar{a}_{in} = \bar{b}_{in};$ $\bar{e}_{in} = \bar{f}_{in};$	$\bar{c}_{im} = \bar{d}_{im};$ $\bar{g}_{im} = \bar{h}_{im}.$

possible, one of which must be introduced in order for the total number of unknowns to become equal to the total number of equations. Table 7 summarizes these cases in the presence of complementary boundary constraints (boundary discontinuities) at both ends, $x_p = 0,\ \bar{x}_p$ ($\bar{x}_p = a$ or b depending on whether $p = 1$ or 2). These six cases, in turn, produce the following $2^{6/2} = 8$ mutually independent combinations of complementary boundary constraints, one of which must be introduced in order for the total number of unknowns to become equal to the total number of equations.

Combination (1): $u_i^{(s)}(x_1, x_2)$, $u_{i,p}^{(s)}(x_1, x_2)$ and $u_{i,pp}^{(s)}(x_1, x_2)$ are not permitted to vanish at an edge $x_p =$ constant, which will result in non-vanishing of the single-barred, single-primed and double-barred coefficients. Vanishing of $u_{i,ppp}^{(s)}(x_1, x_2)$, $u_{i,pppp}^{(s)}(x_1, x_2)$, and $u_{i,ppppp}^{(s)}(x_1, x_2)$ at that edge, that corresponds to vanishing of the double-primed, triple-barred and triple-primed coefficients, constitutes satisfaction of admissible boundary constraints.

Combination (2): $u_i^{(s)}(x_1, x_2)$, $u_{i,p}^{(s)}(x_1, x_2)$ and $u_{i,ppp}^{(s)}$ are not permitted to vanish at an edge $x_p =$ constant, which will result in non-vanishing of the single-barred, single-primed and double-primed coefficients. Vanishing of $u_{i,pp}^{(s)}(x_1, x_2)$, $u_{i,pppp}^{(s)}(x_1, x_2)$, and $u_{i,ppppp}^{(s)}(x_1, x_2)$ at that edge, that corresponds to vanishing of the double-barred, triple-barred and triple-primed coefficients, constitutes satisfaction of admissible boundary constraints.

Combination (3): $u_i^{(s)}(x_1, x_2)$, $u_{i,pp}^{(s)}(x_1, x_2)$ and $u_{i,pppp}^{(s)}(x_1, x_2)$ are not permitted to vanish at an edge $x_p =$ constant, which will result in non-vanishing of the single-barred, double-barred and triple-barred coefficients. Vanishing of $u_{i,p}^{(s)}(x_1, x_2)$, $u_{i,ppp}^{(s)}(x_1, x_2)$, *and* $u_{i,ppppp}^{(s)}(x_1, x_2)$ at that edge, that corresponds to vanishing of the single-primed, double-primed and triple-primed coefficients, constitutes satisfaction of admissible boundary constraints.

TABLE 6

Unsymmetrically placed complementary and admissible boundary constraints for a fourth order PDE at edges $x_p = 0$, $\bar{x}_p$

Combination	Complementary boundary constraint		Admissible boundary constraint		Comments	
	$x_p = 0$	$x_p = \bar{x}_p$	$x_p = 0$	$x_p = \bar{x}_p$	$p = 1$, $t = 2$,	$p = 2$, $t = 1$
1.	$u_i \neq 0; \ldots;$		$u_{i,ppp} = 0;$		$a''_{in} = -\, b''_{in};$	$c''_{im} = -\, d''_{im};$
	$u_{i,ttt} \neq 0$		$u_{i,pp} =$		$g''_{in} = -\, h''_{in};$	$e''_{im} = -f''_{im};$
	$u_{i,p} \neq 0; \ldots;$		$u_{i,ppt} = 0;$		$\bar{\bar{a}}_{in} = -\, \bar{\bar{b}}_{in};$	$\bar{\bar{c}}_{im} = -\, \bar{\bar{d}}_{im};$
	$u_{i,ptt} \neq 0;$				$\bar{\bar{e}}_{in} = -\bar{\bar{f}}_{in};$	$\bar{\bar{g}}_{im} = -\, \bar{\bar{h}}_{im};$
		$u_i \neq 0; \ldots;$		$u_{i,ppp} = 0;$	$a''_{in} = b''_{in};$	$c''_{im} = d''_{im};$
		$u_{i,ttt} \neq 0$		$u_{i,pp} =$	$g''_{in} = h''_{in};$	$e''_{im} = f''_{im};$
		$u_{i,p} \neq 0; \ldots;$		$u_{i,ppt} = 0;$	$\bar{\bar{a}}_{in} = \bar{\bar{b}}_{in};$	$\bar{\bar{c}}_{im} = \bar{\bar{d}}_{im};$
		$u_{i,ptt} \neq 0$			$\bar{\bar{e}}_{in} = \bar{\bar{f}}_{in};$	$\bar{\bar{g}}_{im} = \bar{\bar{h}}_{im};$
2.	$u_i \neq 0; \ldots;$		$u_{i,ppp} = 0;$		$a''_{in} = -\, b''_{in};$	$c''_{im} = -\, d''_{im};$
	$u_{i,ttt} \neq 0;$		$u_{i,p} = u_{i,pt}$		$g''_{in} = -\, h''_{in};$	$e''_{im} = -f''_{im};$
	$u_{i,pp} \neq 0;$		$u_{i,ptt} = 0;$		$a'_{in} = -\, b'_{in};$	$c'_{im} = -\, d'_{im};$
	$u_{i,ppt} \neq 0;$				$g'_{in} = -\, h'_{in};$	$e'_{im} = -f'_{im};$
		$u_i \neq 0; \ldots;$		$u_{i,ppp} = 0;$	$a''_{in} = b''_{in};$	$c''_{im} = d''_{im};$
		$u_{i,ttt} \neq 0$			$g''_{in} = h''_{in};$	$e''_{im} = f''_{im};$
		$u_{i,pp} \neq 0; \ldots;$		$u_{i,p} = u_{i,pt}$	$a'_{in} = b'_{in};$	$c'_{im} = d'_{im};$
		$u_{i,ppt} \neq 0;$		$= u_{i,ptt} = 0;$	$g'_{in} = h'_{in};$	$e'_{im} = f'_{im};$
3.	$u_{i,p} \neq 0; \ldots;$		$u_{i,pp} =$		$\bar{\bar{a}}_{in} = -\, \bar{\bar{b}}_{in};$	$\bar{\bar{c}}_{im} = -\, \bar{\bar{d}}_{im};$
	$u_{i,ptt} \neq 0;$		$u_{i,ppt} = 0$		$\bar{\bar{e}}_{in} = -\bar{\bar{f}}_{in};$	$\bar{\bar{g}}_{im} = -\, \bar{\bar{h}}_{im};$
	$u_{i,ppp} \neq 0;$		$u_i = \cdots$		$\bar{a}_{in} = -\, \bar{b}_{in};$	$\bar{c}_{im} = -\, \bar{d}_{im};$
			$u_{i,ttt} = 0;$		$\bar{e}_{in} = -\bar{f}_{in};$	$\bar{g}_{im} = -\, \bar{h}_{im};$
		$u_{i,p} \neq 0; \ldots;$		$u_{i,pp} =$	$\bar{\bar{a}}_{in} = \bar{\bar{b}}_{in};$	$\bar{\bar{c}}_{im} = \bar{\bar{d}}_{im};$
		$u_{i,ptt} \neq 0$		$u_{i,ppt} = 0$	$\bar{\bar{e}}_{in} = \bar{\bar{f}}_{in};$	$\bar{\bar{g}}_{im} = \bar{\bar{h}}_{im};$
		$u_{i,ppp} \neq 0;$		$u_i = \cdots$	$\bar{a}_{in} = \bar{b}_{in};$	$\bar{c}_{im} = \bar{d}_{im};$
				$u_{i,tt} = 0;$	$\bar{e}_{in} = \bar{f}_{in};$	$\bar{g}_{im} = \bar{h}_{im};$
4.	$u_{i,pp} \neq 0;$		$u_{i,p} = u_{i,pt}$		$a'_{in} = -\, b'_{in};$	$c'_{im} = -\, d'_{im};$
	$u_{i,ppt} \neq 0;$		$= u_{i,ptt} = 0$		$g'_{in} = -\, h'_{in};$	$e'_{im} = -f'_{im};$
	$u_{i,ppp} \neq 0;$		$u_i = \cdots$		$\bar{a}_{in} = -\, \bar{b}_{in};$	$\bar{c}_{im} = -\, \bar{d}_{im};$
			$u_{i,ttt} = 0;$		$\bar{e}_{in} = -\bar{f}_{in};$	$\bar{g}_{im} = -\, \bar{h}_{im};$
		$u_{i,pp} \neq 0;$		$u_{i,p} = u_{i,pt}$	$a'_{in} = b'_{in};$	$c'_{im} = d'_{im};$
		$u_{i,ppt} \neq 0;$		$= u_{i,ptt} = 0$	$g'_{in} = h'_{in};$	$e'_{im} = f'_{im};$
		$u_{i,ppp} \neq 0;$		$u_i = \cdots$	$\bar{a}_{in} = \bar{b}_{in};$	$\bar{c}_{im} = \bar{d}_{im}$
				$u_{i,ttt} = 0;$	$\bar{e}_{in} = \bar{f}_{in};$	$\bar{g}_{im} = \bar{h}_{im}.$

Combination (4): $u_i^{(s)}(x_1, x_2)$, $u_{i,ppp}^{(s)}(x_1, x_2)$ and $u_{i,pppp}^{(s)}(x_1, x_2)$ are not permitted to vanish at an edge $x_p =$ constant, which will result in non-vanishing of the single-barred, double-primed and triple-barred coefficients. Vanishing of $u_{i,p}^{(s)}(x_1, x_2)$, $u_{i,pp}^{(s)}(x_1, x_2)$, and $u_{i,ppppp}^{(s)}(x_1, x_2)$ at that edge, that corresponds to vanishing of the single-primed, double-barred and triple-primed coefficients, constitutes satisfaction of admissible boundary constraints.

TABLE 7

Symmetrically placed complementary and admissible boundary constraints for a sixth order PDE at edges $x_p = 0$, $\bar{x}_p$

Case	Complementary boundary constraints	Admissible boundary constraints	Comments
1.	$u_i \neq 0$; $u_{i,t} \neq 0$; $u_{i,tt} \neq 0$; $u_{i,ttt} \neq 0$; $u_{i,tttt} \neq 0$; $u_{i,ttttt} \neq 0$	$u_{i,ppppp} = 0$	Triple-primed coefficients vanish; Single-barred coefficients non-zero
2.	$u_{i,p} \neq 0$; $u_{i,pt} \neq 0$; $u_{i,ptt} \neq 0$; $u_{i,pttt} \neq 0$; $u_{i,ptttt} \neq 0$	$u_{i,pppp} = u_{i,ppppt} = 0$	Triple-barred coefficients vanish; Single-primed coefficients non-zero
3.	$u_{i,pp} \neq 0$; $u_{i,ppt} \neq 0$; $u_{i,pptt} \neq 0$; $u_{i,ppttt} \neq 0$	$u_{i,ppp} = u_{i,pppt} = u_{i,ppptt} = 0$	Double-primed coefficients vanish; Double-barred coefficients non-zero
4.	$u_{i,ppp} \neq 0$; $u_{i,pppt} \neq 0$; $u_{i,ppptt} \neq 0$	$u_{i,pp} = u_{i,ppt} = u_{i,pptt} = u_{i,ppttt} = 0$	Double-barred coefficients vanish; Double-primed coefficients non-zero
5.	$u_{i,pppp} \neq 0$; $u_{i,ppppt} \neq 0$	$u_{i,p} = u_{i,pt} = u_{i,ptt} = u_{i,pttt} = u_{i,ptttt} = 0$	Single-primed coefficients vanish; Triple-barred coefficients non-zero
6.	$u_{i,ppppp} \neq 0$	$u_i = u_{i,t} = u_{i,tt} = u_{i,ttt} = u_{i,tttt} = u_{i,ttttt} = 0$	Single-barred coefficients vanish; Triple-primed coefficients non-zero

Combination (5): $u^{(s)}_{i,p}(x_1, x_2)$, $u^{(s)}_{i,pp}(x_1, x_2)$ and $u^{(s)}_{i,ppppp}(x_1, x_2)$ are not permitted to vanish at an edge $x_p =$ constant, which will result in non-vanishing of the single-primed, double-barred and triple-primed coefficients. Vanishing of $u^{(s)}_{i}(x_1, x_2)$, $u^{(s)}_{i,ppp}(x_1, x_2)$, and $u^{(s)}_{i,pppp}(x_1, x_2)$ at that edge, that corresponds to vanishing of the single-barred, double-primed and triple-barred coefficients, constitutes satisfaction of admissible boundary constraints.

Combination (6): $u^{(s)}_{i,p}(x_1, x_2)$, $u^{(s)}_{i,ppp}(x_1, x_2)$ and $u^{(s)}_{i,ppppp}(x_1, x_2)$ are not permitted to vanish at an edge $x_p =$ constant, which will result in non-vanishing of the single-primed, double-primed and triple-primed coefficients. Vanishing of $u^{(s)}_{i}(x_1, x_2)$, $u^{(s)}_{i,pp}(x_1, x_2)$, and $u^{(s)}_{i,pppp}(x_1, x_2)$ at that edge, that corresponds to vanishing of the single-barred, double-barred and triple-barred coefficients, constitutes satisfaction of admissible boundary constraints.

Combination (7): $u^{(s)}_{i,pp}(x_1, x_2)$, $u^{(s)}_{i,pppp}(x_1, x_2)$ and $u^{(s)}_{i,ppppp}(x_1, x_2)$ are not permitted to vanish at an edge $x_p =$ constant, which will result in non-vanishing of the double-barred, triple-barred and triple-primed coefficients. Vanishing of $u^{(s)}_{i}(x_1, x_2)$, $u^{(s)}_{i,p}(x_1, x_2)$, and $u^{(s)}_{i,ppp}(x_1, x_2)$ at that edge, that corresponds to vanishing of the single-barred, single-primed and double-primed coefficients, constitutes satisfaction of admissible boundary constraints.

Combination (8): $u^{(s)}_{i,ppp}(x_1, x_2)$, $u^{(s)}_{i,pppp}(x_1, x_2)$ and $u^{(s)}_{i,ppppp}(x_1, x_2)$ are not permitted to vanish at an edge $x_p =$ constant, which will result in non-vanishing of the double-primed, triple-barred and triple-primed coefficients. Vanishing of $u^{(s)}_{i}(x_1, x_2)$, $u^{(s)}_{i,p}(x_1, x_2)$, and $u^{(s)}_{i,pp}(x_1, x_2)$ at that edge, that corresponds to vanishing of the single-barred, single-primed and double-barred coefficients, constitutes satisfaction of admissible boundary constraints.

Special cases of these complementary boundary constraints being assigned at only one of the two opposite ends are summarized in Table 8. As has been mentioned earlier, these

TABLE 8

Unsymmetrically placed complementary and admissible boundary constraints for a sixth order PDE at edges $x_p = 0, \bar{x}_p$

Case	Complementary boundary constraint		Admissible boundary constraint		Comments	
	$x_p = 0$	$x_p = \bar{x}_p$	$x_p = 0$	$x_p = \bar{x}_p$	$p = 1, t = 2$	$p = 2, t = 1$
1.	$u_i \neq 0; \ldots;$ $u_{i,ttttt} \neq 0;$		$u_{i,ppppp} = 0;$		$a'''_{in} = -\,b'''_{in};$ $g'''_{in} = -\,h'''_{in};$	$c'''_{im} = -\,d'''_{im};$ $e'''_{im} = -f'''_{im};$
		$u_i \neq 0; \ldots;$ $u_{i,ttttt} \neq 0;$		$u_{i,ppppp} = 0;$	$a'''_{in} = b'''_{in};$ $g'''_{in} = h'''_{in};$	$c'''_{im} = d'''_{im};$ $e'''_{im} = f'''_{im};$
2.	$u_{i,p} \neq 0; \ldots;$ $u_{i,ptttt} \neq 0;$		$u_{i,pppp} =$ $u_{i,ppppt} = 0;$		$\bar{\bar{\bar{a}}}_{in} = -\,\bar{\bar{\bar{b}}}_{in};$ $\bar{\bar{\bar{e}}}_{in} = -\bar{\bar{\bar{f}}}_{in};$	$\bar{\bar{\bar{c}}}_{im} = -\,\bar{\bar{\bar{d}}}_{im};$ $\bar{\bar{\bar{g}}}_{im} = -\,\bar{\bar{\bar{h}}}_{im};$
		$u_{i,p} \neq 0; \ldots;$ $u_{i,ptttt} \neq 0;$		$u_{i,pppp} =$ $u_{i,ppppt} = 0;$	$\bar{\bar{\bar{a}}}_{in} = \bar{\bar{\bar{b}}}_{in};$ $\bar{\bar{\bar{e}}}_{in} = \bar{\bar{\bar{f}}}_{in};$	$\bar{\bar{\bar{c}}}_{im} = \bar{\bar{\bar{d}}}_{im};$ $\bar{\bar{\bar{g}}}_{im} = \bar{\bar{\bar{h}}}_{im};$
3.	$u_{i,pp} \neq 0; \ldots;$ $u_{i,ppttt} \neq 0;$		$u_{i,ppp} = \cdots =$ $= u_{i,ppptt} = 0;$		$a''_{in} = -\,b''_{in};$ $g''_{in} = -\,h''_{in};$	$c''_{im} = -\,d''_{im}$ $e''_{im} = -f''_{im};$
		$u_{i,pp} \neq 0; \ldots$ $u_{i,ppttt} \neq 0;$		$u_{i,ppp} = \cdots =$ $u_{i,ppptt} = 0;$	$a''_{in} = b''_{in};$ $g''_{in} = h''_{in};$	$c''_{im} = d''_{im};$ $e''_{im} = f''_{im};$
4.	$u_{i,ppp} \neq 0; \ldots;$ $u_{i,ppptt} \neq 0;$		$u_{i,pp} = \ldots;$ $u_{i,ppttt} = 0;$		$\bar{\bar{a}}_{in} = -\,\bar{\bar{b}}_{in};$ $\bar{\bar{e}}_{in} = -\bar{\bar{f}}_{in};$	$\bar{\bar{c}}_{im} = -\,\bar{\bar{d}}_{im};$ $\bar{\bar{g}}_{im} = -\,\bar{\bar{h}}_{im};$
		$u_{i,ppp} \neq 0; \ldots;$ $u_{i,ppptt} \neq 0;$		$u_{i,pp} = \cdots =$ $u_{i,ppttt} = 0;$	$\bar{\bar{a}}_{in} = \bar{\bar{b}}_{in};$ $\bar{\bar{e}}_{in} = \bar{\bar{f}}_{in};$	$\bar{\bar{c}}_{im} = \bar{\bar{d}}_{im};$ $\bar{\bar{g}}_{im} = \bar{\bar{h}}_{im};$
5.	$u_{i,pppp} \neq 0;$ $u_{i,ppppt} \neq 0;$		$u_{i,p} = \cdots =$ $u_{i,ptttt} = 0;$		$a'_{in} = -\,b'_{in};$ $g'_{in} = -\,h'_{in};$	$c'_{im} = -\,d'_{im};$ $e'_{im} = -f'_{im};$
		$u_{i,pppp} \neq 0;$ $u_{i,ppppt} \neq 0;$		$u_{i,p} = \cdots =$ $u_{i,ptttt} = 0;$	$a'_{in} = b'_{in};$ $g'_{in} = h'_{in};$	$c'_{im} = d'_{im};$ $e'_{im} = f'_{im};$
6.	$u_{i,ppppp} \neq 0;$		$u_i = \cdots =$ $u_{i,ttttt} = 0;$		$\bar{a}_{in} = -\,\bar{b}_{in};$ $\bar{e}_{in} = -\bar{f}_{in};$	$\bar{c}_{im} = -\,\bar{d}_{im};$ $\bar{g}_{im} = -\,\bar{h}_{im};$
		$u_{i,ppppp} \neq 0;$		$u_i = \cdots =$ $u_{i,ttttt} = 0;$	$\bar{a}_{in} = \bar{b}_{in};$ $\bar{e}_{in} = \bar{f}_{in};$	$\bar{c}_{im} = \bar{d}_{im};$ $\bar{g}_{im} = \bar{h}_{im}.$

"special" cases permit us to prescribe arbitrary boundary conditions at each of the four edges independent of one another, and thus constitutes the general procedure for solving the most general form of boundary-value problems. This is in contrast to a more "general" case, where assignment of same complementary boundary constraints on two opposite ends proves to be restrictive.

4.2.4. *A system of completely coupled rth* ($r = even$) *order PDEs*

In the case of a boundary-value problem involving a system of completely coupled rth order PDEs, the following r mutually independent cases of complementary boundary constraints are possible. Table 9 summarizes these cases in the presence of complementary

TABLE 9

Symmetrically placed admissible and complementary boundary constraints for an rth order (r = even) PDE at edges $x_p = 0$, $\bar{x}_p$

Complementary boundary constraint	Admissible boundary constraint
1. $u_i \neq 0; u_{i,t} \neq 0; u_{i,tt} \neq 0; \ldots; u_{i,ttt\cdots(r-1)} \neq 0$	$u_{i,ppp\cdots(r-1)} = 0$
2. $u_{i,p} \neq 0; u_{i,pt} \neq 0; \ldots; u_{i,ptt\cdots(r-2)} \neq 0$	$u_{i,pp\cdots(r-2)} = u_{i,tpp\cdots(r-2)} = 0$
3. $u_{i,pp} \neq 0; u_{i,ppt} \neq 0; \ldots; u_{i,ppt\cdots(r-3)} \neq 0$	$u_{i,p\cdots(r-3)} = u_{i,tp\cdots(r-3)} = u_{i,ttp\cdots(r-3)} = 0$
4. $u_{i,ppp} \neq 0; u_{i,pppt} \neq 0: \ldots; u_{i,pppt\cdots(r-4)} \neq 0$	$u_{i,p\cdots(r-4)} = u_{i,tp\cdots(r-4)} = u_{i,ttp\cdots(r-4)} = 0$
--------	--------
--------	--------
$r-2$. $u_{i,p\cdots(r-3)} \neq 0; u_{i,tp\cdots(r-3)} \neq 0; \ldots u_{i,ttp\cdots(r-3)} \neq 0$	$u_{i,pp} = u_{i,ppt} = \cdots = u_{i,ppt\cdots(r-3)} = 0$
$r-1$. $u_{i,pp\cdots(r-2)} \neq 0; u_{i,tpp\cdots(r-2)} \neq 0$	$u_{i,p} = u_{i,ppt} = \cdots = u_{i,ppt\cdots(r-1)} = 0$
r. $u_{i,ppp\cdots(r-1)} \neq 0$	$u_i = u_{i,t} = u_{i,tt} = \cdots = u_{i,ttt\cdots(r-1)} = 0$

boundary constraints (boundary discontinuities) at both ends, $x_p = 0$, $\bar{x}_p$ ($\bar{x}_p = a$ or b depending on whether $p = 1$ or 2). These r cases, in turn, produce $2^{r/2}$ mutually independent combinations of complementary boundary constraints, one of which must be introduced in order for the total number of unknowns to become equal to the total number of equations.

Combination (1): $u_i^{(s)}(x_1, x_2)$, $u_{i,p}^{(s)}(x_1, x_2)$, $u_{i,pp}^{(s)}(x_1, x_2), \ldots, u_{i,ppppp\ldots(r-2)/2}^{(s)}(x_1, x_2)$ are not permitted to vanish at an edge x_p = constant, which will result in non-vanishing of the single-barred, single-primed, double-barred, double-primed,… (i.e., the first half of the barred and primed) coefficients. Vanishing of $u_{i,ppppp\ldots r/2}^{(s)}(x_1, x_2), \ldots, u_{i,ppppp\cdots(r-1)}^{(s)}(x_1, x_2)$ at that edge, that corresponds to vanishing of the latter half of the barred and primed coefficients, constitutes satisfaction of admissible boundary constraints.

⋮

Combination: $u_i^{(s)}(x_1, x_2)$, $u_{i,pp}^{(s)}(x_1, x_2)$, $u_{i,pppp}^{(s)}(x_1, x_2), \ldots,$ are not permitted to vanish at an edge x_p = constant, which will result in non-vanishing of all the barred coefficients. Vanishing of $u_{i,p}^{(s)}(x_1, x_2)$, $u_{i,ppp}^{(s)}(x_1, x_2)$, $u_{i,ppppp}^{(s)}(x_1, x_2), \ldots,$ at that edge, that corresponds to vanishing of all the primed coefficients, constitutes satisfaction of admissible boundary constraints.

⋮

Combination: $u_{i,p}^{(s)}(x_1, x_2)$, $u_{i,ppp}^{(s)}(x_1, x_2)$, $u_{i,ppppp}^{(s)}(x_1, x_2), \ldots,$ are not permitted to vanish at an edge x_p = constant, which will result in non-vanishing of all the primed coefficients. Vanishing of $u_i^{(s)}(x_1, x_2)$, $u_{i,pp}^{(s)}(x_1, x_2)$, $u_{i,pppp}^{(s)}(x_1, x_2), \ldots,$ at that edge, that corresponds to vanishing of all the barred coefficients, constitutes satisfaction of admissible boundary constraints.

⋮

Combination ($2^{r/2}$): $u_{i,ppppp\ldots r/2}^{(s)}(x_1, x_2), \ldots, u_{i,ppppp\ldots(r-1)}^{(s)}(x_1, x_2)$ are not permitted to vanish at an edge x_p = constant, which will result in non-vanishing of the latter half of the barred and primed coefficients. Vanishing of $u_i^{(s)}(x_1, x_2)$, $u_{i,p}^{(s)}(x_1, x_2)$, $u_{i,pp}^{(s)}(x_1, x_2), \ldots, u_{i,ppppp\ldots(r-2)/2}^{(s)}$ (x_1, x_2) at that edge, that corresponds to vanishing of the single-barred, single-primed, double-barred, double-primed, … (i.e., the first half of the barred and primed) coefficients, constitutes satisfaction of admissible boundary constraints.

Special cases of these complementary boundary constraints being assigned at only one of the two opposite ends can be dealt with in a manner similar to $r = 2$, 4, and 6, and will not be presented here in the interest of brevity. As has been mentioned earlier, these "special" cases permit us to prescribe arbitrary boundary conditions at each of the four edges independent of one another, and thus constitutes the general procedure for solving the most general form of boundary-value problems. This is in contrast to a more "general" case, where assignment of same complementary boundary constraints on two opposite ends proves to be restrictive.

5. CONSTRUCTION OF THE COMPLETE FOURIER SOLUTION

Substitution of the correct partial derivatives, as obtained above, into the governing partial differential equations (1) and equating the coefficients of $\sin(\alpha_m x_1)\sin(\beta_n x_2)$, etc., will yield the following, for $i, j = 1, 2, \ldots, N$:

$$
\begin{aligned}
& a_{ij}U^{(1)}_{jmn} - b_{ij1}\alpha_m U^{(4)}_{jmn} - b_{ij2}\beta_n U^{(3)}_{jmn} - c_{ij11}\alpha_m(\alpha_m U^{(1)}_{jmn} + \bar{a}_{jn}\gamma_m + \bar{b}_{jn}\delta_m) \\
&\quad - c_{ij22}\beta_n(\beta_n U^{(1)}_{jmn} + \bar{c}_{jm}\gamma_n + \bar{d}_{jm}\delta_n) + (c_{ij12} + c_{ij21})\alpha_m\beta_n U^{(2)}_{jmn} \\
&\quad + d_{ij111}\alpha_m\{\alpha_m^2 U^{(4)}_{jmn} - (g'_{jn}\gamma_m + h'_{jn}\delta_m)\} + (d_{ij112} + d_{ij121} + d_{ij211})\alpha_m\beta_n\{\alpha_m U^{(3)}_{jmn} \\
&\quad + \alpha_m\beta_n(\bar{g}_{jm}\gamma_n + \bar{h}_{jm}\delta_n)\} + d_{ij222}\{\beta_n^3 U^{(3)}_{jmn} - \beta_n(e'_{jm}\gamma_n + f'_{jm}\delta_n)\} \\
&\quad + e_{ij1111}\{\alpha_m^4 U^{(1)}_{jmn} + \alpha_m^3(\bar{a}_{jn}\gamma_m + \bar{b}_{jn}\delta_m) - \alpha_m(\bar{\bar{a}}_{jn}\gamma_m + \bar{\bar{b}}_{jn}\delta_m)\} \\
&\quad - (e_{ij1112} + e_{ij1121} + e_{ij2111} + e_{ij1211})\{\alpha_m^3\beta_n U^{(2)}_{jmn} - \alpha_m(\bar{\bar{e}}_{jn}\gamma_m + \bar{\bar{f}}_{jn}\delta_m)\} \\
&\quad + (e_{ij1122} + e_{ij1212} + e_{ij2211} + e_{ij2121} + e_{ij2112} + e_{ij1221})\alpha_m\beta_n\{\alpha_m\beta_n U^{(1)}_{jmn} \\
&\quad + \beta_n(\bar{a}_{jn}\gamma_m + \bar{b}_{jn}\delta_m) + \\
&\quad + \alpha_m(\bar{c}_{jm}\gamma_n + \bar{d}_{jm}\delta_n)\} - (e_{ij1222} + e_{ij2122} + e_{ij2212} + e_{ij2221})\beta_n\{\alpha_m\beta_n^2 U^{(2)}_{jmn} \\
&\quad + \bar{\bar{g}}_{jm}\gamma_n + \bar{\bar{h}}_{jm}\delta_n\} \\
&\quad + e_{ij2222}\{\beta_n^4 U^{(1)}_{jmn} + \beta_n^3(\bar{c}_{jm}\gamma_n + \bar{d}_{jm}\delta_n) - \beta_n(\bar{\bar{c}}_{jm}\gamma_n + \bar{\bar{d}}_{jm}\delta_n)\} = Q^{(1)}_{imn} \\
&\quad \text{for } m, n = 1, 2, \ldots, \infty. \qquad (14)
\end{aligned}
$$

Equation (14) and its counterparts, that correspond to equating the coefficients of $\cos(\alpha_m x_1)\cos(\beta_n x_2)$, $\sin(\alpha_m x_1)\cos(\beta_n x_2)$, $\cos(\alpha_m x_1)\sin(\beta_n x_2)$, $\sin(\alpha_m x_1)$, $\sin(\beta_n x_2)$, $\cos(\alpha_m x_1)$, $\cos(\beta_n x_2)$ and the constant terms to zero, will supply $N(4mn + 2m + 2n + 1)$ linear algebraic equations. The remaining equations must be supplied by the prescribed boundary conditions, given by equation (4). Substitution of the appropriate particular solution functions and their appropriately derived partial derivatives up to $(r - 1)$th order into equation (4) will yield

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin(\alpha_m x_1)\sin(\beta_n x_2)\{\bar{a}^{(p)}_{\alpha ij}U^{(1)}_{jmn} - \bar{b}^{(p)}_{\alpha ij1}\alpha_m U^{(4)}_{jmn} - \bar{b}^{(p)}_{\alpha ij2}\beta_n U^{(3)}_{jmn} - \bar{c}^{(p)}_{\alpha ij11}\alpha_m(\alpha_m U^{(1)}_{jmn}$$

$$+ \bar{a}_{jn}\gamma_m + \bar{b}_{jn}\delta_m) + (\bar{c}^{(p)}_{\alpha ij12} + \bar{c}^{(p)}_{\alpha ij21})\alpha_m\beta_n U^{(2)}_{jmn} - \bar{c}^{(p)}_{\alpha ij22}\beta_n(\beta_n U^{(1)}_{jmn} + \bar{c}_{jm}\gamma_n + \bar{d}_{jm}\delta_n)$$

$$+ \bar{d}^{(p)}_{\alpha ij111}\alpha_m(\alpha_m^2 U^{(4)}_{jmn} - g'_{jn}\gamma_m - h'_{jn}\delta_m) + (\bar{d}^{(p)}_{\alpha ij112} + \bar{d}^{(p)}_{\alpha ij121} + \bar{d}^{(p)}_{\alpha ij211})\alpha_m\beta_n(\alpha_m U^{(3)}_{jmn}$$

$$+ \bar{e}_{jn}\gamma_m + \bar{f}_{jn}\delta_m) + (\bar{d}^{(p)}_{\alpha ij212} + \bar{d}^{(p)}_{\alpha ij122} + \bar{d}^{(p)}_{\alpha ij221})\alpha_m\beta_n(\beta_n U^{(4)}_{jmn} + \bar{g}_{jm}\gamma_n + \bar{h}_{jm}\delta_n)$$

$$+ \bar{d}^{(p)}_{\alpha ij222}\beta_n(\beta_n^2 U^{(3)}_{jmn} - e'_{jm}\gamma_n - f'_{jm}\delta_n)\} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\cos(\alpha_m x_1)\cos(\beta_n x_2)\{\cdots\}$$

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin(\alpha_m x_1)\cos(\beta_n x_2)\{\cdots\} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\cos(\alpha_m x_1)\sin(\beta_n x_2)\{\cdots\}$$

$$+ \sum_{m=1}^{\infty}\sin(\alpha_m x_1)\{\cdots\} + \sum_{m=1}^{\infty}\cos(\alpha_m x_1)\{\cdots\} + \sum_{n=1}^{\infty}\sin(\beta_n x_2)\{\cdots\}$$

$$+ \sum_{n=1}^{\infty}\cos(\beta_n x_2)\{\cdots\} + \{\text{constant}\} = 0. \qquad (15)$$

For the rectangular domain $[0, a]x[0, b]$, under consideration, application of the boundary conditions, given by equation (15), at the edges $x_1 = 0, a$ and $x_2 = 0, b$, will, on equating the coefficients of $\sin(\alpha_m x_1)$, etc., to zero, yield $4N(2n + 1)$ and $4N(2m + 1)$ linear algebraic equations, respectively, for a system of fourth order completely coupled PDEs. Finally, a system of $N(4mn + 10m + 10n + 9)$ linear algebraic equations in like number of unknown Fourier coefficients will be solved. Generalization of the above to the case of the boundary-value problem involving a system of rth (r = even) order completely coupled PDEs subjected to satisfaction of admissible boundary conditions/constraints and associated complementary boundary constraints would result in a system of $N\{4mn + 2(r + 1)(m + n) + 2r + 1\}$ linear algebraic equations in like number of unknown Fourier coefficients.

6. APPLICATIONS TO CLASSICAL LAMINATION THEORY (CLT) BASED LAMINATED THIN SHELLS—PROBLEM STATEMENT

A doubly curved panel of rectangular planform is shown in Figure 1, where the reference surface-parallel orthogonal co-ordinate-axes, x_1 and x_2, representing the lines of principal curvature, are placed at the midsurface of the shell, with the x_3-axis remaining parallel to its normal. R_i ($i = 1, 2$) represents the principal radii of curvature of the shell's middle surface. a and b represent the curved span lengths in the x_1 and x_2 directions, respectively, while h denotes the total thickness. The thickness of each layer is denoted by $h^{(k)} = |x_3^{(k-1)} - x_3^{(k)}|$, in which $x_3^{(k-1)}$ and $x_3^{(k)}$, $k = 1, \ldots, N$, are the distances from the reference surface to the bottom and top face of each lamina, respectively, with N being the total number of layers. The simplifying assumptions are: (1) shallowness, (2) transverse inextensibility, (3) classical

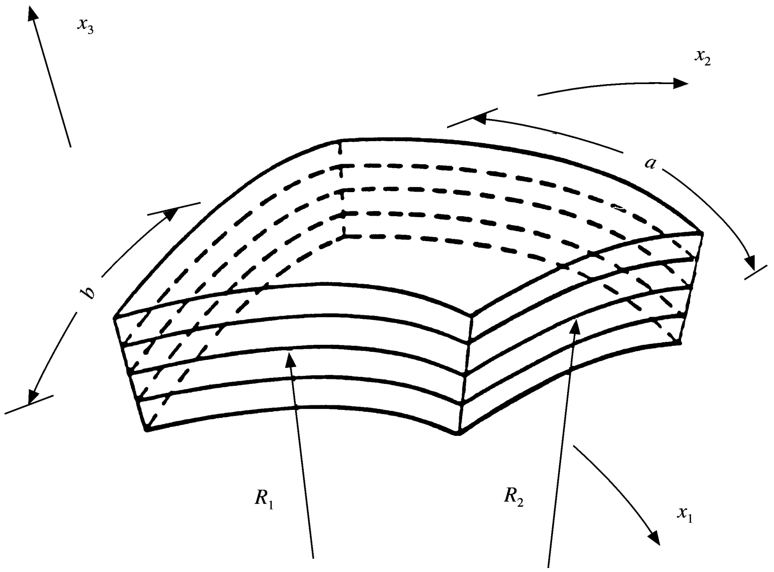

Figure 1. A thin laminated double curved panel of rectangular planform.

lamination theory (CLT) and (4) negligibility of geodesic curvatures of the lines of curvature co-ordinates [17]. Based on these assumptions, generally valid for the case of relatively thin shells, the equations of equilibrium yield the following system of highly coupled fourth order partial differential equations (see e.g., reference [17]):

$$b_{i31}u_{3,1} + b_{i32}u_{3,2} + c_{i111}u_{1,11} + c_{i112}u_{1,12} + c_{i122}u_{1,22} + c_{i211}u_{2,11}$$
$$+ c_{i212}u_{2,12} + c_{i222}u_{2,22} + d_{i3111}u_{3,111} + d_{i3112}u_{3,112} + d_{i3122}u_{3,122}$$
$$+ d_{i322}u_{3,222} = C_i - q_i, \quad i = 1, 2, \qquad (16a, b)$$

$$a_{33}u_3 + b_{311}u_{1,1} + b_{312}u_{1,2} + b_{321}u_{2,1} + b_{322}u_{2,2} + c_{3311}u_{3,11} + c_{3312}u_{3,12}$$
$$+ c_{3322}u_{3,22} + d_{31111}u_{1,111} + d_{31112}u_{1,112} + d_{31122}u_{1,122} + d_{31222}u_{1,222}$$
$$+ d_{32111}u_{2,111} + d_{32112}u_{2,112} + d_{32122}u_{2,122} + d_{32222}u_{2,222} + e_{331111}u_{3,1111}$$
$$+ e_{331112}u_{3,1112} + d_{331122}u_{3,1122} + e_{331222}u_{3,1222} + e_{332222}u_{3,2222} = C_3 - q_3, \qquad (16c)$$

where the constant coefficients are as provided by equations (C1) in Appendix C, and C_i, $i = 1, 2, 3$, is given as

$$C_i = -\omega^2\left(P_1 + \frac{2P_{12}}{R_1}\right)u_i, i = 1, 2, \quad C_3 = -\omega^2 P_1 u_3, \qquad (17a, b)$$

wherein both the surface-parallel and transverse inertias are included, and where P_i, $i = 1, 2$, is defined as

$$(P_1, P_2) = \sum_{k=1}^{N} \int_{x_3^{(k-1)}}^{x_3^{(k)}} \rho^{(k)}(1, x_3)\,dx_3 \tag{18}$$

in which $\rho^{(k)}$ is the density of the kth layer.

The following boundary conditions, prescribed at an edge $x_1 = 0$ or $\bar{x}_1$, and their counterparts prescribed at one or both of the other two edges, form a self-adjoint differential system along with the governing PDEs (16).

Simply supported edge.

$$\text{SS1}: u_3 = M_1 = N_1 = N_6 = 0, \tag{19a}$$

$$\text{SS2}: u_3 = M_1 = u_1 = N_6 = 0, \tag{19b}$$

$$\text{SS3}: u_3 = M_1 = N_1 = u_2 = 0, \tag{19c}$$

$$\text{SS4}: u_3 = M_1 = u_1 = u_2 = 0. \tag{19d}$$

Clamped edge:

$$\text{C1}: u_3 = u_{3,1} = N_1 = N_6 = 0, \tag{20a}$$

$$\text{C2}: u_3 = u_{3,1} = u_1 = N_6 = 0, \tag{20b}$$

$$\text{C3}: u_3 = u_{3,1} = N_1 = u_2 = 0, \tag{20c}$$

$$\text{C4}: u_3 = u_{3,1} = u_1 = u_2 = 0. \tag{20d}$$

Free edge:

$$\text{F1}: Q1 + M_{6,2} = M_1 = N_1 = N_6 = 0, \tag{21a}$$

$$\text{F2}: Q1 + M_{6,2} = M_1 = u_1 = N_6 = 0, \tag{21b}$$

$$\text{F3}: Q1 + M_{6,2} = M_1 = N_1 = u_2 = 0, \tag{21c}$$

$$\text{F4}: Q1 + M_{6,2} = M_1 = u_1 = u_2 = 0. \tag{21d}$$

Roller-skate edge:

$$\text{RS1}: Q1 + M_{6,2} = u_{3,1} = N_1 = N_6 = 0, \tag{22a}$$

$$\text{RS2}: Q1 + M_{6,2} = u_{3,1} = u_1 = N_6 = 0, \tag{22b}$$

$$\text{RS3}: Q1 + M_{6,2} = u_{3,1} = N_1 = u_2 = 0, \tag{22c}$$

$$\text{RS4}: Q1 + M_{6,2} = u_{3,1} = u_1 = u_2 = 0, \tag{22d}$$

wherein N_i, M_i $(i = 1, 2, 6)$ for an arbitrarily laminated anisotropic shell can be expressed in terms of the midsurface strains and changes in curvature as follows [21]:

$$N_1 = \left\{A_{11} + \frac{B_{11}}{R_1}\right\}u_{1,1} + \{A_{16} + cB_{16}\}u_{1,2} + \left\{A_{12} + \frac{B_{12}}{R_2}\right\}u_{2,2}$$

$$+ \{A_{16} - cB_{16}\}u_{2,1} + \left\{\frac{A_{11}}{R_1} + \frac{A_{12}}{R_2}\right\}u_3 - B_{11}u_{3,11} - 2B_{16}u_{3,12} - B_{12}u_{3,22}, \tag{23}$$

where A_{ij}, B_{ij}, and D_{ij} $(i, j = 1, 2, 6)$ are extensional, coupling, and bending rigidities, respectively, while A_{ij} $(i, j = 4, 5)$ denotes transverse shear rigidities. N_2 and M_2 can be obtained from the expressions for N_1 and M_1, respectively, by replacing 1 by 2, c by $-c$, and *vice versa.*

7. FORCED UNDAMPED VIBRATION OF THIN LAMINATED SHELLS/PLATES

In what follows, the dynamic response of thin laminated shells and plates subjected to periodic loading is investigated. The following cases are considered to illustrate the present solution technique.

7.1. AN ARBITRARY LAMINATED THIN ANISOTROPIC DOUBLY CURVED GENERAL PANEL SUBJECTED TO GENERAL TRANSVERSE PERIODIC LOADING

A curved panel of rectangular planform, but otherwise of arbitrary geometry, is considered:

$$R_1 \neq R_2, \quad c \neq 0.$$

7.1.1. *Particular solutions*

The particular solution functions, which can be initially assumed in the form of equations (6, 7), will be dependent on the applied loading. Without any loss of generality, the following distributed periodic loading in equations (16) is assumed [1]:

$$q_1(x_1, x_2) = q_2(x_1, x_2) = 0; q_3(x_1, x_2) = p_0 + p_1x_1/a + p_2x_2/b + p_3x_1x_2/(ab), \tag{24}$$

which is expanded in the form of double Fourier series as follows:

$$p_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Q_{3mn}^{(1)} \sin(\alpha_m x_1)\sin(\beta_n x_2), \tag{25a}$$

$$p_1x_1/a = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} Q_{3mn}^{(4)} \cos(\alpha_m x_1)\sin(\beta_n x_2), \tag{25b}$$

$$p_2 x_2/b = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} Q_{3mn}^{(3)} \sin(\alpha_m x_1) \cos(\beta_n x_2), \tag{25c}$$

$$p_3 x_1 x_2/(ab) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} Q_{3mn}^{(2)} \cos(\alpha_m x_1) \cos(\beta_n x_2), \tag{25d}$$

in which

$$\begin{aligned} Q_{3mn}^{(1)} &= 16p_0/(\pi^2 mn) \quad \text{for } m, n = 1, 3, \ldots, \infty, \\ &= 0 \quad \text{for } m, n = 2, 4, \ldots, \infty, \end{aligned} \tag{26a}$$

$$\begin{aligned} Q_{3mn}^{(2)} &= 16p_3/(\pi^2 mn)^2 \quad \text{for } m, n = 1, 3, \ldots, \infty, \\ &= 0 \quad \text{for } m, n = 2, 4, \ldots, \infty, \end{aligned} \tag{26b}$$

$$\begin{aligned} Q_{3m0}^{(2)} &= -\,2p_3/(\pi m)^2 \quad \text{for } m, n = 1, 3, \ldots, \infty, \\ &= 0 \quad \text{for } m = 2, 4, \ldots, \infty, \end{aligned} \tag{26c}$$

$$\begin{aligned} Q_{30n}^{(2)} &= -\,2p_3/(\pi n)^2 \quad \text{for } n = 1, 3, \ldots, \infty, \\ &= 0 \quad \text{for } n = 2, 4, \ldots, \infty, \end{aligned} \tag{26d}$$

$$\begin{aligned} Q_{3mn}^{(3)} &= -\,16p_2/(\pi^3 mn^2) \quad \text{for } m, n = 1, 3, \ldots, \infty, \\ &= 0 \quad \text{for } m, n = 2, 4, \ldots, \infty, \end{aligned} \tag{26e}$$

$$\begin{aligned} Q_{3m0}^{(3)} &= 2p_2/(\pi n) \quad \text{for } m = 1, 3, \ldots, \infty, \\ &= 0 \quad \text{for } m = 2, 4, \ldots, \infty, \end{aligned} \tag{26f}$$

$$\begin{aligned} Q_{3mn}^{(4)} &= -\,16p_1/(\pi^3 m^2 n) \quad \text{for } m, n = 1, 3, \ldots, \infty, \\ &= 0 \quad \text{for } m, n = 2, 4, \ldots, \infty, \end{aligned} \tag{26g}$$

$$\begin{aligned} Q_{30n}^{(4)} &= 2p_1/(\pi n) \quad \text{for } n = 1, 3, \ldots, \infty, \\ &= 0 \quad \text{for } n = 2, 4, \ldots, \infty. \end{aligned} \tag{26h}$$

Substitution of equations (25, 26) into equation (16) and equating the coefficients of $\sin(\alpha_m x_1)\sin(\beta_n x_2)$, etc., will reveal that $U_{3mn}^{(1)}, U_{3mn}^{(2)}, U_{1mn}^{(3)}, U_{1mn}^{(4)}, U_{2mn}^{(3)}$, and $U_{2mn}^{(4)}$ are coupled through the presence of $Q_{3mn}^{(1)}$ and $Q_{3mn}^{(2)}$. Additionally, $U_{3mn}^{(3)}, U_{3mn}^{(4)}, U_{1mn}^{(1)}, U_{1mn}^{(2)}, U_{2mn}^{(1)}$ and $U_{2mn}^{(2)}$ are coupled through the presence of $U_{3mn}^{(3)}$, and $U_{3mn}^{(4)}$. Therefore, in the case of an arbitrarily laminated anisotropic doubly curved shell, subjected to the transverse periodic loading, given by equations (2b), (5) and (24–26), the appropriate particular solution functions will be given by equations (6, 7) with $u_j, j = 1, \ldots, N = 3$.

7.1.2. *Complementary solution*

The complementary or boundary Fourier solutions to the system of partial differential equations for a thin arbitrarily laminated doubly curved panel given by equation (16), subjected to various combinations of transverse and in-plane boundary conditions, given by equations 19(a–d)–22(a–d), prescribed symmetrically or asymmetrically (with respect to panel centerlines) at opposite edges can easily be obtained using the present approach. Tables 10, 12 and 14 summarize the non-zero boundary Fourier coefficients for various symmetrically prescribed transverse and surface-parallel boundary conditions. Similarly,

TABLE 10

Symmetrically placed complementary and admissible boundary constraints for the transverse displacement u_3 of a thin laminated doubly curved panel at edges $x_p = 0, \bar{x}_p$

Boundary condition (transverse)	Complementary boundary constraint	Admissible boundary constraint	Vanishing coefficients		Non-vanishing coefficients	
			$p = 1$, $t = 2$	$p = 2$, $t = 1$	$p = 1$, $t = 2$	$p = 2$, $t = 1$
Free (F):	$u_3 \neq 0; \ldots;$ $u_{3,ttt} \neq 0;$	$u_{3,ppp} = 0;$	$a''_{3n} = b''_{3n} =$ $g''_{3n} = h''_{3n} = 0$	$c''_{3m} = d''_{3m} =$ $e''_{3m} = f''_{3m} = 0$	$\bar{a}_{3n}; \bar{b}_{3n};$ $\bar{e}_{3n}; \bar{f}_{3n} \neq 0$	$\bar{c}_{3m}; \bar{d}_{3m};$ $\bar{g}_{3m}; \bar{h}_{3m} \neq 0$
	$u_{3,p} \neq 0; u_{3,pt} \neq 0;$ $u_{3,ptt} \neq 0$	$u_{3,pp} =$ $u_{3,ppt} = 0$	$\bar{\bar{a}}_{3n} = \bar{\bar{b}}_{3n} =$ $\bar{\bar{e}}_{3n} = \bar{\bar{f}}_{3n} = 0$	$\bar{\bar{c}}_{3m} = \bar{\bar{d}}_{3m} =$ $\bar{\bar{g}}_{3m} = \bar{\bar{h}}_{3m} = 0$	$a'_{3n}; b'_{3n};$ $g'_{3n}; h'_{3n} \neq 0$	$c'_{3m}; d'_{3m};$ $e'_{3m}; f'_{3m} \neq 0$
Roller-skate (RS):	$u_3 \neq 0; \ldots;$ $u_{3,ttt} \neq 0;$	$u_{3,ppp} = 0;$	$a''_{3n} = b''_{3n} =$ $g''_{3n} = h''_{3n} = 0$	$c''_{3m} = d''_{3m} =$ $e''_{3m} = f''_{3m} = 0$	$\bar{a}_{3n}; \bar{b}_{3n};$ $\bar{e}_{3n}; \bar{f}_{3n} \neq 0$	$\bar{c}_{3m}; \bar{d}_{3m};$ $\bar{g}_{3m}; \bar{h}_{3m} \neq 0$
	$u_{3,pp} \neq 0;$ $u_{3,ppt} \neq 0;$	$u_{3,p} = u_{3,pt}$ $= u_{3,ptt} = 0;$	$a'_{3n} = b'_{3n} =$ $g'_{3n} = h'_{3n} = 0$	$c'_{3m} = d'_{3m} =$ $e'_{3m} = f'_{3m} = 0$	$\bar{\bar{a}}_{3n}; \bar{\bar{b}}_{3n};$ $\bar{\bar{e}}_{3n}; \bar{\bar{f}}_{3n} \neq 0$	$\bar{\bar{c}}_{3m}; \bar{\bar{d}}_{3m};$ $\bar{\bar{g}}_{3m}; \bar{\bar{h}}_{3m} \neq 0$
Simply supported (SS):	$u_{3,p} \neq 0; u_{3,pt} \neq 0;$ $u_{3,ptt} \neq 0;$	$u_{3,pp} =$ $u_{3,ppt} = 0;$	$\bar{\bar{a}}_{3n} = \bar{\bar{b}}_{3n} =$ $\bar{\bar{e}}_{3n} = \bar{\bar{f}}_{3n} = 0$	$\bar{\bar{c}}_{3m} = \bar{\bar{d}}_{3m} =$ $\bar{\bar{g}}_{3m} = \bar{\bar{h}}_{3m} = 0$	$a'_{3n}; b'_{3n};$ $g'_{3n}; h'_{3n} \neq 0$	$c'_{3m}; d'_{3m};$ $e'_{3m}; f'_{3m} \neq 0$
	$u_{3,ppp} \neq 0$	$u_3 = \cdots$ $= u_{3,ttt} = 0$	$\bar{a}_{3n} = \bar{b}_{3n} =$ $\bar{e}_{3n} = \bar{f}_{3n} = 0$	$\bar{c}_{3m} = \bar{d}_{3m} =$ $\bar{g}_{3m} = \bar{h}_{3m} = 0$	$a''_{3n}; b''_{3n};$ $g''_{3n}; h''_{3n} \neq 0$	$c''_{3m}; d''_{3m};$ $e''_{3m}; f''_{3m} \neq 0$
Clamped (C):	$u_{3,pp} \neq 0;$ $u_{3,ppt} \neq 0;$	$u_{3,p} = u_{3,pt}$ $= u_{3,ptt} = 0;$	$a'_{3n} = b'_{3n} =$ $g'_{3n} = h'_{3n} = 0$	$c'_{3m} = d'_{3m} =$ $e'_{3m} = f'_{3m} = 0$	$\bar{\bar{a}}_{3n}; \bar{\bar{b}}_{3n};$ $\bar{\bar{e}}_{3n}; \bar{\bar{f}}_{3n} \neq 0$	$\bar{\bar{c}}_{3m}; \bar{\bar{d}}_{3m};$ $\bar{\bar{g}}_{3m}; \bar{\bar{h}}_{3m} \neq 0$
	$u_{3,ppp} \neq 0$	$u_3 = \cdots$ $u_{3,ttt} = 0$	$\bar{a}_{3n} = \bar{b}_{3n} =$ $\bar{e}_{3n} = \bar{f}_{3n} = 0$	$\bar{c}_{3m} = \bar{d}_{3m} =$ $\bar{g}_{3m} = \bar{h}_{3m} = 0$	$a''_{3n}; b''_{3n};$ $g''_{3n}; h''_{3n} \neq 0$	$c''_{3m}; d''_{3m};$ $e''_{3m}; f''_{3m} \neq 0.$

Table 11

Unsymmetrically placed complementary and admissible boundary constraints for the transverse displacement u_3 of a thin laminated doubly curved panel at edges $x_p = 0$, $\bar{x}_p$

Combination	Complementary boundary constraint		Admissible boundary constraint		Comments	
	At $x_p = 0$ (only)	At $x_p = \bar{x}_p$ (only)	At $x_p = 0$ (only)	At $x_p = \bar{x}_p$ (only)	$p = 1$, $t = 2$	$p = 2$, $t = 1$
Free (F):	$u_3 \neq 0; \ldots;$ $u_{3,ttt} \neq 0;$ $u_{3,p} \neq 0; \ldots;$ $u_{3,ptt} \neq 0;$		$u_{3,ppp} = 0;$ $u_{3,pp} =$ $u_{3,ppt} = 0;$		$a''_{3n} = -\, b''_{3n};$ $g''_{3n} = -\, h''_{3n};$ $\bar{\bar{a}}_{3n} = -\, \bar{\bar{b}}_{3n};$ $\bar{\bar{e}}_{3n} = -\bar{\bar{f}}_{3n};$	$c''_{3m} = -\, d''_{3m};$ $e''_{3m} = -f''_{3m};$ $\bar{\bar{c}}_{3m} = -\, \bar{\bar{d}}_{3m};$ $\bar{\bar{g}}_{3m} = -\, \bar{\bar{h}}_{3m};$
		$u_3 \neq 0; \ldots;$ $u_{3,ttt} \neq 0;$ $u_{3,p} \neq 0; \ldots;$ $u_{3,ptt} \neq 0;$		$u_{3,ppp} = 0;$ $u_{3,pp} =$ $u_{3,ppt} = 0;$	$a''_{3n} = b''_{3n};$ $g''_{3n} = h''_{3n};$ $\bar{\bar{a}}_{3n} = \bar{\bar{b}}_{3n};$ $\bar{\bar{e}}_{3n} = \bar{\bar{f}}_{3n};$	$c''_{3m} = d''_{3m};$ $e''_{3m} = f''_{3m}$ $\bar{\bar{c}}_{3m} = \bar{\bar{d}}_{3m};$ $\bar{\bar{g}}_{3m} = \bar{\bar{h}}_{3m};$
Roller-skate (RS):	$u_3 \neq 0; \ldots;$ $u_{3,ttt} \neq 0;$ $u_{3,pp} \neq 0;$ $u_{3,ppt} \neq 0;$		$u_{3,ppp} = 0;$ $u_{3,p} = u_{3,pt}$ $u_{3,ptt} = 0;$		$a''_{3n} = -\, b''_{3n};$ $g''_{3n} = -\, h''_{3n};$ $a'_{3n} = -\, b'_{3n};$ $g'_{3n} = -\, h'_{3n};$	$c''_{3m} = -\, d''_{3m};$ $e''_{3m} = -f''_{3m};$ $c'_{3m} = -\, d'_{3m};$ $e'_{3m} = -f'_{3m};$
		$u_3 \neq 0; \ldots;$ $u_{3,ttt} \neq 0$ $u_{3,pp} \neq 0;$ $u_{3,ppt} \neq 0;$		$u_{3,ppp} = 0$ $u_{3,p} = u_{3,pt}$ $= u_{3,ptt} = 0;$	$a''_{3n} = b''_{3n};$ $g''_{3n} = h''_{3n};$ $a'_{3n} = b'_{3n};$ $g'_{3n} = h'_{3n}$	$c''_{3m} = d''_{3m};$ $e''_{3m} = f''_{3m};$ $c'_{3m} = d'_{3m};$ $e'_{3m} = f'_{3m};$
Simply supported (SS):	$u_{3,p} \neq 0; \ldots;$ $u_{3,ptt} \neq 0;$ $u_{3,ppp} \neq 0;$		$u_{3,pp} =$ $u_{3,pptt} = 0$ $u_3 = \cdots$ $u_{3,ttt} = 0;$		$\bar{\bar{a}}_{3n} = -\, \bar{\bar{b}}_{3n};$ $\bar{\bar{e}}_{3n} = -\bar{\bar{f}}_{3n};$ $\bar{a}_{3n} = -\, \bar{b}_{3n};$ $\bar{e}_{3n} = -\bar{f}_{3n};$	$\bar{\bar{c}}_{3m} = -\, \bar{\bar{d}}_{3m};$ $\bar{\bar{g}}_{3m} = -\, \bar{\bar{h}}_{3m};$ $\bar{c}_{3m} = -\, \bar{d}_{3m};$ $\bar{g}_{3m} = -\, \bar{h}_{3m};$
		$u_{3,p} \neq 0; \ldots;$ $u_{3,ptt} \neq 0$ $u_{3,ppp} \neq 0;$		$u_{3,pp} =$ $u_{3,ppt} = 0$ $u_3 = \cdots$ $u_{3,ttt} = 0;$	$\bar{\bar{a}}_{3n} = \bar{\bar{b}}_{3n};$ $\bar{\bar{e}}_{3n} = \bar{\bar{f}}_{3n};$ $\bar{a}_{3n} = \bar{b}_{3n};$ $\bar{e}_{3n} = \bar{f}_{3n};$	$\bar{\bar{c}}_{3m} = \bar{\bar{d}}_{3m};$ $\bar{\bar{g}}_{3m} = \bar{\bar{h}}_{3m};$ $\bar{c}_{3m} = \bar{d}_{3m};$ $\bar{g}_{3m} = \bar{h}_{3m};$
Clamped (C):	$u_{3,pp} \neq 0;$ $u_{3,ppt} \neq 0;$ $u_{3,ppp} \neq 0;$		$u_{3,p} = u_{3,pt}$ $= u_{3,ptt} = 0;$ $u_3 = \cdots$ $u_{3,ttt} = 0;$		$a'_{3n} = -\, b'_{3n};$ $g'_{3n} = -\, h'_{3n};$ $\bar{a}_{3n} = -\, \bar{b}_{3n};$ $\bar{e}_{3n} = -\bar{f}_{3n};$	$c'_{3m} = -\, d'_{3m};$ $e'_{3m} = -f'_{3m};$ $\bar{c}_{3m} = -\, \bar{d}_{3m};$ $\bar{g}_{3m} = -\, \bar{h}_{3m};$
		$u_{3,pp} \neq 0;$ $u_{3,ppt} \neq 0;$ $u_{3,ppp} \neq 0;$		$u_{3,p} = u_{3,pt}$ $= u_{3,ptt} = 0;$ $u_3 = \cdots$ $u_{3,ttt} = 0;$	$a'_{3n} = b'_{3n};$ $g'_{3n} = h'_{3n};$ $\bar{a}_{3n} = \bar{b}_{3n};$ $\bar{e}_{3n} = \bar{f}_{3n};$	$c'_{3m} = d'_{3m};$ $e'_{3m} = f'_{3m};$ $\bar{c}_{3m} = \bar{d}_{3m};$ $\bar{g}_{3m} = \bar{h}_{3m}.$

Tables 11, 13 and 15 summarize the non-zero boundary Fourier coefficients for various asymmetrically prescribed transverse and surface-parallel boundary conditions. The following examples are provided for the purpose of illustration.

TABLE 12

Symmetrically placed complementary and admissible boundary constraints for the in-plane displacements u_1 of a thin laminated doubly curved panel at edges

Boundary condition (in plane)	Complementary boundary constraint	Admissible boundary constraint	Comments	
			Vanishing coefficients	Non-vanishing coefficients
(a) $x_1 = 0, \bar{x}_1 = a$				
1 or 3	$u_1 \neq 0; u_{1,2} \neq 0;$	$u_{1,1} = 0$	$a'_{1n} = b'_{1n} = g'_{1n} = h'_{1n} = 0$	$\bar{a}_{1n}; \bar{b}_{1n}; \bar{e}_{1n}; \bar{f}_{1n} \neq 0$
2 or 4	$u_{1,1} \neq 0$	$u_1 = u_{1,2} = 0$	$\bar{a}_{1n} = \bar{b}_{1n} = \bar{e}_{1n} = \bar{f}_{1n} = 0$	$a'_{1n}; b'_{1n}; g'_{1n}; h'_{1n} \neq 0.$
(b) $x_2 = 0, \bar{x}_2 = b$				
1 or 2	$u_1 \neq 0; u_{1,1} \neq 0$	$u_{1,2} = 0$	$c'_{1m} = d'_{1m} = e'_{1m} = f'_{1m} = 0$	$\bar{c}_{1m}; \bar{d}_{1m}; \bar{g}_{1m}; \bar{h}_{1m} \neq 0$
3 or 4	$u_{1,2} \neq 0$	$u_1 = u_{1,1} = 0$	$\bar{c}_{1m} = \bar{d}_{1m} = \bar{g}_{1m} = \bar{h}_{1m} = 0$	$c'_{1m}; d'_{1m}; e'_{1m}; f'_{1m} \neq 0.$

7.1.2.1. *Example* 1: *clamped C4 at all four edges*. For clamped boundary condition prescribed at all four edges, the transverse complementary boundary constraints, $u_{3,11} \neq 0$ and $u_{3,111} \neq 0$ at $x_1 = 0$, $\bar{x}_1(= a)$, while $u_{3,22} \neq 0$; and $u_{3,222} \neq 0$ at the other two, dictate that the boundary Fourier coefficients—$\bar{\bar{a}}_{3n}$; $\bar{\bar{b}}_{3n}$; $\bar{\bar{e}}_{3n}$; $\bar{\bar{f}}_{3n}$; a''_{3n}; b''_{3n}; g''_{3n}; h''_{3n}; $\bar{\bar{c}}_{3m}$; $\bar{\bar{d}}_{3m}$; $\bar{\bar{g}}_{3m}$; $\bar{\bar{h}}_{3m}$; c''_{3m}; d''_{3m}; e''_{3m} and f''_{3m}—be non-zero (Table 10). Additionally, for the type 4 surface-parallel boundary condition prescribed at all four edges, the surface-parallel complementary boundary constraints, $u_{1,1} \neq 0$ and $u_{2,1} \neq 0$ at two opposite edges, $x_1 = 0$, $\bar{x}_1(= a)$, while $u_{1,2} \neq 0$ and $u_{2,2} \neq 0$ at the other two, dictate that the boundary Fourier coefficients—a'_{1n}; b'_{1n}; g'_{1n}; h'_{1n}; a'_{2n}; b'_{2n}; g'_{2n}; h'_{2n}; c'_{1m}; d'_{1m}; e'_{1m}; f'_{1m}; c'_{2m}; d'_{2m}; e'_{2m} and f'_{2m}—be non-zero (Tables 12, 14). The corresponding non-zero displacements or their normal derivatives at the edges are then given as follows:

$$\{u^{(1)}_{3,11}(0, x_2);\ u^{(1)}_{3,11}(a, x_2)\} = \frac{a}{4}\sum_{n=0}^{\infty}(\mp\ \bar{\bar{a}}_{3n} - \bar{\bar{b}}_{3n})\sin(\beta_n x_2), \tag{27a, b}$$

$$\{u^{(3)}_{3,11}(0, x_2);\ u^{(3)}_{3,11}(a, x_2)\} = \frac{a}{4}\sum_{n=0}^{\infty}(\mp\ \bar{\bar{e}}_{3n} - \bar{\bar{f}}_{3n})\cos(\beta_n x_2), \tag{27c, d}$$

$$\{u^{(1)}_{3,22}(x_1, 0);\ u^{(1)}_{3,22}(x_1, b)\} = \frac{b}{4}\sum_{m=0}^{\infty}(\mp\ \bar{\bar{c}}_{3m} - \bar{\bar{d}}_{3m})\sin(\alpha_m x_1), \tag{27e, f}$$

$$\{u^{(4)}_{3,22}(x_1, 0);\ u^{(4)}_{3,22}(x_1, b)\} = \frac{b}{4}\sum_{m=0}^{\infty}(\mp\ \bar{\bar{g}}_{3m} - \bar{\bar{h}}_{3m})\cos(\alpha_m x_1), \tag{27g, h}$$

$$\{u^{(2)}_{3,111}(0, x_2);\ u^{(2)}_{3,111}(a, x_2)\} = \frac{a}{4}\sum_{n=0}^{\infty}(\mp\ a''_{3n} - b''_{3n})\cos(\beta_n x_2), \tag{27i, j}$$

TABLE 13

Unsymmetrically placed complementary and admissible boundary constraints for the in-plane displacement u_1 of a thin laminated doubly curved panel at edges

Boundary condition (in plane)	Complementary boundary constraint		Admissible boundary constraint		Comments
(a) $x_1 = 0, \bar{x}_1$					
	At $x_1 = 0$ (only)	At $x_1 = \bar{x}_1$ (only)	At $x_1 = 0$ (only)	At $x_1 = \bar{x}_1$ (only)	
1 or 3	$u_1 \neq 0$; $u_{1,2} \neq 0$		$u_{1,1} = 0$		$a'_{1n} = -b'_{1n}$; $g'_{1n} = -h'_{1n}$;
2 or 4		$u_{1,1} \neq 0$		$u_1 = u_{1,2} = 0$	$\bar{a}_{1n} = \bar{b}_{1n}$; $\bar{e}_{1n} = \bar{f}_{1n}$;
2 or 4	$u_{1,1} \neq 0$		$u_1 = u_{1,2} = 0$		$\bar{a}_{1n} = -\bar{b}_{1n}$; $\bar{e}_{1n} = -\bar{f}_{1n}$;
1 or 3		$u_1 \neq 0$; $u_{1,2} \neq 0$		$u_{1,1} = 0$	$a'_{1n} = b'_{1n}$; $g'_{1n} = h'_{1n}$.
(b) $x_2 = 0, \bar{x}_1$					
	At $x_2 = 0$ (only)	At $x_2 = \bar{x}_2$ (only)	At $x_2 = 0$ (only)	At $x_2 = \bar{x}_2$ (only)	
1 or 2	$u_1 \neq 0$; $u_{1,1} \neq 0$		$u_{1,2} = 0$		$c'_{1m} = -d'_{1m}$; $e'_{1m} = -f'_{1m}$;
3 or 4		$u_{1,2} \neq 0$		$u_1 = u_{1,1} = 0$	$\bar{c}_{1m} = \bar{d}_{1m}$; $\bar{g}_{1m} = \bar{h}_{1m}$;
3 or 4	$u_{1,2} \neq 0$		$u_1 = u_{1,1} = 0$		$\bar{c}_{1m} = -\bar{d}_{1m}$; $\bar{g}_{1m} = -\bar{h}_{1m}$;
1 or 2		$u_1 \neq 0$; $u_{1,1} \neq 0$		$u_{1,2} = 0$	$c'_{1m} = d'_{1m}$; $e'_{1m} = f'_{1m}$.

$$\{u^{(4)}_{3,111}(0, x_2);\ u^{(4)}_{3,111}(a, x_2)\} = \frac{a}{4}\sum_{n=0}^{\infty} (\mp g''_{3n} - h''_{3n}) \sin(\beta_n x_2), \qquad (27\text{k, l})$$

$$\{u^{(2)}_{3,222}(x_1, 0);\ u^{(2)}_{i,222}(x_1, b)\} = \frac{b}{4}\sum_{m=0}^{\infty} (\mp c''_{3m} - d''_{3m}) \cos(\alpha_m x_1), \qquad (27\text{m, n})$$

$$\{u^{(3)}_{3,222}(x_1, 0);\ u^{(3)}_{3,222}(x_1, b)\} = \frac{b}{4}\sum_{m=0}^{\infty} (\mp e''_{3m} - f''_{3m}) \sin(\alpha_m x_1), \qquad (27\text{o, p})$$

$$\{u^{(2)}_{i,1}(0, x_2);\ u^{(2)}_{i,1}(a, x_2)\} = \frac{a}{4}\sum_{n=0}^{\infty} (\mp a'_{in} - b'_{in}) \cos(\beta_n x_2), \quad i = 1, 2, \qquad (28\text{a, b})$$

$$\{u^{(4)}_{i,1}(0, x_2);\ u^{(4)}_{i,1}(a, x_2)\} = \frac{a}{4}\sum_{n=0}^{\infty} (\mp g'_{in} - h'_{in}) \sin(\beta_n x_2), \quad i = 1, 2, \qquad (28\text{c, d})$$

TABLE 14

Symmetrically placed complementary and admissible boundary constraints for the in-plane displacement u_2 of a thin laminated doubly curved panel at edges

Boundary condition (in plane)	Complementary boundary constraint	Admissible boundary constraint	Comments	
			Vanishing coefficients	Non-vanishing coefficients
(a) $x_1 = 0, \bar{x}_1 = a$				
1 or 2	$u_2 \neq 0$; $u_{2,2} \neq 0$;	$u_{2,1} = 0$	$a'_{2n} = b'_{2n} = g'_{2n} = h'_{2n} = 0$	$\bar{a}_{2n}$; $\bar{b}_{2n}$; $\bar{e}_{2n}$;$\bar{f}_{2n} \neq 0$
3 or 4	$u_{2,1} \neq 0$	$u_2 = u_{2,2} = 0$	$\bar{a}_{2n} = \bar{b}_{2n} = \bar{e}_{2n} = \bar{f}_{2n} = 0$	a'_{2n}; b'_{2n}; g'_{2n}; $h'_{2n} \neq 0.$
(b) $x_2 = 0, \bar{x}_2 = b$				
1 or 3	$u_2 \neq 0$; $u_{2,1} \neq 0$	$u_{2,2} = 0$	$c'_{2m} = d'_{2m} = e'_{2m} = f'_{2m} = 0$	$\bar{c}_{2m}$; $\bar{d}_{2m}$; $\bar{g}_{2m}$; $\bar{h}_{2m} \neq 0$
2 or 4	$u_{2,2} \neq 0$	$u_2 = u_{2,1} = 0$	$\bar{c}_{2m} = \bar{d}_{2m} = \bar{g}_{2m} = \bar{h}_{2m} = 0$	c'_{2m}; d'_{2m}; e'_{2m};$f'_{2m} \neq 0$

$$\{u_{i,2}^{(2)}(x_1, 0); u_{i,2}^{(2)}(x_1, b)\} = \frac{b}{4} \sum_{m=0}^{\infty} (\mp c'_{im} - d'_{im}) \cos(\alpha_m x_1), \quad i = 1, 2, \tag{28e, f}$$

$$\{u_{i,2}^{(3)}(x_1, 0); u_{i,2}^{(3)}(x_1, b)\} = \frac{b}{4} \sum_{m=0}^{\infty} (\mp e'_{im} - f'_{im}) \sin(\alpha_m x_1), \quad i = 1, 2. \tag{28g, h}$$

7.1.2.2. *Example* 2: *free F*1 *at all four edges.* For free boundary conditions prescribed at all four edges, the transverse complementary boundary constraints, $u_3 \neq 0$ and $u_{3,1} \neq 0$ at $x_1 = 0, \bar{x}_1 (= a)$, while $u_3 \neq 0$ and $u_{3,2} \neq 0$ at $x_2 = 0, \bar{x}_2$ $(= b)$, dictate that the boundary Fourier coefficients—$\bar{a}_{3n}$; $\bar{b}_{3n}$; $\bar{e}_{3n}$;$\bar{f}_{3n}$; a'_{3n}; b'_{3n}; g'_{3n}; h'_{3n}; $\bar{c}_{3m}$; $\bar{d}_{3m}$; $\bar{g}_{3m}$; $\bar{h}_{3m}$; c'_{3m}; d'_{3m}; e'_{3m} and f'_{3m}—be non-zero (Table 10). Additionally, for the type 1 surface-parallel boundary condition prescribed at all four edges, the surface-parallel complementary boundary constraints, $u_1 \neq 0$ and $u_2 \neq 0$ at all four edges, dictate that the boundary Fourier coefficients—$\bar{a}_{1n}$; $\bar{b}_{1n}$; $\bar{e}_{1n}$;$\bar{f}_{1n}$; $\bar{a}_{2n}$; $\bar{b}_{2n}$; $\bar{e}_{2n}$;$\bar{f}_{2n}$; $\bar{c}_{1m}$; $\bar{d}_{1m}$; $\bar{g}_{1m}$; $\bar{h}_{1m}$; $\bar{c}_{2m}$; $\bar{d}_{2m}$; $\bar{g}_{2m}$ and $\bar{h}_{2m}$—be non-zero (Tables 12, 14). The corresponding non-zero displacements or their normal derivatives at the edges are then given as follows:

$$\{u_3^{(1)}(0, x_2); u_3^{(1)}(a, x_2)\} = \frac{a}{4} \sum_{n=0}^{\infty} (\mp \bar{a}_{3n} - \bar{b}_{3n}) \sin(\beta_n x_2), \tag{29a, b}$$

$$\{u_3^{(3)}(0, x_2); u_3^{(3)}(a, x_2)\} = \frac{a}{4} \sum_{n=0}^{\infty} (\mp \bar{e}_{3n} - \bar{f}_{3n}) \cos(\beta_n x_2), \tag{29c, d}$$

$$\{u_3^{(1)}(x_1, 0); u_3^{(1)}(x_1, b)\} = \frac{b}{4} \sum_{m=0}^{\infty} (\mp \bar{c}_{3m} - \bar{d}_{3m}) \sin(\alpha_m x_1), \tag{29e, f}$$

TABLE 15

Unsymmetrically placed complementary and admissible boundary constraints for the in-plane displacement u_2 of a thin laminated doubly curved panel at edges

Boundary condition (in plane)	Complementary boundary constraint		Admissible boundary constraint		Comments
(a) $x_1 = 0, \bar{x}_1$					
	At $x_1 = 0$ (only)	At $x_1 = \bar{x}_1$ (only)	At $x_1 = 0$ (only)	At $x_1 = \bar{x}_1$ (only)	
1 or 2	$u_2 \neq 0$; $u_{2,2} \neq 0$		$u_{2,1} = 0$		$a'_{2n} = -b'_{2n}$; $g'_{2n} = -h'_{2n}$;
3 or 4		$u_{2,1} \neq 0$		$u_2 = u_{2,2} = 0$	$\bar{a}_{2n} = \bar{b}_{2n}$; $\bar{e}_{2n} = \bar{f}_{2n}$;
3 or 4	$u_{2,1} \neq 0$		$u_2 = u_{2,2} = 0$		$\bar{a}_{2n} = -\bar{b}_{2n}$; $\bar{e}_{2n} = -\bar{f}_{2n}$;
1 or 2		$u_2 \neq 0$; $u_{2,2} \neq 0$		$u_{2,1} = 0$	$a'_{2n} = b'_{2n}$; $g'_{2n} = h'_{2n}$.
(b) $x_2 = 0, \bar{x}_2$					
	At $x_2 = 0$ (only)	At $x_2 = \bar{x}_2$ (only)	At $x_2 = 0$ (only)	At $x_2 = \bar{x}_2$ (only)	
1 or 3	$u_2 \neq 0$; $u_{2,1} \neq 0$		$u_{2,2} = 0$		$c'_{2m} = -d'_{2m}$; $e'_{2m} = -f'_{2m}$;
2 or 4		$u_{2,2} \neq 0$		$u_2 = u_{2,1} = 0$	$\bar{c}_{2m} = \bar{d}_{2m}$; $\bar{g}_{2m} = \bar{h}_{2m}$;
2 or 4	$u_{2,2} \neq 0$		$u_2 = u_{2,1} = 0$		$\bar{c}_{2m} = -\bar{d}_{2m}$; $\bar{g}_{2m} = -\bar{h}_{2m}$;
1 or 3		$u_2 \neq 0$; $u_{2,1} \neq 0$		$u_{2,2} = 0$	$c'_{2m} = d'_{2m}$; $e'_{2m} = f'_{2m}$.

$$\{u_3^{(4)}(x_1, 0); u_3^{(4)}(x_1, b)\} = \frac{b}{4} \sum_{m=0}^{\infty} (\mp \bar{g}_{3m} - \bar{h}_{3m}) \cos(\alpha_m x_1), \qquad (29\text{g, h})$$

$$\{u_{3,1}^{(2)}(0, x_2); u_{3,1}^{(2)}(a, x_2)\} = \frac{a}{4} \sum_{n=0}^{\infty} (\mp a'_{3n} - b'_{3n}) \cos(\beta_n x_2), \qquad (29\text{i, j})$$

$$\{u_{3,1}^{(4)}(0, x_2); u_{3,1}^{(4)}(a, x_2)\} = \frac{a}{4} \sum_{n=0}^{\infty} (\mp g'_{3n} - h'_{3n}) \sin(\beta_n x_2), \qquad (29\text{k, l})$$

$$\{u_{3,2}^{(2)}(x_1, 0); u_{i,2}^{(2)}(x_1, b)\} = \frac{b}{4} \sum_{m=0}^{\infty} (\mp c'_{3m} - d'_{3m}) \cos(\alpha_m x_1), \qquad (29\text{m, n})$$

$$\{u_{3,2}^{(3)}(x_1, 0); u_{3,2}^{(3)}(x_1, b)\} = \frac{b}{4} \sum_{m=0}^{\infty} (\mp e'_{3m} - f'_{3m}) \sin(\alpha_m x_1), \qquad (29\text{o, p})$$

$$\{u_i^{(1)}(0, x_2); u_i^{(1)}(a, x_2)\} = \frac{a}{4} \sum_{n=0}^{\infty} (\mp \bar{a}_{in} - \bar{b}_{in}) \sin(\beta_n x_2), \quad i = 1, 2; \qquad (30\text{a, b})$$

$$\{u_i^{(3)}(0,x_2);u_i^{(3)}(a,x_2)\}=\frac{a}{4}\sum_{n=0}^{\infty}(\mp\bar{e}_{in}-\bar{f}_{in})\cos(\beta_n x_2),\quad i=1,2; \tag{30c, d}$$

$$\{u_i^{(1)}(x_1,0);u_i^{(1)}(x_1,b)\}=\frac{b}{4}\sum_{m=0}^{\infty}(\mp\bar{c}_{im}-\bar{d}_{im})\sin(\alpha_m x_1),\quad i=1,2; \tag{30e, f}$$

$$\{u_i^{(4)}(x_1,0);u_i^{(4)}(x_1,b)\}=\frac{b}{4}\sum_{m=0}^{\infty}(\mp\bar{g}_{im}-\bar{h}_{im})\cos(\alpha_m x_1). \tag{30g, h}$$

7.1.2.3. *Example* 3: *simply supported SS2 at all four edges.* For simply supported boundary condition prescribed at all four edges, the transverse complementary boundary constraints, $u_{3,1}\neq 0$ and $u_{3,111}\neq 0$ at $x_1=0,\bar{x}_1(=a)$, while $u_{3,2}\neq 0$ and $u_{3,222}\neq 0$ at the other two, dictate that the boundary Fourier coefficients—a'_{3n}; b'_{3n}; g'_{3n}; h'_{3n}; a''_{3n}; b''_{3n}; g''_{3n}; h''_{3n}; c'_{3m}; d'_{3m}; e'_{3m}; f'_{3m}; c''_{3m}; d''_{3m}; e''_{3m}; f''_{3m}—be non-zero (Table 10). Additionally, for the type 2 surface-parallel boundary condition prescribed at all four edges, the surface-parallel complementary boundary constraints, $u_{1,1}\neq 0$ and $u_{2,1}\neq 0$ at two opposite edges, $x_1=0,\bar{x}_1(=a)$, while $u_1\neq 0$ and $u_2\neq 0$, at the other two opposite edges, $x_2=0,\bar{x}_2(=b)$, dictate that the boundary Fourier coefficients—a'_{1n}; b'_{1n}; g'_{1n}; h'_{1n}; $\bar{a}_{2n}$; $\bar{b}_{2n}$; $\bar{e}_{2n}$; $\bar{f}_{2n}$; $\bar{c}_{1m}$; $\bar{d}_{1m}$; $\bar{g}_{1m}$; $\bar{h}_{1m}$; c'_{2m}; d'_{2m}; e'_{2m}; f'_{2m}—be non-zero (Tables 12 and 14). The corresponding non-zero displacements or their normal derivatives at the edges are then given as follows:

$$\{u_{3,1}^{(2)}(0,x_2);u_{3,1}^{(2)}(a,x_2)\}=\frac{a}{4}\sum_{n=0}^{\infty}(\mp a'_{3n}-b'_{3n})\cos(\beta_n x_2), \tag{31a, b}$$

$$\{u_{3,1}^{(4)}(0,x_2);u_{3,1}^{(4)}(a,x_2)\}=\frac{a}{4}\sum_{n=0}^{\infty}(\mp g'_{3n}-h'_{3n})\sin(\beta_n x_2), \tag{31c, d}$$

$$\{u_{3,2}^{(2)}(x_1,0);u_{3,2}^{(2)}(x_1,b)\}=\frac{b}{4}\sum_{m=0}^{\infty}(\mp c'_{3m}-d'_{3m})\cos(\alpha_m x_1), \tag{31e, f}$$

$$\{u_{3,2}^{(3)}(x_1,0);u_{3,2}^{(3)}(x_1,b)\}=\frac{b}{4}\sum_{m=0}^{\infty}(\mp e'_{3m}-f'_{3m})\sin(\alpha_m x_1), \tag{31g, h}$$

$$\{u_{3,111}^{(2)}(0,x_2);u_{3,111}^{(2)}(a,x_2)\}=\frac{a}{4}\sum_{n=0}^{\infty}(\mp a''_{3n}-b''_{3n})\cos(\beta_n x_2), \tag{31i, j}$$

$$\{u_{3,111}^{(4)}(0,x_2);u_{3,111}^{(4)}(a,x_2)\}=\frac{a}{4}\sum_{n=0}^{\infty}(\mp g''_{3n}-h''_{3n})\sin(\beta_n x_2), \tag{31k, l}$$

$$\{u_{3,222}^{(2)}(x_1,0);u_{3,222}^{(2)}(x_1,b)\}=\frac{b}{4}\sum_{m=0}^{\infty}(\mp c''_{3m}-d''_{3m})\cos(\alpha_m x_1), \tag{31m, n}$$

$$\{u_{3,222}^{(3)}(x_1,0);u_{3,222}^{(3)}(x_1,b)\}=\frac{b}{4}\sum_{m=0}^{\infty}(\mp e''_{3m}-f''_{3m})\sin(\alpha_m x_1), \tag{31o, p}$$

$$\{u_{1,1}^{(2)}(0,x_2);u_{1,1}^{(2)}(a,x_2)\}=\frac{a}{4}\sum_{n=0}^{\infty}(\mp a'_{1n}-b'_{1n})\cos(\beta_n x_2), \tag{32a, b}$$

$$\{u_{1,1}^{(4)}(0,x_2);u_{1,1}^{(4)}(a,x_2)\}=\frac{a}{4}\sum_{n=0}^{\infty}(\mp g'_{1n}-h'_{1n})\sin(\beta_n x_2), \tag{32c, d}$$

$$\{u_1^{(1)}(x_1,0);u_1^{(1)}(x_1,b)\}=\frac{b}{4}\sum_{m=0}^{\infty}(\mp \bar{c}_{1m}-\bar{d}_{1m})\sin(\alpha_m x_1), \tag{32e, f}$$

$$\{u_1^{(4)}(x_1,0);u_1^{(4)}(x_1,b)\}=\frac{b}{4}\sum_{m=0}^{\infty}(\mp \bar{g}_{1m}-\bar{h}_{1m})\cos(\alpha_m x_1), \tag{32g, h}$$

$$\{u_2^{(1)}(0,x_2);u_2^{(1)}(a,x_2)\}=\frac{a}{4}\sum_{n=0}^{\infty}(\mp \bar{a}_{2n}-\bar{b}_{2n})\sin(\beta_n x_2), \tag{32i, j}$$

$$\{u_2^{(3)}(0,x_2);u_2^{(3)}(a,x_2)\}=\frac{a}{4}\sum_{n=0}^{\infty}(\mp \bar{e}_{2n}-\bar{f}_{2n})\cos(\beta_n x_2), \tag{32k, l}$$

$$\{u_{2,2}^{(2)}(x_1,0);u_{2,2}^{(2)}(x_1,b)\}=\frac{b}{4}\sum_{m=0}^{\infty}(\mp c'_{2m}-d'_{2m})\cos(\alpha_m x_1), \tag{32m, n}$$

$$\{u_{2,2}^{(3)}(x_1,0);u_{2,2}^{(3)}(x_1,b)\}=\frac{b}{4}\sum_{m=0}^{\infty}(\mp e'_{2m}-f'_{2m})\sin(\alpha_m x_1). \tag{32o, p}$$

7.1.2.4. *Example* 4: *clamped C*4 *at edges* $x_1=0$, $\bar{x}_1(=a)$ *and simply supported SS*2 *at edges* $x_2=0$, $\bar{x}_2\ (=b)$. For clamped boundary condition prescribed at two opposite edges, $x_1=0$, $\bar{x}_1(=a)$, the transverse complementary boundary constraints, $u_{3,11}\neq 0$ and $u_{3,111}\neq 0$, dictate that the boundary Fourier coefficients—$\bar{a}_{3n}$; $\bar{\bar{b}}_{3n}$; $\bar{e}_{3n}$; $\bar{\bar{f}}_{3n}$; a''_{3n}; b''_{3n}; g''_{3n}; h''_{3n}—be non-zero (Table 10). Additionally, for the type 4 surface-parallel boundary condition prescribed at the two opposite edges, $x_1=0$, $\bar{x}_1(=a)$, the surface-parallel complementary boundary constraints, $u_{1,1}\neq 0$ and $u_{2,1}\neq 0$, dictate that the boundary Fourier coefficients—a'_{1n}; b'_{1n}; g'_{1n}; h'_{1n}; a'_{2n}; b'_{2n}; g'_{2n}; h'_{2n}—be non-zero (Tables 12, 14). Likewise, for simply supported boundary condition prescribed at the two opposite edges, $x_2=0$, $\bar{x}_2(=b)$, the transverse complementary boundary constraints, $u_{3,2}\neq 0$ and $u_{3,222}\neq 0$, dictate that the boundary Fourier coefficients—c'_{3m}; d'_{3m}; e'_{3m};f'_{3m}; c''_{3m}; d''_{3m}; e''_{3m}; f''_{3m}—be non-zero (Table 10). Additionally, for the type 2 surface-parallel boundary condition prescribed at the two opposite edges, $x_2=0$, $\bar{x}_2(=b)$, the surface-parallel complementary boundary constraints, $u_1\neq 0$ and $u_{2,2}\neq 0$, dictate that the boundary Fourier coefficients—$\bar{c}_{1m}$; $\bar{d}_{1m}$; $\bar{g}_{1m}$; $\bar{h}_{1m}$; c'_{2m}; d'_{2m}; e'_{2m};f'_{2m}—be non-zero (Tables 12 and 14). The corresponding non-zero displacements or their normal derivatives at the edges are then given as follows:

$$\{u_{3,11}^{(1)}(0,x_2);u_{3,11}^{(1)}(a,x_2)\}=\frac{a}{4}\sum_{n=0}^{\infty}(\mp \bar{a}_{3n}-\bar{\bar{b}}_{3n})\sin(\beta_n x_2), \tag{33a, b}$$

$$\{u_{3,11}^{(3)}(0,x_2);u_{3,11}^{(3)}(a,x_2)\}=\frac{a}{4}\sum_{n=0}^{\infty}(\mp \bar{e}_{3n}-\bar{\bar{f}}_{3n})\cos(\beta_n x_2), \tag{33c, d}$$

$$\{u_{3,111}^{(2)}(0,x_2);u_{3,111}^{(2)}(a,x_2)\}=\frac{a}{4}\sum_{n=0}^{\infty}(\mp a''_{3n}-b''_{3n})\cos(\beta_n x_2), \tag{33e, f}$$

$$\{u^{(4)}_{3,111}(0, x_2); u^{(4)}_{3,111}(a, x_2)\} = \frac{a}{4}\sum_{n=0}^{\infty}(\mp g''_{3n} - h''_{3n})\sin(\beta_n x_2), \tag{33g, h}$$

$$\{u^{(2)}_{3,2}(x_1, 0); u^{(2)}_{3,2}(x_1, b)\} = \frac{b}{4}\sum_{m=0}^{\infty}(\mp c'_{3m} - d'_{3m})\cos(\alpha_m x_1), \tag{33i, j}$$

$$\{u^{(3)}_{3,2}(x_1, 0); u^{(2)}_{3,2}(x_1, b)\} = \frac{b}{4}\sum_{m=0}^{\infty}(\mp e'_{3m} - f'_{3m})\sin(\alpha_m x_1), \tag{33k, l}$$

$$\{u^{(2)}_{3,222}(x_1, 0); u^{(2)}_{i,222}(x_1, b)\} = \frac{b}{4}\sum_{m=0}^{\infty}(\mp c''_{3m} - d''_{3m})\cos(\alpha_m x_1), \tag{33m, n}$$

$$\{u^{(2)}_{3,222}(x_1, 0); u^{(3)}_{3,222}(x_1, b)\} = \frac{b}{4}\sum_{m=0}^{\infty}(\mp e''_{3m} - f''_{3m})\sin(\alpha_m x_1), \tag{33o, p}$$

$$\{u^{(2)}_{1,1}(0, x_2); u^{(2)}_{1,1}(a, x_2)\} = \frac{a}{4}\sum_{n=0}^{\infty}(\mp a'_{1n} - b'_{1n})\cos(\beta_n x_2), \tag{34a, b}$$

$$\{u^{(4)}_{1,1}(0, x_2); u^{(4)}_{1,1}(a, x_2)\} = \frac{a}{4}\sum_{n=0}^{\infty}(\mp g'_{1n} - h'_{1n})\sin(\beta_n x_2), \tag{34c, d}$$

$$\{u^{(2)}_{2,1}(0, x_2); u^{(2)}_{2,1}(a, x_2)\} = \frac{a}{4}\sum_{n=0}^{\infty}(\mp a'_{2n} - b'_{2n})\cos(\beta_n x_2), \tag{34e, f}$$

$$\{u^{(4)}_{2,1}(0, x_2); u^{(4)}_{2,1}(a, x_2)\} = \frac{a}{4}\sum_{n=0}^{\infty}(\mp g'_{2n} - h'_{2n})\sin(\beta_n x_2), \tag{34g, h}$$

$$\{u^{(1)}_{1}(x_1, 0); u^{(1)}_{1}(x_1, b)\} = \frac{b}{4}\sum_{m=0}^{\infty}(\mp \bar{c}_{1m} - \bar{d}_{1m})\sin(\alpha_m x_1), \tag{34i, j}$$

$$\{u^{(4)}_{1}(x_1, 0); u^{(4)}_{1}(x_1, b)\} = \frac{b}{4}\sum_{m=0}^{\infty}(\mp \bar{g}_{1m} - \bar{h}_{1m})\cos(\alpha_m x_1), \tag{34k, l}$$

$$\{u^{(2)}_{2,2}(x_1, 0); u^{(2)}_{2,2}(x_1, b)\} = \frac{b}{4}\sum_{m=0}^{\infty}(\mp c'_{2m} - d'_{2m})\cos(\alpha_m x_1), \tag{34m, n}$$

$$\{u^{(3)}_{2,2}(x_1, 0); u^{(3)}_{2,2}(x_1, b)\} = \frac{b}{4}\sum_{m=0}^{\infty}(\mp e'_{2m} - f'_{2m})\sin(\alpha_m x_1). \tag{34o, p}$$

7.1.2.5. *Example 5: clamped C4 at edge $x_1 = 0$, free F1 at edge $x_1 = \bar{x}_1(= a)$, roller-skate RS3 at $x_2 = 0$ and simply supported SS2 at edge $x_2 = \bar{x}_2(= b)$.* For clamped boundary condition prescribed at the edge, $x_1 = 0$, the transverse complementary boundary constraints, $u_{3,11} \neq 0$ and $u_{3,111} \neq 0$, dictate that the sum of boundary Fourier coefficients—$\bar{\bar{a}}_{3n} + \bar{\bar{b}}_{3n}$; $\bar{\bar{e}}_{3n} + \bar{\bar{f}}_{3n}$; $a''_{3n} + b''_{3n}$; and $g''_{3n} + h''_{3n}$—be non-zero. Furthermore, at the edge $x_1 = 0$, the transverse boundary constraints, $u_3 = 0$ and $u_{3,1} = 0$, dictate that the boundary Fourier coefficients—$\bar{a}_{3n}(= -\bar{b}_{3n})$; $\bar{e}_{3n}(= -\bar{f}_{3n})$; $a'_{3n}(= -b'_{3n})$ and $g'_{3n}(= -h'_{3n})$—be non-zero (Tables 11). For the free boundary condition prescribed at the edge, $x_1 = \bar{x}_1(= a)$, the transverse complementary boundary constraints, $u_3 \neq 0$ and $u_{3,1} \neq 0$, dictate that the sum of boundary Fourier coefficients—$\bar{a}_{3n} - \bar{b}_{3n}$;

$\bar{e}_{3n} - \bar{f}_{3n}$; $a'_{3n} - b'_{3n}$; and $g'_{3n} - h'_{3n}$—be non-zero. Additionally, at the edge, $x_1 = \bar{x}_1(=a)$, the transverse boundary constraints, $u_{3,11} = 0$ and $u_{3,111} = 0$, dictate that the boundary Fourier coefficients—$\bar{\bar{a}}_{3n}(=\bar{\bar{b}}_{3n})$; $\bar{\bar{e}}_{3n}(=\bar{\bar{f}}_{3n})$; $a''_{3n}(=b''_{3n})$ and $g''_{3n}(=h''_{3n})$—be non-zero (Table 11). The corresponding non-zero transverse displacements or their normal derivatives at the edges $x_1 = 0$, $\bar{x}_1(=a)$, are then given as follows:

$$u^{(1)}_{3,11}(0, x_2) = -\frac{a}{2}\sum_{n=0}^{\infty} \bar{\bar{a}}_{3n}\sin(\beta_n x_2), \quad u^{(3)}_{3,11}(0, x_2) = -\frac{a}{2}\sum_{n=0}^{\infty} \bar{\bar{e}}_{3n}\cos(\beta_n x_2), \tag{35a, b}$$

$$u^{(2)}_{3,111}(0, x_2) = -\frac{a}{2}\sum_{n=0}^{\infty} a''_{3n}\cos(\beta_n x_2), \quad u^{(4)}_{3,111}(0, x_2) = -\frac{a}{2}\sum_{n=0}^{\infty} g''_{3n}\sin(\beta_n x_2), \tag{35c, d}$$

$$u^{(1)}_{3}(a, x_2) = \frac{a}{2}\sum_{n=0}^{\infty} \bar{a}_{3n}\sin(\beta_n x_2), \quad u^{(3)}_{3}(a, x_2) = \frac{a}{2}\sum_{n=0}^{\infty} \bar{e}_{3n}\cos(\beta_n x_2), \tag{35e, f}$$

$$u^{(2)}_{3,1}(a, x_2) = \frac{a}{2}\sum_{n=0}^{\infty} a'_{3n}\cos(\beta_n x_2), \quad u^{(4)}_{3,1}(a, x_2) = \frac{a}{2}\sum_{n=0}^{\infty} g'_{3n}\sin(\beta_n x_2). \tag{35g, h}$$

For the type 4 surface-parallel boundary condition prescribed at the edge $x_1 = 0$, the surface-parallel complementary boundary constraints, $u_{1,1} \neq 0$ and $u_{2,1} \neq 0$, dictate that the sum of boundary Fourier coefficients—$a'_{1n} + b'_{1n}$; $a'_{2n} + b'_{2n}$; $g'_{1n} + h'_{1n}$; and $g'_{2n} + h'_{2n}$—be non-zero. Furthermore, at the edge $x_1 = 0$, the boundary constraints, $u_1 = u_2 = 0$, dictate that the boundary Fourier coefficients—$\bar{a}_{1n}(=-\bar{b}_{1n})$; $\bar{e}_{1n}(=-\bar{f}_{1n})$; $\bar{a}_{2n}(=-\bar{b}_{2n})$; $\bar{e}_{2n}(=-\bar{f}_{2n})$—be non-zero (Tables 13(a), 15(a)). Likewise, for the type 1 surface-parallel boundary condition prescribed at the edge $x_1 = a$, the surface-parallel complementary boundary constraints, $u_1 \neq 0$ and $u_2 \neq 0$, dictate that the algebraic sum of boundary Fourier coefficients—$\bar{a}_{1n} - \bar{b}_{1n}$; $\bar{e}_{1n} - \bar{f}_{1n}$; $\bar{a}_{2n} - \bar{b}_{2n}$ and $\bar{e}_{2n} - \bar{f}_{2n}$—be non-zero. Additionally, at the edge $x_1 = a$, the surface-parallel boundary constraints, $u_{1,1} = u_{2,1} = 0$, dictate that the boundary Fourier coefficients—$a'_{1n}(=b'_{1n})$; $a'_{2n}(=b'_{2n})$; $g'_{1n}(=h'_{1n})$ and $g'_{2n}(=h'_{2n})$—be non-zero (Tables 13(a), 15(a)). Finally, the corresponding non-zero surface-parallel displacements or their normal derivatives at the edges $x_1 = 0$, $\bar{x}_1(=a)$, are then given as follows:

$$u^{(2)}_{1,1}(0, x_2) = -\frac{a}{2}\sum_{n=0}^{\infty} a'_{1n}\cos(\beta_n x_2), \quad u^{(4)}_{1,1}(0, x_2) = -\frac{a}{2}\sum_{n=0}^{\infty} g'_{1n}\sin(\beta_n x_2), \tag{36a, b}$$

$$u^{(2)}_{2,1}(0, x_2) = -\frac{a}{2}\sum_{n=0}^{\infty} a'_{2n}\cos(\beta_n x_2), \quad u^{(4)}_{2,1}(0, x_2) = -\frac{a}{2}\sum_{n=0}^{\infty} g'_{2n}\sin(\beta_n x_2), \tag{36c, d}$$

$$u^{(1)}_{1}(a, x_2) = \frac{a}{2}\sum_{n=0}^{\infty} \bar{a}_{1n}\sin(\beta_n x_2), \quad u^{(3)}_{1}(a, x_2) = \frac{a}{2}\sum_{n=0}^{\infty} \bar{e}_{1n}\cos(\beta_n x_2), \tag{36e, f}$$

$$u^{(4)}_{2,1}(0, x_2) = -\frac{a}{2}\sum_{n=0}^{\infty} g'_{2n}\sin(\beta_n x_2), \quad u^{(1)}_{2}(a, x_2) = \frac{a}{2}\sum_{n=0}^{\infty} \bar{a}_{2n}\sin(\beta_n x_2). \tag{36g, h}$$

For the roller-skate boundary condition prescribed at the edge $x_2 = 0$, the transverse complementary boundary constraints, $u_3 \neq 0$ and $u_{3,22} \neq 0$, dictate that the sum of boundary Fourier coefficients—$\bar{c}_{3m} + \bar{d}_{3m}$; $\bar{g}_{3m} + \bar{h}_{3m}$; $\bar{\bar{c}}_{3m} + \bar{\bar{d}}_{3m}$ and $\bar{\bar{g}}_{3m} + \bar{\bar{h}}_{3m}$—be non-zero. Furthermore, at the edge $x_2 = 0$, the transverse boundary constraints,

$u_{3,2} = u_{3,222} = 0$, dictate that the boundary Fourier coefficients—$c'_{3m}(= -d'_{3m})$; $e'_{3m}(= -f'_{3m})$; $c''_{3m}(= -d''_{3m})$ and $e''_{3m}(= -f''_{3m})$—be non-zero (Table 11). For the simply supported boundary condition prescribed at the edge $x_2 = 0$, the transverse complementary boundary constraints, $u_{3,2} \neq 0$ and $u_{3,222} \neq 0$, dictate that the sum of boundary Fourier coefficients—$c'_{3m} + d'_{3m}$; $e'_{3m} + f'_{3m}$; $c''_{3m} + d''_{3m}$ and $e''_{3m} + f''_{3m}$—be non-zero. Additionally, at this edge, the transverse boundary constraints, $u_3 = u_{3,22} = 0$, dictate that the boundary Fourier coefficients—$\bar{c}_{3m}(= \bar{d}_{3m})$; $\bar{g}_{3m}(= \bar{h}_{3m})$; $\bar{\bar{c}}_{3m}(= \bar{\bar{d}}_{3m})$ and $\bar{\bar{g}}_{3m}(= \bar{\bar{h}}_{3m})$—be non-zero (Table 11). The corresponding non-zero transverse displacements or their normal derivatives at the edges $x_2 = 0, b$, are then given as follows:

$$u_3^{(1)}(x_1, 0) = -\frac{b}{2}\sum_{m=0}^{\infty} \bar{c}_{3m}\sin(\alpha_m x_1), \quad u_3^{(4)}(x_1, 0) = -\frac{b}{2}\sum_{m=0}^{\infty} \bar{g}_{3m}\cos(\alpha_m x_1), \qquad (37a, b)$$

$$u_{3,22}^{(1)}(x_1, 0) = -\frac{b}{2}\sum_{m=0}^{\infty} \bar{\bar{c}}_{3m}\sin(\alpha_m x_1), \quad u_{3,22}^{(4)}(x_1, 0) = -\frac{b}{2}\sum_{m=0}^{\infty} \bar{\bar{g}}_{3m}\cos(\alpha_m x_1), \qquad (37c, d)$$

$$u_{3,2}^{(2)}(x_1, b) = \frac{b}{2}\sum_{m=0}^{\infty} c'_{3m}\cos(\alpha_m x_1), \quad u_{3,2}^{(3)}(x_1, b) = \frac{b}{2}\sum_{m=0}^{\infty} e'_{3m}\sin(\alpha_m x_1), \qquad (37e, f)$$

$$u_{3,222}^{(2)}(x_1, b) = \frac{b}{2}\sum_{m=0}^{\infty} c''_{3m}\cos(\alpha_m x_1), \quad u_{3,222}^{(3)}(x_1, b) = \frac{b}{2}\sum_{m=0}^{\infty} e''_{3m}\sin(\alpha_m x_1). \qquad (37g, h)$$

For the type 3 surface-parallel boundary condition prescribed at the edge $x_2 = 0$, the surface-parallel complementary boundary constraints, $u_{1,2} \neq 0$ and $u_{2,2} \neq 0$, dictate that the sum of boundary Fourier coefficients—$c'_{1m} + d'_{1m}$; $e'_{1m} + f'_{1m}$; $c'_{2m} + d'_{2m}$ and $e'_{2m} + f'_{2m}$—be non-zero. Furthermore, at this edge the surface-parallel boundary constraints, $u_1 = u_2 = 0$, dictate that the boundary Fourier coefficients—$\bar{c}_{1m}(= -\bar{d}_{1m})$; $\bar{g}_{1m}(= -\bar{h}_{1m})$; $\bar{c}_{2m}(= -\bar{d}_{2m})$ and $\bar{g}_{2m}(= -\bar{h}_{2m})$—be non-zero (Tables 13(b), 15(b)). Likewise, for the type 2 surface-parallel boundary condition prescribed at the edge $x_2 = b$, the surface-parallel complementary boundary constraints, $u_1 \neq 0$; $u_2 \neq 0$; dictate that the algebraic sum of boundary Fourier coefficients—$\bar{c}_{1m} - \bar{d}_{1m}$; $\bar{g}_{1m} - \bar{h}_{1m}$; $\bar{c}_{2m} - \bar{d}_{2m}$ and $\bar{g}_{2m} - \bar{h}_{2m}$—be non-zero. Additionally, at this edge, the surface-parallel boundary constraints, $u_{1,2} = u_{2,2} = 0$, dictate that the boundary Fourier coefficients—$c'_{1m}(= d'_{1m})$; $e'_{1m}(= f'_{1m})$; $c'_{2m}(= d'_{2m})$ and $e'_{2m}(= f'_{2m})$—be non-zero (Tables 13(b), 15(b)). Finally, the corresponding non-zero surface-parallel displacements or their normal derivatives at the edges $x_2 = 0, b$, are then given as follows:

$$u_{1,2}^{(2)}(x_1, 0) = -\frac{b}{2}\sum_{m=0}^{\infty} c'_{1m}\cos(\alpha_m x_1), \quad u_{1,2}^{(3)}(x_1, 0) = -\frac{b}{2}\sum_{m=0}^{\infty} e'_{1m}\sin(\alpha_m x_1), \qquad (38a, b)$$

$$u_{2,2}^{(2)}(x_1, 0) = -\frac{b}{2}\sum_{m=0}^{\infty} c'_{2m}\cos(\alpha_m x_1), \quad u_{2,2}^{(3)}(x_1, 0) = -\frac{b}{2}\sum_{m=0}^{\infty} e'_{2m}\sin(\alpha_m x_1), \qquad (38c, d)$$

$$u_1^{(1)}(x_1, b) = \frac{b}{2}\sum_{m=0}^{\infty} \bar{c}_{1m}\sin(\alpha_m x_1), \quad u_1^{(4)}(x_1, b) = \frac{b}{2}\sum_{m=0}^{\infty} \bar{g}_{1m}\cos(\alpha_m x_1), \qquad (38e, f)$$

$$u_2^{(1)}(x_1, b) = \frac{b}{2}\sum_{m=0}^{\infty} \bar{c}_{2m}\sin(\alpha_m x_1), \quad u_2^{(4)}(x_1, b) = \frac{b}{2}\sum_{m=0}^{\infty} \bar{g}_{2m}\cos(\alpha_m x_1). \qquad (38g, h)$$

7.1.2.6. *Example* 6: *clamped C4 at edge* $x_1 = 0$, *simply supported SS2 at edge* $x_2 = \bar{x}_2 (= b)$, *roller-skate RS3 at* $x_2 = 0$ *and free F1 at edge* $x_2 = \bar{x}_2(= b)$. For the combination of clamped boundary condition prescribed at the edge, $x_1 = 0$, and the simply supported one at $x_1 = \bar{x}_1(= a)$, the transverse complementary boundary constraint, $u_{3,111} \neq 0$, is symmetrically placed, which, consequently, demands that the boundary Fourier coefficients—a''_{3n}, b''_{3n}, g''_{3n} and h''_{3n}—be non-zero. Furthermore, for clamped boundary condition prescribed at the edge, $x_1 = 0$, the transverse complementary boundary constraint, $u_{3,11} \neq 0$, dictates that the sum of boundary Fourier coefficients—$\bar{\bar{a}}_{3n} + \bar{\bar{b}}_{3n}$ and $\bar{\bar{e}}_{3n} + \bar{\bar{f}}_{3n}$—be non-zero. Additionally, at the edge, $x_1 = \bar{x}_1(= a)$, the transverse boundary constraint, $u_{3,11} = 0$, demands that the boundary Fourier coefficients—$\bar{\bar{a}}_{3n}(= \bar{\bar{b}}_{3n})$ and $\bar{\bar{e}}_{3n}(= \bar{\bar{f}}_{3n})$—be non-zero (Table 11). The corresponding non-zero transverse displacements or their normal derivatives at the edges $x_1 = 0$, $\bar{x}_1(= a)$, are then given as follows:

$$\{u^{(2)}_{3,111}(0, x_2); u^{(2)}_{3,111}(a, x_2)\} = \frac{a}{4} \sum_{n=0}^{\infty} (\mp a''_{3n} - b''_{3n}) \cos(\beta_n x_2), \tag{39a, b}$$

$$\{u^{(4)}_{3,111}(0, x_2); u^{(4)}_{3,111}(a, x_2)\} = \frac{a}{4} \sum_{n=0}^{\infty} (\mp g''_{3n} - h''_{3n}) \sin(\beta_n x_2), \tag{39c, d}$$

$$u^{(1)}_{3,11}(0, x_2) = -\frac{a}{2} \sum_{n=0}^{\infty} \bar{\bar{a}}_{3n} \sin(\beta_n x_2), \quad u^{(3)}_{3,11}(0, x_2) = -\frac{a}{2} \sum_{n=0}^{\infty} \bar{\bar{e}}_{3n} \cos(\beta_n x_2), \tag{39e, f}$$

$$u^{(2)}_{3,1}(a, x_2) = \frac{a}{2} \sum_{n=0}^{\infty} a'_{3n} \cos(\beta_n x_2), \quad u^{(4)}_{3,1}(a, x_2) = \frac{a}{2} \sum_{n=0}^{\infty} g'_{3n} \sin(\beta_n x_2). \tag{39g, h}$$

For the combination of type 4 surface-parallel boundary condition prescribed at the edge, $x_1 = 0$, and the type 2 one at $x_1 = 0$, $\bar{x}_1(= a)$, the surface-parallel complementary boundary constraints, $u_{1,1} \neq 0$; $u_{2,1} \neq 0$, dictate that the boundary Fourier coefficients—a'_{1n}, b'_{1n}, g'_{1n}, h'_{1n}, a'_{2n}, b'_{2n}, g'_{2n} and h'_{2n}—be non-zero. Finally, the corresponding non-zero surface-parallel displacements or their normal derivatives at the edges $x_1 = 0$, $\bar{x}_1(= a)$, are then given as follows:

$$\{u^{(2)}_{i,1}(0, x_2); u^{(2)}_{i,1}(a, x_2)\} = \frac{a}{4} \sum_{n=0}^{\infty} (\mp a'_{in} - b'_{in}) \cos(\beta_n x_2), \quad i = 1, 2, \tag{40a, b}$$

$$\{u^{(4)}_{i,1}(0, x_2); u^{(4)}_{i,1}(a, x_2)\} = \frac{a}{4} \sum_{n=0}^{\infty} (\mp g'_{in} - h'_{in}) \sin(\beta_n x_2), \quad i = 1, 2. \tag{40c, d}$$

For the combination of roller-skate boundary condition prescribed at the edge, $x_2 = 0$, and free boundary condition at $x_2 = b$, the transverse complementary boundary constraint, $u_3 \neq 0$, demands that the boundary Fourier coefficients—$\bar{c}_{3m}$, $\bar{d}_{3m}$, $\bar{g}_{3m}$ and $\bar{h}_{3m}$—be non-zero. Additionally, for the roller-skate boundary condition prescribed at the edge $x_2 = 0$, the transverse complementary boundary constraint, $u_{3,22} \neq 0$, dictates that the sum of boundary Fourier coefficients—$\bar{\bar{c}}_{3m} + \bar{\bar{d}}_{3m}$ and $\bar{\bar{g}}_{3m} + \bar{\bar{h}}_{3m}$—be non-zero. For the free boundary condition prescribed at the edge $x_2 = b$, the tranverse boundary constraint, $u_{3,22} = 0$, dictates that the boundary Fourier coefficients—$\bar{\bar{c}}_{3m}(= \bar{\bar{d}}_{3m})$ and $\bar{\bar{g}}_{3m}(= \bar{\bar{h}}_{3m})$—be non-zero. Additionally, at the edge $x_2 = b$, the transverse complementary boundary constraint, $u_{3,2} \neq 0$, dictates that the algebraic sum of boundary Fourier coefficients—$c'_{3m} - d'_{3m}$ and $e'_{3m} - f'_{3m}$—be non-zero. Furthermore, at the edge $x_2 = 0$, the transverse boundary constraint, $u_{3,2} = 0$, dictates that the boundary Fourier coefficients—$c'_{3m} = -d'_{3m}$; $e'_{3m} = -f'_{3m}$—be non-zero (Table 11). The corresponding non-zero transverse displacements or their normal derivatives at the edges $x_2 = 0, b$, are then given as follows:

$$\{u_3^{(1)}(x_1, 0); u_3^{(1)}(x_1, b)\} = \frac{b}{4}\sum_{m=0}^{\infty} (\mp \bar{c}_{3m} - \bar{d}_{3m}) \sin(\alpha_m x_1), \tag{41a, b}$$

$$\{u_3^{(4)}(x_1, 0); u_3^{(4)}(x_1, b)\} = \frac{b}{4}\sum_{m=0}^{\infty} (\mp \bar{g}_{3m} - \bar{h}_{3m}) \cos(\alpha_m x_1), \tag{41c, d}$$

$$u_{3,22}^{(1)}(x_1, 0) = -\frac{b}{2}\sum_{m=0}^{\infty} \bar{\bar{c}}_{3m} \sin(\alpha_m x_1), \quad u_{3,22}^{(4)}(x_1, 0) = -\frac{b}{2}\sum_{m=0}^{\infty} \bar{\bar{g}}_{3m} \cos(\alpha_m x_1), \tag{41e, f}$$

$$u_{3,2}^{(2)}(x_1, b) = \frac{b}{2}\sum_{m=0}^{\infty} c'_{3m} \cos(\alpha_m x_1), \quad u_{3,2}^{(3)}(x_1, b) = \frac{b}{2}\sum_{m=0}^{\infty} e'_{3m} \sin(\alpha_m x_1). \tag{41g, h}$$

For the type 3 surface-parallel boundary condition prescribed at the edge, $x_2 = 0$, the in-plane complementary boundary constraints, $u_{1,2} \neq 0$ and $u_{2,2} \neq 0$, dictate that the sum of boundary Fourier coefficients—$c'_{1m} + d'_{1m}$; $e'_{1m} + f'_{1m}$; $c'_{2m} + d'_{2m}$ and $e'_{2m} + f'_{2m}$—be non-zero. Furthermore, at this edge the surface-parallel boundary constraints, $u_1 = u_2 = 0$, dictate that the boundary Fourier coefficients—$\bar{c}_{1m}(= -\bar{d}_{1m})$; $\bar{g}_{1m}(= -\bar{h}_{1m})$; $\bar{c}_{2m}(= -\bar{d}_{2m})$ and $\bar{g}_{2m}(= -\bar{h}_{2m})$—be non-zero (Tables 13(b), 15(b)). Likewise, for the type 1 surface-parallel boundary condition prescribed at the edge $x_2 = b$, the surface-parallel complementary boundary constraints, $u_1 \neq 0$, and $u_2 \neq 0$, dictate that the algebraic sum of boundary Fourier coefficients—$\bar{c}_{1m} - \bar{d}_{1m}$; $\bar{g}_{1m} - \bar{h}_{1m}$; $\bar{c}_{2m} - \bar{d}_{2m}$ and $\bar{g}_{2m} - \bar{h}_{2m}$—be non-zero. Additionally, at this edge the surface-parallel boundary constraints, $u_{1,2} = u_{2,2} = 0$, dictate that the boundary Fourier coefficients—$c'_{1m}(= d'_{1m})$; $e'_{1m}(= f'_{1m})$; $c'_{2m}(= d'_{2m})$ and $e'_{2m}(= f'_{2m})$—be non-zero (Tables 13(b), 15(b)). Finally, the corresponding non-zero surface-parallel displacements or their normal derivatives at the edges $x_2 = 0, b$, are then given as follows:

$$u_{1,2}^{(2)}(x_1, 0) = -\frac{b}{2}\sum_{m=0}^{\infty} c'_{1m} \cos(\alpha_m x_1), \quad u_{1,2}^{(3)}(x_1, 0) = -\frac{b}{2}\sum_{m=0}^{\infty} e'_{1m} \sin(\alpha_m x_1), \tag{42a, b}$$

$$u_{2,2}^{(2)}(x_1, 0) = -\frac{b}{2}\sum_{m=0}^{\infty} c'_{2m} \cos(\alpha_m x_1), \quad u_{2,2}^{(3)}(x_1, 0) = -\frac{b}{2}\sum_{m=0}^{\infty} e'_{2m} \sin(\alpha_m x_1), \tag{42c, d}$$

$$u_1^{(1)}(x_1, b) = \frac{b}{2}\sum_{m=0}^{\infty} \bar{c}_{1m} \sin(\alpha_m x_1), \quad u_1^{(4)}(x_1, b) = \frac{b}{2}\sum_{m=0}^{\infty} \bar{g}_{1m} \cos(\alpha_m x_1), \tag{42e, f}$$

$$u_2^{(1)}(x_1, b) = \frac{b}{2}\sum_{m=0}^{\infty} \bar{c}_{2m} \sin(\alpha_m x_1), \quad u_2^{(4)}(x_1, b) = \frac{b}{2}\sum_{m=0}^{\infty} \bar{g}_{2m} \cos(\alpha_m x_1). \tag{42g, h}$$

7.2. AN ARBITRARILY LAMINATED THIN ANISOTROPIC DOUBLY CURVED GENERAL PANEL SUBJECTED TO UNIFORM TRANSVERSE PERIODIC LOADING

This is a special case of the example, considered in the preceding section. The transverse periodic loading in equations (16) is given by

$$q_1 = q_2 = 0, \quad q_3(x_1, x_2) = p_0, \tag{43}$$

where p_0 is given by equations (25a) and (26a).

Following an identical procedure as above, an examination of equations (16) reveals that $U^{(1)}_{3mn}$, $U^{(2)}_{3mn}$, $U^{(3)}_{1mn}$, $U^{(4)}_{1mn}$, $U^{(3)}_{2mn}$, $U^{(4)}_{2mn}$ are coupled through the presence of $Q^{(1)}_{3mn}$ and the corresponding particular solution functions will be necessary and sufficient to furnish the appropriate particular solution to the system of three PDEs given by equations (16). The remaining Fourier coefficients will drop out.

With regard to the complementary solution, the boundary Fourier coefficients are limited to $\bar{a}_{3n}$, $\bar{b}_{3n}$; a'_{3n}, b'_{3n}; $\bar{c}_{3m}$, $\bar{d}_{3m}$; c'_{3m}, d'_{3m}; $\bar{\bar{a}}_{3n}$, $\bar{\bar{b}}_{3n}$; a''_{3n}, b''_{3n}; $\bar{\bar{c}}_{3m}$, $\bar{\bar{d}}_{3m}$; c''_{3m}, d''_{3m}; g'_{1n}, h'_{1n}; g'_{2n}, h'_{2n}; e'_{1m}, f'_{1m}; e'_{2m}, f'_{2m}; $\bar{e}_{1n}$, $\bar{f}_{1n}$; $\bar{e}_{2n}$, $\bar{f}_{2n}$; $\bar{g}_{1m}$, $\bar{h}_{1m}$; $\bar{g}_{2m}$ and $\bar{h}_{2m}$ (see Tables 10–15). For the six examples of boundary-value problems considered in section 7.1, only non-vanishing boundary transverse and surface-parallel displacements and their normal derivatives in equations (27)–(42) are those that correspond to these boundary Fourier coefficients. The remaining boundary Fourier coefficients will vanish, and so will the corresponding transverse and in-plane boundary displacements, and their normal derivatives in equations (27)–(42).

Various special cases of lamination will be considered below.

7.2.1. *Antisymmetric angle-ply doubly curved panel*

For this type of lamination,

$$A_{16} = A_{26} = B_{11} = B_{12} = B_{22} = B_{66} = D_{16} = D_{26} = 0. \tag{44}$$

Substitution of equation (44) into equations (16), followed by an examination of the reduced equation reveals that $U^{(1)}_{3mn}$ is coupled to $U^{(3)}_{1mn}$, $U^{(4)}_{1mn}$, $U^{(3)}_{2mn}$ and $U^{(4)}_{2mn}$ through the presence of $Q^{(1)}_{3mn}$. In addition, $U^{(3)}_{1mn}$, $U^{(4)}_{1mn}$, $U^{(3)}_{2mn}$ and $U^{(4)}_{2mn}$ are coupled to $U^{(2)}_{3mn}$, even though $Q^{(2)}_{3mn} = 0$. The corresponding solution functions will be necessary and sufficient to furnish the appropriate particular solution to the system of three PDEs given by equations (16). The remaining Fourier coefficients will drop out. The complementary solutions for various combinations of prescribed boundary conditions are the same as those for the arbitrarily laminated case discussed above.

7.2.2. *Symmetric angle-ply doubly curved panel*

For this lamination,

$$A_{45} = B_{ij} = 0, \; i, j = 1, 2, 6. \tag{45}$$

Substitution of equation (45) into equations (16), followed by an examination of the reduced equation will reveal that $U^{(1)}_{3mn}$ is coupled to $U^{(3)}_{1mn}$, $U^{(4)}_{1mn}$, $U^{(3)}_{2mn}$ and $U^{(4)}_{2mn}$ through the presence of $Q^{(1)}_{3mn}$. In addition, $U^{(3)}_{1mn}$, $U^{(4)}_{1mn}$, $U^{(3)}_{2mn}$ and $U^{(4)}_{2mn}$ are coupled to $U^{(2)}_{3mn}$, even though $Q^{(2)}_{3mn} = 0$. The corresponding solution functions will be necessary and sufficient to furnish the appropriate particular solution to the system of three PDEs given by equations (16). The remaining Fourier coefficients will drop out. The complementary solutions for various combinations of prescribed boundary conditions are the same as those for the arbitrarily laminated case discussed above.

7.2.3. *General (unsymmetric) cross-ply doubly curved panel*

For this lamination,

$$A_{16} = A_{26} = A_{45} = B_{16} = B_{26} = D_{16} = D_{26} = 0. \tag{46}$$

Substitution of equation (43) into equations (16), followed by an examination of the reduced equation reveals that $U^{(1)}_{3mn}$, $U^{(4)}_{1mn}$, $U^{(4)}_{1mn}$ and $U^{(3)}_{2mn}$ are coupled through the presence of $Q^{(1)}_{3mn}$. The corresponding solution functions will be necessary and sufficient to furnish the appropriate particular solution to the system of three PDEs given by equations (16). The remaining Fourier coefficients will drop out.

With regard to the complementary solution, the boundary Fourier coefficients are limited to $\bar{a}_{3n}, \bar{b}_{3n}$; $\bar{c}_{3m}, \bar{d}_{3m}$; $\bar{\bar{a}}_{3n}, \bar{\bar{b}}_{3n}$; $\bar{\bar{c}}_{3m}, \bar{\bar{d}}_{3m}$; g'_{1n}, h'_{1n}; $\bar{e}_{2n}, \bar{f}_{2n}$; $\bar{g}_{1m}, \bar{h}_{1m}$; e'_{2m}, and f'_{2m} (see Tables 10–15). For the six examples of boundary-value problems considered in section 7.1, only one-vanishing boundary transverse and surface-parallel displacements and their normal derivatives in equations (27)–(42) are those that correspond to these boundary Fourier coefficients. The remaining boundary Fourier coefficients will vanish, and so will the corresponding transverse and in-plane boundary displacements, and their normal derivatives in equations (27)–(42).

A solution to this special case, including numerical results, has been presented by Chaudhuri and Kabir [18] for the SS2-type simply supported boundary conditions prescribed at all four edges. Furthermore, solutions to the special cases of isotropic doubly curved and cylindrical panels for the same boundary condition have also been presented by Chaudhuri and Kabir [19], and Kabir and Chaudhuri [20] respectively.

In addition, it is noteworthy that the assumed double Fourier series solutions, corresponding to the above Fourier coefficients will satisfy the SS3-type simply supported boundary conditions, prescribed at all the four edges and given by equation (19c). Finally, the present solution, for an isotropic cylindrical panel with the SS3-type simply supported boundary conditions prescribed at all the four edges, reduces to the corresponding Navier solution given by Timoshenko and Woinowsky–Krieger [3].

7.3. AN ARBITRARILY LAMINATED THIN ANISOTROPIC RECTANGULAR PLATE SUBJECTED TO UNIFORM TRANSVERSE PERIODIC LOADING

An arbitrarily laminated anisotropic plate can be treated as a special case of the corresponding doubly curved panel by substituting

$$1/R_1 = 1/R_2 = c = 0. \tag{47}$$

The uniformly distributed transverse load is given by equations (43), (25a) and (26a). Substitution of equation (47) into equations (16), followed by an examination of the reduced equation reveals that $U^{(1)}_{3mn}$ and $U^{(2)}_{3mn}$ are coupled through the presence of $Q^{(1)}_{3mn}$. Additionally, $U^{(3)}_{1mn}$, $U^{(4)}_{1mn}$, $U^{(3)}_{2mn}$ and $U^{(4)}_{2mn}$ are coupled. The corresponding solution functions will be necessary and sufficient to furnish the appropriate particular solution to the system of three PDEs given by equations (16). The remaining Fourier coefficients will drop out. The non-vanishing complementary solutions for various combinations of prescribed boundary conditions are the same as their counterparts for arbitrarily laminated doubly curved panels discussed in section 7.2.

The following special cases of lamination will be considered.

7.3.1. *Antisymmetric angle-ply rectangular plate*

Substitution of equations (44) and (47) into equations (16), followed by an examination of the reduced equation reveals that $U^{(1)}_{3mn}$ is needed because of the presence of $Q^{(1)}_{3mn}$. Additionally, $U^{(1)}_{3mn}$ is coupled to $U^{(3)}_{1mn}$ and $U^{(4)}_{2mn}$. The corresponding solution functions will be necessary and sufficient to furnish the appropriate particular solution to the system of three PDEs given by equations (16). The remaining Fourier coefficients will drop out.

The non-vanishing complementary solutions for various combinations of prescribed boundary conditions are limited to $\bar{a}_{3n}, \bar{b}_{3n}$; $\bar{c}_{3m}, \bar{d}_{3m}$; $\bar{\bar{a}}_{3n}, \bar{\bar{b}}_{3n}$; $\bar{\bar{c}}_{3m}, \bar{\bar{d}}_{3m}$; $\bar{e}_{1n}, \bar{f}_{1n}$; e'_{1m}, f'_{1m}; g'_{2n}, h'_{2n}; $\bar{g}_{2m}$ and $\bar{h}_{2m}$ (see Tables 10–15). For the six examples of boundary-value problems considered in section 7.1, only non-vanishing boundary transverse and surface-parallel displacements and their normal derivatives in equations (27)–(42) are those that correspond to these boundary Fourier coefficients. The remaining boundary Fourier coefficients will vanish, and so will the corresponding transverse and in-plane boundary displacements, and their normal derivatives in equations (27)–(42).

Whitney and Leissa [21] have shown that the assumed double Fourier series particular solution functions, corresponding to the aforementioned Fourier coefficients, will satisfy the SS2 (S3 according to the nomenclature used by Jones [2]) type boundary conditions, given by equation (19b).

7.3.2. *Symmetric angle-ply rectangular plate*

Substitution of equations (45) and (47) into equations (16), followed by an examination of the reduced equation reveals that $U^{(1)}_{3mn}$ and $U^{(2)}_{3mn}$ are needed because of the presence of $Q^{(1)}_{3mn}$. The corresponding solution functions will be necessary and sufficient to furnish the appropriate particular solution to the PDE given by equation (16c). The remaining Fourier coefficients will drop out. It may be noted that unlike their counterparts for a shell, the in-plane displacement components vanish at the reference (middle) surface of a symmetrically laminated plate, and hence play no role in either the governing PDE or the boundary conditions.

The non-vanishing complementary solutions (transverse displacements or deflections only) for various combinations of prescribed boundary conditions are similar to their counterparts for symmetric angle-ply doubly curved panels discussed in section 7.2.2. Solution to this special case, including numerical results, has been presented by Whitney [15] for the clamped boundary conditions prescribed at all four edges.

7.3.3. *General (unsymmetric) cross-ply rectangular plate*

Substitution of equations (46) and (47) into equations (16), followed by an examination of the reduced equation reveals that $U^{(1)}_{3mn}$, $U^{(4)}_{1mn}$ and $U^{(3)}_{2mn}$ are coupled through the presence of $Q^{(1)}_{3mn}$. The corresponding solution functions will be necessary and sufficient to furnish the appropriate particular solution to the system of three PDEs given by equations (16). The remaining Fourier coefficients will drop out. The complementary solutions for various combinations of prescribed boundary conditions are same as their doubly curved panel counterparts discussed in section 7.2.3. In addition, it is noteworthy that the assumed double Fourier series solutions, corresponding to the above Fourier coefficients will satisfy the SS3-type simply supported boundary conditions, prescribed at all the four edges and given by equation (19c).

In the case of a symmetrically laminated cross-ply plate, the in-plane displacement components, as expected, vanish at the middle surface. Only $U^{(1)}_{3mn}$ is needed because of the presence of $Q^{(1)}_{3mn}$. The present solution for a homogeneous isotropic clamped plate reduces to its counterpart given by Green [12]. Finally, the present solution for an isotropic simply supported plate reduces to the corresponding Navier solution (see reference [3]).

8. FREE UNDAMPED VIBRATION OF THIN LAMINATED SHELLS/PLATES

Substitution of $q_i = 0$ into equations (16) renders the problem to that of eigen-BVP. Here the primary goal is to determine the eigenvalues (natural frequencies) and the corresponding eigenfunctions (mode shapes). In some sense, this class of solutions is more fundamental than its forced vibration counterparts, because once the eigenfunctions are known, they can serve as the bases for the expansion of unknown solution functions for the forced vibration problem discussed above.

8.1. FREE UNDAMPED VIBRATION OF THIN ARBITRARILY LAMINATED DOUBLY CURVED PANELS

Case 1. Examination of equations (16) in conjunction with $q_i = 0$ reveals that $U^{(1)}_{3mn}$, $U^{(2)}_{3mn}$, $U^{(3)}_{1mn}$, $U^{(4)}_{1mn}$, $U^{(3)}_{2mn}$, $U^{(4)}_{2mn}$ are coupled, and the corresponding particular solution functions will be necessary and sufficient to furnish an appropriate particular solution to the system of three PDEs given by equations (16). The remaining Fourier coefficients will drop out. This case is similar to section 7.2 already discussed. The same holds for the complementary solution.

Case 2. Examination of equations (16) in conjunction with $q_i = 0$ reveals that an alternate set of eigenfunctions are also possible. For an arbitrarily laminated doubly curved panel, $U^{(3)}_{3mn}$, $U^{(4)}_{3mn}$, $U^{(1)}_{1mn}$, $U^{(2)}_{1mn}$, $U^{(1)}_{2mn}$ and $U^{(2)}_{2mn}$ are coupled, and the corresponding particular solution functions will be necessary and sufficient to furnish an alternative particular solution to the system of three PDEs given by equations (16). The remaining Fourier coefficients will drop out.

With regard to the complementary solution, the boundary Fourier coefficients $\bar{a}_{3n},\bar{b}_{3n}$; a'_{3n},b'_{3n}; $\bar{c}_{3m},\bar{d}_{3m}$; c'_{3m},d'_{3m}; $\bar{\bar{a}}_{3n},\bar{\bar{b}}_{3n}$; a''_{3n},b''_{3n}; $\bar{\bar{c}}_{3m},\bar{\bar{d}}_{3m}$; c''_{3m},d''_{3m}; g'_{1n},h'_{1n}; g'_{2n},h'_{2n}; e'_{1m},f'_{1m}; e'_{2m},f'_{2m}; $\bar{e}_{1n},\bar{f}_{1n}$; $\bar{e}_{2n},\bar{f}_{2n}$; $\bar{g}_{1m},\bar{h}_{1m}$; $\bar{g}_{2m}$ and $\bar{h}_{2m}$ that correspond to vanishing particular solutions will automatically be zero (see Tables 10–15). The remaining boundary Fourier coefficients and the corresponding transverse and surface-parallel boundary displacements and their normal derivatives will be non-zero depending on the prescribed boundary conditions. These are as given by their counterparts discussed in section 7.1.

The following special cases of lamination are considered.

8.1.1. *Antisymmetric angle-ply doubly curved panel*

Examination of equations (16) in conjunction with $q_i = 0$ and equation (44) reveals that two cases are possible, which are similar to their arbitrary lamination counterparts discussed above (see also section 7.2 for Case 1).

8.1.2. *Symmetric angle-ply doubly curved panel*

Examination of equations (16) in conjunction with $q_i = 0$ and equation (45) reveals that two cases are possible, which are similar to their arbitrary lamination and antisymmetric angle-ply counterparts discussed above (see also section 7.2 for Case 1).

8.1.3. *General (unsymmetric) cross-ply doubly curved panel*

Examination of equations (16) in conjunction with $q_i = 0$ and equation (46) reveals that four cases are possible. Case 1 is similar to its forced vibration counterpart pertaining to uniform transverse periodic loading discussed in section 7.2 and has been utilized (see reference [18] for solution and numerical results for the SS2-type boundary conditions prescribed at all four edges). Case 2 involves coupling of $U^{(2)}_{3mn}$ with $U^{(3)}_{1mn}$ and $U^{(4)}_{2mn}$. Case 3 is characterized by coupling of $U^{(3)}_{3mn}$with $U^{(2)}_{1mn}$ and $U^{(1)}_{2mn}$, while Case 4 involves coupling of $U^{(4)}_{3mn}$ with $U^{(1)}_{1mn}$ and $U^{(2)}_{2mn}$.

With regard to the complementary solution, the boundary Fourier coefficients, in the four cases, are limited to the following (see Tables 10–15):

Case 1. $\bar{a}_{3n},\bar{b}_{3n}$; $\bar{c}_{3m},\bar{d}_{3m}$; $\bar{\bar{a}}_{3n},\bar{\bar{b}}_{3n}$; $\bar{\bar{c}}_{3m},\bar{\bar{d}}_{3m}$; g'_{1n},h'_{1n}; $\bar{e}_{2n},\bar{f}_{2n}$; $\bar{g}_{1m},\bar{h}_{1m}$; e'_{2m} and f'_{2m}.
Case 2. a'_{3n},b'_{3n}; c'_{3m},d'_{3m}; a''_{3n},b''_{3n}; c''_{3m},d''_{3m}; g'_{2n},h'_{2n}; e'_{1m},f'_{1m}; $\bar{e}_{1n},\bar{f}_{1n}$; $\bar{g}_{2m}$ and $\bar{h}_{2m}$.
Case 3. $\bar{e}_{3n},\bar{f}_{3n}$; $\bar{g}_{3m},\bar{h}_{3m}$; $\bar{\bar{e}}_{3n},\bar{\bar{f}}_{3n}$; $\bar{\bar{g}}_{3m},\bar{\bar{h}}_{3m}$; c'_{1n},d'_{1n}; $\bar{a}_{2n},\bar{b}_{2n}$; $\bar{c}_{1m},\bar{d}_{1m}$; a'_{2m} and b'_{2m}.
Case 4. e'_{3n},f'_{3n}; g'_{3m},h'_{3m}; e''_{3n},f''_{3n}; g''_{3m},h''_{3m}; c'_{2n},d'_{2n}; a'_{1m},b'_{1m}; $\bar{a}_{1n},\bar{b}_{1n}$; $\bar{c}_{2m}$ and $\bar{d}_{2m}$.

For the six examples of boundary-value problems considered in section 7.1, only non-vanishing boundary transverse and surface-parallel displacements and their normal derivatives in equations (27)–(42) are those that correspond to these boundary Fourier coefficients. The remaining boundary Fourier coefficients will vanish, and so will the corresponding transverse and in-plane boundary displacements, and their normal derivatives in equations (27)–(42).

8.2. FREE UNDAMPED VIBRATION OF THIN ARBITRARILY LAMINATED RECTANGULAR PLATES

Examination of equations (16) in conjunction with $q_i = 0$ and equation (47) reveals that two cases are possible. Case 1 is similar to its forced vibration counterpart pertaining to uniform transverse periodic loading discussed in section 7.3. Both the cases are similar to their doubly curved shell counterparts discussed in section 8.1.

The following special cases of lamination are considered.

8.2.1. *Antisymmetric angle-ply rectangular plate*

Substitution of $q_i = 0$ in conjunction with equations (44) and (47) into equations (16), followed by an examination of the reduced equation reveals that four cases are possible. Case 1 is similar to its forced vibration counterpart pertaining to uniform transverse periodic loading discussed in section 7.3 (Here $U^{(1)}_{3mn}$ is coupled to $U^{(3)}_{1mn}$ and $U^{(4)}_{2mn}$). Case 2 involves coupling of $U^{(2)}_{3mn}$ with $U^{(4)}_{1mn}$ and $U^{(3)}_{2mn}$. Case 3 is characterized by coupling of $U^{(3)}_{3mn}$ with $U^{(1)}_{1mn}$ and $U^{(2)}_{2mn}$, while Case 4 involves coupling of $U^{(4)}_{3mn}$ with $U^{(2)}_{1mn}$ and $U^{(1)}_{2mn}$.

With regard to the complementary solution, the boundary Fourier coefficients, in the four cases, are limited to the following (see Tables 10–15):

Case 1. $\bar{a}_{3n},\bar{b}_{3n}$; $\bar{c}_{3m},\bar{d}_{3m}$; $\bar{\bar{a}}_{3n},\bar{\bar{b}}_{3n}$; $\bar{\bar{c}}_{3m},\bar{\bar{d}}_{3m}$; g'_{2n},h'_{2n}; e'_{1m},f'_{1m}; $\bar{e}_{1n},\bar{f}_{1n}$; $\bar{g}_{2m}$ and $\bar{h}_{2m}$.
Case 2. a'_{3n},b'_{3n}; c'_{3m},d'_{3m}; a''_{3n},b''_{3n}; c''_{3m},d''_{3m}; g'_{1n},h'_{1n}; $\bar{e}_{2n},\bar{f}_{2n}$; $\bar{g}_{1m},\bar{h}_{1m}$; e'_{2m} and f'_{2m}.
Case 3. $\bar{e}_{3n},\bar{f}_{3n}$; $\bar{g}_{3m},\bar{h}_{3m}$; $\bar{\bar{e}}_{3n},\bar{\bar{f}}_{3n}$; $\bar{\bar{g}}_{3m},\bar{\bar{h}}_{3m}$; c'_{2n},d'_{2n}; a'_{1m},b'_{1m}; $\bar{a}_{1n},\bar{b}_{1n}$; $\bar{c}_{2m}$ and $\bar{d}_{2m}$.
Case 4. e'_{3n},f'_{3n}; g'_{3m},h'_{3m}; e''_{3n},f''_{3n}; g''_{3m},h''_{3m}; c'_{1n},d'_{1n}; $\bar{a}_{2n},\bar{b}_{2n}$; $\bar{c}_{1m},\bar{d}_{1m}$; a'_{2m} and b'_{2m}.

For the six examples of boundary-value problems considered in section 7.1, only non-vanishing boundary transverse and surface-parallel displacements and their normal derivatives in equations (27)–(42) are those that correspond to these boundary Fourier coefficients. The remaining boundary Fourier coefficients will vanish, and so will the corresponding transverse and in-plane boundary displacements, and their normal derivatives in equations (27)–(42).

8.2.2. *Symmetric angle-ply rectangular plate*

Substitution of equations (45) and (47) and $q_i = 0$ into equations (16), followed by an examination of the reduced equation reveals that two cases of transverse vibration are possible (it may be noted that transverse and in-plane motions are uncoupled here). Case 1 is similar to its forced vibration counterpart pertaining to uniform transverse periodic loading discussed in section 7.3. Case 2 involves coupling of $U^{(3)}_{3mn}$ and $U^{(4)}_{3mn}$.

With regard to the complementary solution, the boundary Fourier coefficients, in the two cases (transverse vibration only), are limited to the following (see Tables 10–15):

Case 1. $\bar{a}_{3n}, \bar{b}_{3n}; \bar{c}_{3m}, \bar{d}_{3m}; \bar{\bar{a}}_{3n}, \bar{\bar{b}}_{3n}; \bar{\bar{c}}_{3m}, \bar{\bar{d}}_{3m}; a'_{3n}, b'_{3n}; c'_{3m}, d'_{3m}; a''_{3n}, b''_{3n}; c''_{3m}$ and d''_{3m}.
Case 2. $\bar{e}_{3n}, \bar{f}_{3n}; \bar{g}_{3m}, \bar{h}_{3m}; \bar{\bar{e}}_{3n}, \bar{\bar{f}}_{3n}; \bar{\bar{g}}_{3m}, \bar{\bar{h}}_{3m}; e'_{3n}, f'_{3n}; g'_{3m}, h'_{3m}; e''_{3n}, f''_{3n}; g''_{3m}$ and h''_{3m}.

For the six examples of boundary-value problems considered in section 7.1, only non-vanishing boundary transverse displacements and their normal derivatives in equations (27)–(42) are those that correspond to these boundary Fourier coefficients. The remaining boundary Fourier coefficients will vanish, and so will the corresponding transverse and in-plane boundary displacements, and their normal derivatives in equations (27)–(42).

8.2.3. *General (unsymmetric) cross-ply rectangular plate*

Substitution of equations (46) and (47) and $q_i = 0$ into equations (16), followed by an examination of the reduced equation reveals that four cases are possible similar to its doubly curved counterpart discussed above (Case 1 is similar to its forced vibration counterpart pertaining to uniform transverse periodic loading discussed in section 7.3).

In the case of a symmetrically laminated cross-ply plate, the in-plane displacement components, as expected, vanish at the middle surface. Only $U^{(1)}_{3mn}$ is non-zero, which are same as their doubly curved panel counterparts.

The present solution for a symmetrically laminated cross-ply plate directly reduces to that for a homogeneous orthotropic or isotropic plate without any difficulty.

9. STATIC DEFLECTION OF THIN LAMINATED SHELLS/PLATES

Substitution of $C_i = 0$ into equations (16) reduces the forced vibration BVPs investigated in sections 7.1–7.3 to their counterparts. Clearly, the same solutions as are obtained for various laminations, geometries and boundary conditions considered in sections 7.1–7.3 are valid here ($e^{i\omega\tau} = 1$), and will not be repeated in the interest of brevity of presentation.

10. SUMMARY AND CONCLUSIONS

A heretofore unavailable double Fourier series based approach, for obtaining non-separable solution to a system of completely coupled linear rth order partial differential equations with constant coefficients and subjected to general (completely coupled) boundary conditions, has been presented. The method has been successfully implemented to solve a class of hitherto unsolved boundary-value problems, pertaining to free and forced vibrations of arbitrarily laminated anisotropic doubly curved thin panels of rectangular planform, with arbitrarily prescribed (both symmetric and asymmetric with respect to the panel centerlines) admissible boundary conditions and subjected to general transverse loading.

Existing solutions such as those due to Navier or Levy are based on the well-known method of separation of variables. Such solutions represent particular solutions whenever the method of separation of variables works, and when these particular solution functions fortuitously satisfy the boundary conditions. The method of separation of variables for obtaining particular solutions does not work even for a symmetric angle-ply plate because of the presence of bending–twisting coupling rigidities, let alone arbitrarily laminated plates and shells with the exception of cross-ply curved panels. This is because the variables are, in general, not separable, and more important, boundary conditions are not satisfied *a priori*. The present investigation bridges this long-standing analytical gap.

For the derivation of the complementary solution, the complementary boundary constraints, which are inequalities, play as important a role as the (prescribed) admissible boundary conditions, which are equalities. The complementary boundary constraints enter into the picture through boundary discontinuities of some of the particular solution functions and their partial derivatives. Such discontinuities form sets of measure zero. The admissible boundary constraints, which are equalities, are conjugates of the associated complementary boundary constraints, which are inequalities. They are selected at an edge in a direction normal to that edge in order to guarantee the self-adjointness of the corresponding one-dimensional differential system.

In the most general case, the particular solutions satisfy $N(4mn + 2m + 2n + 1)$ equations for arbitrary m, n in terms of as many unknown Fourier coefficients. In order for this method to furnish a complete solution to the self-adjoint differential system given by equations (1, 4), $2rN(m + n + 1)$ additional unknown (boundary Fourier) coefficients must be furnished by the complementary boundary constraints. For a system of fourth order completely coupled PDEs, this number reduces to $8N(m + n + 1)$ additional unknown coefficients.

In the case of a boundary-value problem involving a system of completely coupled rth order PDEs, r mutually independent cases of complementary boundary constraints are possible. These r cases, in turn, produce $2^{r/2}$ mutually independent combinations of complementary boundary constraints, one of which must be introduced in order for the total number of unknowns to become equal to the total number of equations.

Special cases of these complementary boundary constraints being assigned at only one of the two opposite ends are easily handled in the present approach. These "special" cases permit us to prescribe arbitrary boundary conditions at each of the four edges independent of one another, and thus constitute the general procedure for solving the most general form of boundary-value problems. This is in contrast to the more "general" case, where assignment of same complementary boundary constraints on two opposite ends proves to be restrictive.

Such specific cases of lamination as antisymmetric angle-ply, symmetric angle-ply and general cross-ply, such particular case of loading as uniformly distributed transverse periodic loading and free vibration, and such specific case of geometry as a rectangular plate, can be obtained as special cases of the above. Six sets of boundary conditions are used to illustrate the present method for derivation of complementary solutions. In addition, this method is shown to reproduce the available boundary-continuous solutions for antisymmetric cross-ply plates and doubly curved shells with SS3-type simply supported boundary conditions and antisymmetric angle-ply plates with SS2 type simply supported boundary conditions.

Overall, this investigation provides complete Fourier solutions to laminated plate/shell boundary-value problems in the frequency domain that have never been attempted by earlier investigators. Although the method is illustrated here using a set of example problems pertaining to thin arbitrarily laminated anisotropic doubly curved and flat panels, it is equally applicable to their thick shell counterparts, such as those based on higher order shear deformation theory (HSDT). This generalized double Fourier series approach has, as a first step, been applied by Kabir and Chaudhuri [20], and Chaudhuri and Kabir [19, 18] to the analysis of thin cylindrical and doubly curved isotropic and cross-ply panels, and also to thick cross-ply doubly curved panels [22], subjected to symmetrical (with respect to panel central lines) boundary conditions. Numerical results for thin and thick laminated anisotropic doubly curved and flat panels, computed using the present solutions, are currently under way at the University of Utah, and will be reported in a forthcoming paper.

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APPENDIX A: ILLUSTRATION OF PROCEDURE

The following will illustrate the procedure of (partial) differentiation of the assumed double Fourier series, given by equations (6, 7), in the presence of "ordinary" discontinuities (resulting from the above hypothesis) for the general (or mixed) types of prescribed boundary conditions, which will be assumed identical at two opposite edges. The generalization of arbitrary (i.e., unsymmetric with respect to the centerlines of the panels) mix of boundary conditions will also be investigated in this study.

$$u_i^{(1)}(x_1, x_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{imn}^{(1)} \sin(\alpha_m x_1) \sin(\beta_n x_2), \quad 0 < x_1 < a, \quad 0 < x_2 < b, \tag{A1a}$$

$$u_{i,1}^{(1)}(x_1, x_2) = \frac{1}{2} \sum_{n=1}^{\infty} \bar{a}_{in} \sin(\beta_n x_2) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\alpha_m U_{imn}^{(1)} + \bar{a}_{in}\gamma_m + \bar{b}_{in}\delta_m) \cos(\alpha_m x_1) \sin(\beta_n x_2),$$
$$0 \leqslant x_1 \leqslant a, \quad 0 < x_2 < b, \tag{A1b}$$

$$u_{i,2}^{(1)}(x_1, x_2) = \frac{1}{2} \sum_{m=1}^{\infty} \bar{c}_{im} \sin(\alpha_m x_1) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\beta_n U_{imn}^{(1)} + \bar{c}_{im}\gamma_n + \bar{d}_{im}\delta_n) \sin(\alpha_m x_1) \cos(\beta_n x_2),$$
$$0 < x_1 < a, \quad 0 \leqslant x_2 \leqslant b, \tag{A1c}$$

$$u_{i,12}^{(1)}(x_1, x_2) = u_{i,21}^{(2)}(x_1, x_2) = \frac{1}{2} \sum_{m=1}^{\infty} \alpha_m \bar{c}_{im} \cos(\alpha_m x_1)$$
$$+ \frac{1}{2} \sum_{n=1}^{\infty} \beta_n \bar{a}_{in} \cos(\beta_n x_2) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{\alpha_m \beta_n U_{imn}^{(1)} + \beta_n(\bar{a}_{in}\gamma_m + b_{in}\delta_m)$$
$$+ \alpha_m(\bar{c}_{im}\gamma_n + \bar{d}_{im}\delta_n)\} \cos(\alpha_m x_1) \cos(\beta_n x_2), \quad 0 \leqslant x_1 \leqslant a, \quad 0 \leqslant x_2 \leqslant b, \tag{A1d}$$

$$u_{i,111}^{(1)}(x_1, x_2) = \frac{1}{2} \sum_{n=1}^{\infty} \bar{\bar{a}}_{in} \sin(\beta_n x_2) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [-\alpha_m^2(\alpha_m U_{imn}^{(1)} + \bar{a}_{in}\gamma_m + \bar{b}_{in}\delta_m)$$
$$+ \bar{\bar{a}}_{in}\gamma_m + \bar{\bar{b}}_{in}\delta_m] \cos(\alpha_m x_1) \sin(\beta_n x_2), \quad 0 \leqslant x_1 \leqslant a, \quad 0 \leqslant x_2 \leqslant b, \tag{A1e}$$

$$u_{i,222}^{(1)}(x_1, x_2) = \frac{1}{2} \sum_{m=1}^{\infty} \bar{\bar{c}}_{im} \sin(\alpha_m x_1) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [-\beta_n^2(\beta_n U_{imn}^{(1)} + \bar{c}_{im}\gamma_n + \bar{d}_{im}\delta_n)$$
$$+ \bar{\bar{c}}_{im}\gamma_n + \bar{\bar{d}}_{im}\delta_n] \sin(\alpha_m x_1) \cos(\beta_n x_2), \; 0 < x_1 < a, \quad 0 \leqslant x_2 \leqslant b, \tag{A1f}$$

$$u^{(1)}_{i,1211}(x_1, x_2) = u^{(1)}_{i,1112}(x_1, x_2) = u^{(1)}_{i,2111}(x_1, x_2) = u^{(1)}_{i,1121}(x_1, x_2)$$

$$= -\frac{1}{2}\sum_{m=1}^{\infty} \alpha_m^3 \bar{c}_{im}\cos(\alpha_m x_1) + \frac{1}{2}\sum_{n=1}^{\infty} \beta_n \bar{\bar{a}}_{in}\cos(\beta_n x_2) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[-\alpha_m^2\{\alpha_m\beta_n U^{(1)}_{imn}$$

$$+ \beta_n(\bar{a}_{in}\gamma_m + \bar{b}_{in}\delta_m) + \alpha_m(\bar{c}_{im}\gamma_n + \bar{d}_{im}\delta_n)\} + \beta_n(\bar{\bar{a}}_{in}\gamma_m + \bar{\bar{b}}_{in}\delta_m)]\cos(\alpha_m x_1)\cos(\beta_n x_2),$$

$$0 \leqslant x_1 \leqslant a,\ 0 \leqslant x_2 \leqslant b, \tag{A1g}$$

$$u^{(1)}_{i,2221}(x_1, x_2) = u^{(1)}_{i,1222}(x_1, x_2) = u^{(1)}_{i,2122}(x_1, x_2) = u^{(1)}_{i,2212}(x_1, x_2)$$

$$= -\frac{1}{2}\sum_{n=1}^{\infty} \beta_n^3 \bar{a}_{in}\cos(\beta_n x_2) + \frac{1}{2}\sum_{n=1}^{\infty} \alpha_m \bar{\bar{c}}_{im}\cos(\alpha_m x_1)$$

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[-\beta_n^2\{\alpha_m\beta_n U^{(1)}_{imn} + \alpha_m(\bar{c}_{im}\gamma_n + \bar{d}_{im}\delta_n) + \beta_n(\bar{a}_{in}\gamma_m + \bar{b}_{in}\delta_m)\}$$

$$+ \alpha_m(\bar{\bar{c}}_{im}\gamma_n + \bar{\bar{d}}_{im}\delta_n)\}\cos(\alpha_m x_1)\cos(\beta_n x_2), \quad 0 \leqslant x_1 \leqslant a,\ 0 \leqslant x_2 \leqslant b, \tag{A1h}$$

in which

$$(\gamma_m, \delta_m) = \begin{cases}(0, 1) & \text{for } m = \text{odd},\\ (1, 0) & \text{for } m = \text{even}.\end{cases} \tag{A2}$$

The remaining partial derivatives can be obtained by termwise differentiation.

$$u^{(2)}_i(x_1, x_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} U^{(2)}_{imn}\cos(\alpha_m x_1)\cos(\beta_n x_2), \quad 0 \leqslant x_1 \leqslant a,\ 0 \leqslant x_2 \leqslant b, \tag{A3a}$$

$$u^{(2)}_{i,1}(x_1, x_2) = -\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \alpha U^{(2)}_{imn}\sin(\alpha_m x_1)\cos(\beta_n x_2), \quad 0 < x_1 < a,\ 0 \leqslant x_2 \leqslant b, \tag{A3b}$$

$$u^{(2)}_{i,2}(x_1, x_2) = -\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \beta_n U^{(2)}_{imn}\cos(\alpha_m x_1)\sin(\beta_n x_2), \quad 0 \leqslant x_1 \leqslant a,\ 0 < x_2 < b, \tag{A3c}$$

$$u^{(2)}_{i,11}(x_1, x_2) = \frac{1}{4}a'_{i0} + \frac{1}{2}\sum_{n=1}^{\infty} b'_{in}\cos(\beta_n x_2) + \sum_{m=1}^{\infty}\{-\alpha_m^2 U^{(2)}_{im0} + \frac{1}{2}(a'_{i0}\gamma_m + b'_{i0}\delta_m)\}$$

$$\times\cos(\alpha_m x_1) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-\alpha_m^2 U^{(2)}_{imn} + a'_{in}\gamma_m + b'_{in}\delta_m)\cos(\alpha_m x_1)\cos(\beta_n x_2),$$

$$0 \leqslant x_1 \leqslant a,\ 0 \leqslant x_2 \leqslant b, \tag{A3d}$$

$$u^{(2)}_{i,22}(x_1,x_2)=\frac{1}{4}c'_{i0}+\frac{1}{2}\sum_{m=1}^{\infty}c'_{im}\cos(\alpha_m x_1)+\sum_{n=1}^{\infty}\left\{-\beta_n^2U^{(2)}_{i0n}+\frac{1}{2}(c'_{i0}\gamma_n+d'_{i0}\delta_n)\right\}\cos(\beta_n x_2)$$
$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-\beta_n^2U^{(2)}_{imn}+c'_{im}\gamma_n+d'_{im}\delta_n)\cos(\alpha_m x_1)\cos(\beta_n x_2),\quad 0\leqslant x_1\leqslant a,\ 0\leqslant x_2\leqslant b,\tag{A3e}$$

$u^{(2)}_{i,12}(x_1,x_2)=u^{(2)}_{i,21}(x_1,x_2)$ can be obtained by termwise differentiation,

$$u^{(2)}_{i,1111}(x_1,x_2)=\frac{1}{4}a''_{i0}+\frac{1}{2}\sum_{n=1}^{\infty}a''_{in}\cos(\beta_n x_2)+\sum_{m=1}^{\infty}\left[\alpha_m^2\left\{\alpha_m^2U^{(2)}_{im0}-\frac{1}{2}(a'_{i0}\gamma_m+\beta'_{i0}\delta_m)\right\}\right.$$
$$\left.+a''_{i0}\gamma_m+b''_{i0}\delta_m\right]\cos(\alpha_m x_1)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[\alpha_m^2\{\alpha_m^2U^{(2)}_{imn}-(a'_{in}\gamma_m+b'_{in}\delta_m)\}$$
$$+a''_{in}\gamma_m+b''_{in}\delta_m]\cos(\alpha_m x_1)\cos(\beta_n x_2),\quad 0\leqslant x_1\leqslant a,\ 0\leqslant x_2\leqslant b,\tag{A3f}$$

$$u^{(2)}_{i,2222}(x_1,x_2)=\frac{1}{4}c''_{i0}+\frac{1}{2}\sum_{m=1}^{\infty}c''_{im}\cos(\alpha_m x_1)+\sum_{n=1}^{\infty}\left[\beta_n^2\left\{\beta_n^2U^{(2)}_{i0n}-\frac{1}{2}(c'_{i0}\gamma_n+d'_{i0}\delta_n)\right\}\right.$$
$$\left.+\frac{1}{2}(c''_{i0}\gamma_n+d''_{i0}\delta_n\right]\cos(\beta_n x_2)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[\beta_n^2\{\beta_n^2U^{(2)}_{imn}-(c'_{im}\gamma_n+d_{im}\delta_n)\}$$
$$+c''_{im}\gamma_n+d''_{im}\delta_n]\cos(\alpha_m x_1)\cos(\beta_n x_2),\quad 0\leqslant x_1\leqslant a,\ 0\leqslant x_2\leqslant b,\tag{A3g}$$

$$u^{(2)}_{i,1122}(x_1,x_2)=u^{(2)}_{i,2211}(x_1,x_2)=u^{(2)}_{i,1212}(x_1,x_2)=u^{(2)}_{i,2121}(x_1,x_2)$$
$$=-\frac{1}{2}\sum_{m=1}^{\infty}\alpha_m^2c'_{im}\cos(\alpha_m x_1)-\frac{1}{2}\sum_{n=1}^{\infty}\beta_n^2b'_{in}\cos(\beta_n x_2)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[\{\alpha_m^2\beta_n^2U^{(2)}_{imn}$$
$$-\beta_n^2(a'_{in}\gamma_m+b'_{in}\delta_m)-\alpha_m^2(c'_{im}\gamma_n+d'_{im}\delta_n)]\cos(\alpha_m x_1)\cos(\beta_n x_2),$$
$$0\leqslant x_1\leqslant a,\ 0\leqslant x_2\leqslant b,\tag{A3h}$$

$$u^{(3)}_i(x_1,x_2)=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}U^{(3)}_{imn}\sin(\alpha_m x_1)\cos(\beta_n x_2),\quad 0<x_1<a,\ 0\leqslant x_2\leqslant b,\tag{A4a}$$

$$u^{(3)}_{i,1}(x_1,x_2)=\frac{1}{4}\bar{e}_{i0}+\sum_{m=1}^{\infty}\left\{\alpha_mU^{(3)}_{im0}+\frac{1}{2}(\bar{e}_{i0}\gamma_m+\bar{f}_{i0}\delta_m)\right\}\cos(\alpha_m x_1)+\frac{1}{2}\sum_{n=1}^{\infty}\bar{e}_{in}\cos(\beta_n x_2)$$
$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\alpha_mU^{(3)}_{imn}+\bar{e}_{in}\gamma_m+\bar{f}_{in}\delta_m)\cos(\alpha_m x_1)\cos(\beta_n x_2),\quad 0\leqslant x_1\leqslant a,\ 0\leqslant x_2\leqslant b,\tag{A4b}$$

$$u_{i,2}^{(3)}(x_1, x_2) = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \beta_n U_{imn}^{(3)} \sin(\alpha_m x_1)\sin(\beta_n x_2), \quad 0 < x_1 < a,\ 0 < x_2 < b, \tag{A4c}$$

$$u_{i,22}^{(3)}(x_1, x_2) = \frac{1}{2}\sum_{m=1}^{\infty} e'_{im}\sin(\alpha x_1) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-\beta^2 U_{imn}^{(3)} + e'_{im}\gamma_n + f'_{im}\delta_n)\sin(\alpha_m x_1)\cos(\beta_n x_2),$$
$$0 < x_1 < a,\ 0 \leqslant x_2 \leqslant b \tag{A4d}$$

$$u_{i,111}^{(3)}(x_1, x_2) = \frac{1}{4}\bar{\bar{e}}_{i0} - \sum_{m=1}^{\infty}\left[\alpha_m^3 U_{im0}^{(3)} + \frac{1}{2}\alpha_m^2(\bar{e}_{i0}\gamma_m + \bar{f}_{i0}\delta_m) - \frac{1}{2}(\bar{\bar{e}}_{i0}\gamma_m + \bar{\bar{f}}_{i0}\delta_m)\right]\cos(\alpha_m x_1)$$
$$+ \frac{1}{2}\sum_{n=1}^{\infty}\bar{\bar{e}}_{in}\cos(\beta_n x_2) - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\alpha_m^3 U_{imn}^{(3)} + \alpha_m^2(\bar{e}_{in}\gamma_m + \bar{f}_{in}\delta_m)\right.$$
$$\left. - (\bar{\bar{e}}_{in}\gamma_m + \bar{\bar{f}}_{in}\delta_m)\right]\cos(\alpha_m x_1)\cos(\beta_n x_2), \quad 0 \leqslant x_1 \leqslant a,\ 0 \leqslant x_2 \leqslant b \tag{A4e}$$

$$u_{i,122}^{(3)}(x_1, x_2) = u_{i,212}^{(3)}(x_1, x_2) = u_{i,221}^{(3)}(x_1, x_2) = \frac{1}{2}\sum_{m=1}^{\infty}\alpha_m e'_{im}\cos(\alpha_m x_1)$$
$$-\frac{1}{2}\sum_{n=1}^{\infty}\beta_n^2\bar{e}_{in}\cos(\beta_n x_2) - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\alpha_m\beta_n^2 U_{imn}^{(3)} - \alpha_m(e'_{im}\gamma_n + f'_{im}\delta_n)\right.$$
$$\left. + \beta_n^2(\bar{e}_{in}\gamma_m + \bar{f}_{in}\delta_m)\right]\cos(\alpha_m x_1)\cos(\beta_n x_2), \quad 0 \leqslant x_1 \leqslant a,\ 0 \leqslant x_2 \leqslant b \tag{A4f}$$

$$u_{i,2222}^{(3)}(x_1, x_2) = \frac{1}{2}\sum_{m=1}^{\infty} e''_{im}\sin(\alpha_m x_1) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\beta_n^4 U_{imn}^{(3)} - \beta_n^2(e'_{im}\gamma_n + f'_{im}\delta_n)\right.$$
$$\left. + e''_{im}\gamma_n + f''_{im}\delta_n\right]\sin(\alpha_m x_1)\cos(\beta_n x_2), \quad 0 < x_1 < a,\ 0 \leqslant x_2 \leqslant b. \tag{A4g}$$

The remaining partial derivatives can be obtained by termwise differentiation.

$$u_i^{(4)}(x_1, x_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} U_{imn}^{(4)}\cos(\alpha_m x_1)\sin(\beta_n x_2), \quad 0 \leqslant x_1 \leqslant a,\ 0 < x_2 < b, \tag{A5a}$$

$$u_{i,1}^{(4)}(x_1, x_2) = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha_m U_{imn}^{(4)}\sin(\alpha_m x_1)\sin(\beta_n x_2), \quad 0 < x_1 < a,\ 0 < x_2 < b, \tag{A5b}$$

$$u_{i,2}^{(4)}(x_1, x_2) = \frac{1}{4}\bar{g}_{i0} + \frac{1}{2}\sum_{m=1}^{\infty}\bar{g}_{im}\cos(\alpha_m x_1) + \frac{1}{2}\sum_{n=1}^{\infty}(\beta_n U_{imn}^{(4)} + \bar{g}_{i0}\gamma_n + \bar{h}_{i0}\delta_n)\cos(\beta_n x_2)$$
$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\beta_n U_{imn}^{(4)} + \bar{g}_{im}\gamma_n + \bar{h}_{im}\delta_n)\cos(\alpha_m x_1)\cos(\beta_n x_2),$$
$$0 \leqslant x_1 \leqslant a,\ 0 \leqslant x_2 \leqslant b, \tag{A5c}$$

$$u^{(4)}_{i,11}(x_1,x_2)=\frac{1}{2}\sum_{n=1}^{\infty} g'_{in}\sin(\beta_n x_2)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-\alpha_m^2 U^{(4)}_{imn}+g'_{in}\gamma_m+h'_{in}\delta_m)\cos(\alpha_m x_1)\sin(\beta_n x_2);$$

$$0\leqslant x_1\leqslant a,\ 0<x_2<b \quad \text{(A5d)}$$

$$u^{(4)}_{i,1111}(x_1,x_2)=\frac{1}{2}\sum_{n=1}^{\infty} g''_{in}\sin(\beta_n x_2)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[\alpha_m^4 U^{(4)}_{imn}-\alpha_m^2(g'_{in}\gamma_m+h'_{in}\delta_m)$$

$$+g''_{in}\gamma_m+h''_{in}\delta_m]\cos(\alpha_m x_1)\sin(\beta_n x_2),\quad 0\leqslant x_1\leqslant a,\ 0<x_2<b \quad \text{(A5e)}$$

$$u^{(4)}_{i,112}(x_1,x_2)=u^{(4)}_{i,121}(x_1,x_2)=u^{(4)}_{i,211}(x_1,x_2)=-\frac{1}{2}\sum_{m=1}^{\infty}\alpha_m^2\bar{g}_{im}\cos(\alpha_m x_1)$$

$$+\frac{1}{2}\sum_{n=1}^{\infty}\beta_n g'_{in}\cos(\beta_n x_2)-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[\alpha_m^2\beta_n U^{(4)}_{imn}-\beta_n(g'_{in}\gamma_m+h'_{in}\delta_m)+\alpha_m^2(\bar{g}_{im}\gamma_n+\bar{h}_{im}\delta_n)]$$

$$\cos(\alpha_m x_1)\cos(\beta_n x_2),\quad 0\leqslant x_1\leqslant a,\ 0\leqslant x_2\leqslant b \quad \text{(A5f)}$$

$$u^{(4)}_{i,222}(x_1,x_2)=\frac{1}{4}\bar{\bar{g}}_{i0}+\frac{1}{2}\sum_{m=1}^{\infty}\bar{\bar{g}}_{im}\cos(\alpha_m x_1)-\sum_{n=1}^{\infty}[\beta_n^3 U^{(4)}_{i0n}+\frac{1}{2}\beta_n^2(\bar{g}_{i0}\gamma_n+\bar{h}_{i0}\delta_n)$$

$$-\frac{1}{2}(\bar{\bar{g}}_{i0}\gamma_n+\bar{\bar{h}}_{i0}\delta_n)]\cos(\beta_n x_2)-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[\beta_n^3 U^{(4)}_{imn}+\beta_n^2(\bar{g}_{im}\gamma_n+\bar{h}_{im}\delta_n)+\bar{\bar{g}}_{im}\gamma_n$$

$$+\bar{\bar{h}}_{im}\delta_n]\cos(\alpha_m x_1)\cos(\beta_n x_2),\quad 0\leqslant x_1\leqslant a,\ 0\leqslant x_2\leqslant b \quad \text{(A5g)}$$

The remaining partial derivatives of $u_i^{(4)}(x_1,x_2)$ can be obtained by termwise differentiation.

APPENDIX B: BOUNDARY FOURIER COEFFICIENTS

The boundary Fourier coefficients, referred to in the text, are defined as follows:

$$(\bar{a}_{in};\bar{b}_{in})=\frac{4}{ab}\int_0^b\{\pm u_i^{(1)}(a,x_2)-u_i^{(1)}(0,x_2)\}\sin(\beta_n x_2)\,\mathrm{d}x_2, \quad \text{(B1a, b)}$$

$$(\bar{\bar{a}}_{in};\bar{\bar{b}}_{in})=\frac{4}{ab}\int_0^b\{\pm u^{(1)}_{i,11}(a,x_2)-u^{(1)}_{i,11}(0,x_2)\}\sin(\beta_n x_2)\,\mathrm{d}x_2, \quad \text{(B1c, d)}$$

$$(\bar{c}_{im};\bar{d}_{im})=\frac{4}{ab}\int_0^a\{\pm u_i^{(1)}(x_1,b)-u_i^{(1)}(x_1,0)\}\sin(\alpha_m x_1)\,\mathrm{d}x_1, \quad \text{(B1e, f)}$$

$$(\bar{\bar{c}}_{im};\bar{\bar{d}}_{im})=\frac{4}{ab}\int_0^a\{\pm u^{(1)}_{i,22}(x_1,b)-u^{(1)}_{i,22}(x_1,0)\}\sin(\alpha_m x_1)\,\mathrm{d}x_1, \quad \text{(B1g, h)}$$

$$(\bar{e}_{in};\bar{f}_{in}) = \frac{4}{ab}\int_0^b \{\pm u_i^{(3)}(a, x_2) - u_i^{(3)}(0, x_2)\}\cos(\beta_n x_2)\,\mathrm{d}x_2, \tag{B1i, j}$$

$$(\bar{\bar{e}}_{in};\bar{\bar{f}}_{in}) = \frac{4}{ab}\int_0^b \{\pm u_{i,11}^{(3)}(a, x_2) - u_{i,11}^{(3)}(0, x_2)\}\cos(\beta_n x_2)\,\mathrm{d}x_2, \tag{B1k, l}$$

$$(\bar{g}_{im};\bar{h}_{im}) = \frac{4}{ab}\int_0^a \{\pm u_i^{(4)}(x_1, b) - u_i^{(4)}(x_1, 0)\}\cos(\alpha_m x_1)\,\mathrm{d}x_1, \tag{B1m, n}$$

$$(\bar{\bar{g}}_{im};\bar{\bar{h}}_{im}) = \frac{4}{ab}\int_0^a \{\pm u_{i,22}^{(4)}(x_1, b) - u_{i,22}^{(4)}(x_1, 0)\}\sin(\alpha_m x_1)\,\mathrm{d}x_1, \tag{B1o, p}$$

$$(a'_{in};b'_{in}) = \frac{4}{ab}\int_0^b \{\pm u_{i,1}^{(2)}(a, x_2) - u_{i,1}^{(2)}(0, x_2)\}\cos(\beta_n x_2)\,\mathrm{d}x_2, \tag{B2a, b}$$

$$(a''_{in};b''_{in}) = \frac{4}{ab}\int_0^b \{\pm u_{i,111}^{(2)}(a, x_2) - u_{i,111}^{(2)}(0, x_2)\}\cos(\beta_n x_2)\,\mathrm{d}x_2, \tag{B2c, d}$$

$$(c'_{im};d'_{im}) = \frac{4}{ab}\int_0^a \{\pm u_{i,2}^{(2)}(x_1, b) - u_{i,2}^{(2)}(x_1, 0)\}\cos(\alpha_m x_1)\,\mathrm{d}x_1, \tag{B2e, f}$$

$$(c''_{im};d''_{im}) = \frac{4}{ab}\int_0^a \{\pm u_{i,222}^{(2)}(x_1, b) - u_{i,222}^{(2)}(x_1, 0)\}\cos(\alpha_m x_1)\,\mathrm{d}x_1, \tag{B2g, h}$$

$$(e'_{im};f'_{im}) = \frac{4}{ab}\int_0^a \{\pm u_{i,2}^{(3)}(x_1, b) - u_{i,2}^{(3)}(x_1, 0)\}\sin(\alpha_m x_1)\,\mathrm{d}x_1, \tag{B2i, j}$$

$$(e''_{im};f''_{im}) = \frac{4}{ab}\int_0^a \{\pm u_{i,222}^{(3)}(x_1, b) - u_{i,222}^{(3)}(x_1, 0)\}\sin(\alpha_m x_1)\,\mathrm{d}x_1, \tag{B2k, l}$$

$$(g'_{in};h'_{in}) = \frac{4}{ab}\int_0^b \{\pm u_{i,1}^{(4)}(a, x_2) - u_{i,1}^{(4)}(0, x_2)\}\sin(\beta_n x_2)\,\mathrm{d}x_2, \tag{B2m, n}$$

$$(g''_{in};h''_{in}) = \frac{4}{ab}\int_0^b \{\pm u_{i,111}^{(4)}(a, x_2) - u_{i,111}^{(4)}(0, x_2)\}\sin(\beta_n x_2)\,\mathrm{d}x_2. \tag{B2o, p}$$

The non-zero boundary displacements and their derivatives for symmetrically placed boundary conditions are given as follows:

$$\{u_i^{(1)}(0, x_2); u_i^{(1)}(a, x_2)\} = \frac{a}{4}\sum_{n=0}^{\infty}(\mp\,\bar{a}_{in} - \bar{b}_{in})\sin(\beta_n x_2), \tag{B3a, b}$$

$$\{u^{(1)}_{i,11}(0,x_2);\, u^{(1)}_{i,11}(a,x_2)\} = \frac{a}{4}\sum_{n=0}^{\infty}(\mp \bar{\bar{a}}_{in} - \bar{\bar{b}}_{in})\sin(\beta_n x_2), \tag{B3c, d}$$

$$\{u^{(1)}_{i}(x_1,0);\, u^{(1)}_{i}(x_1,b)\} = \frac{b}{4}\sum_{m=0}^{\infty}(\mp \bar{c}_{im} - \bar{d}_{im})\sin(\alpha_m x_1), \tag{B3e, f}$$

$$\{u^{(1)}_{i,22}(x_1,0);\, u^{(1)}_{i,22}(x_1,b)\} = \frac{b}{4}\sum_{m=0}^{\infty}(\mp \bar{\bar{c}}_{im} - \bar{\bar{d}}_{im})\sin(\alpha_m x_1), \tag{B3g, h}$$

$$\{u^{(3)}_{i}(0,x_2);\, u^{(3)}_{i}(a,x_2)\} = \frac{a}{4}\sum_{n=0}^{\infty}(\mp \bar{e}_{in} - \bar{f}_{in})\cos(\beta_n x_2), \tag{B3i, j}$$

$$\{u^{(3)}_{i,11}(0,x_2);\, u^{(3)}_{i,11}(a,x_2)\} = \frac{a}{4}\sum_{n=0}^{\infty}(\mp \bar{\bar{e}}_{in} - \bar{\bar{f}}_{in})\cos(\beta_n x_2), \tag{B3k, l}$$

$$\{u^{(4)}_{i}(x_1,0);\, u^{(4)}_{i}(x_1,b)\} = \frac{b}{4}\sum_{m=0}^{\infty}(\mp \bar{g}_{im} - \bar{h}_{im})\cos(\alpha_m x_1), \tag{B3m, n}$$

$$\{u^{(4)}_{i,22}(x_1,0);\, u^{(4)}_{i,22}(x_1,b)\} = \frac{b}{4}\sum_{m=0}^{\infty}(\mp \bar{\bar{g}}_{im} - \bar{\bar{h}}_{im})\cos(\alpha_m x_1), \tag{B3o, p}$$

$$\{u^{(2)}_{i,1}(0,x_2);\, u^{(2)}_{i,1}(a,x_2)\} = \frac{a}{4}\sum_{n=0}^{\infty}(\mp a'_{in} - b'_{in})\cos(\beta_n x_2), \tag{B4a, b}$$

$$\{u^{(2)}_{i,111}(0,x_2);\, u^{(2)}_{i,111}(a,x_2)\} = \frac{a}{4}\sum_{n=0}^{\infty}(\mp a''_{in} - b''_{in})\cos(\beta_n x_2), \tag{B4c, d}$$

$$\{u^{(2)}_{i,2}(x_1,0);\, u^{(2)}_{i,2}(x_1,b)\} = \frac{b}{4}\sum_{m=0}^{\infty}(\mp c'_{im} - d'_{im})\cos(\alpha_m x_1), \tag{B4e, f}$$

$$\{u^{(2)}_{i,222}(x_1,0);\, u^{(2)}_{i,222}(x_1,b)\} = \frac{b}{4}\sum_{m=0}^{\infty}(\mp c''_{im} - d''_{im})\cos(\alpha_m x_1), \tag{B4g, h}$$

$$\{u^{(3)}_{i,2}(x_1,0);\, u^{(3)}_{i,2}(x_1,b)\} = \frac{b}{4}\sum_{m=0}^{\infty}(\mp e'_{im} - f'_{im})\sin(\alpha_m x_1), \tag{B4i, j}$$

$$\{u^{(3)}_{i,222}(x_1,0);\, u^{(3)}_{i,222}(x_1,b)\} = \frac{b}{4}\sum_{m=0}^{\infty}(\mp e''_{im} - f''_{im})\sin(\alpha_m x_1), \tag{B4k, l}$$

$$\{u^{(4)}_{i,1}(0,x_2);\, u^{(4)}_{i,1}(a,x_2)\} = \frac{a}{4}\sum_{n=0}^{\infty}(\mp g'_{in} - h'_{in})\sin(\beta_n x_2), \tag{B4m, n}$$

$$\{u^{(4)}_{i,111}(0,x_2);\, u^{(4)}_{i,111}(a,x_2)\} = \frac{a}{4}\sum_{n=0}^{\infty}(\mp g''_{in} - h''_{in})\sin(\beta_n x_2). \tag{B4o, p}$$

APPENDIX C: NON-ZERO COEFFICIENTS FOR CLT-BASED FORMULATION

The non-zero coefficients, for a CLT-based formulation, are as furnished below

$$a_{33} = -\left(\frac{A_{11}}{R_1^2} + 2\frac{A_{12}}{R_1 R_2} + \frac{A_{22}}{R_2^2}\right), \quad b_{131} = \frac{A_{11}}{R_1} + \frac{A_{12}}{R_2}, \quad b_{132} = b_{231} = \frac{A_{16}}{R_1} + \frac{A_{26}}{R_2},$$

$$b_{232} = \frac{A_{12}}{R_1} + \frac{A_{22}}{R_2}, \quad b_{311} = -\left(\frac{A_{11}}{R_1} + \frac{B_{11}}{R_1^2} + \frac{A_{12}}{R_2} + \frac{B_{12}}{R_1 R_2}\right),$$

$$b_{312} = -\left(\frac{A_{16}}{R_1} + \frac{B_{16}}{R_1^2} + c\frac{B_{16}}{R_1} + \frac{A_{26}}{R_2} + \frac{B_{26}}{R_1 R_2} + c\frac{B_{26}}{R_2}\right),$$

$$b_{321} = -\left(\frac{A_{16}}{R_1} + \frac{B_{16}}{R_1 R_2} + c\frac{B_{16}}{R_1^2} + \frac{A_{26}}{R_2} + \frac{B_{26}}{R_2^2} + c\frac{B_{26}}{R_2}\right),$$

$$b_{322} = \left(\frac{A_{11}}{R_1} + \frac{B_{11}}{R_1^2} + \frac{A_{12}}{R_2} + \frac{B_{12}}{R_1 R_2}\right), \quad c_{1111} = A_{11} + \frac{B_{11}}{R_1},$$

$$c_{1112} = 2A_{16} + 2\frac{B_{16}}{R_1} + 2cB_{16} + c\frac{D_{16}}{R_1},$$

$$c_{1122} = A_{66} + \frac{B_{66}}{R_1} + 2cB_{66} + c\frac{D_{66}}{R_1} + c^2 D_{66}, \quad c_{1211} = A_{16} + \frac{B_{16}}{R_2} - cB_{16},$$

$$c_{1212} = A_{12} + \frac{B_{12}}{R_1} + A_{66} + \frac{B_{66}}{R_2} + c\frac{D_{66}}{R_1} - c^2 D_{66}, \quad c_{1222} = A_{26} + \frac{B_{26}}{R_2},$$

$$c_{2111} = A_{16} - cB_{16} + \frac{B_{16} - cD_{16}}{R_1}, \quad c_{2112} = A_{12} + \frac{B_{12}}{R_1} + A_{66} + \frac{B_{66}}{R_1} - \frac{cD_{66}}{R_1} - c^2 D_{66},$$

$$c_{2122} = A_{26} + \frac{B_{26}}{R_1} + cB_{26}, \quad c_{2211} = A_{66} + \frac{B_{66}}{R_2} - 2cB_{66} - c\frac{D_{66}}{R_1} + c^2 D_{66},$$

$$c_{2212} = 2A_{26} + 2\frac{B_{26}}{R_2} - 2cB_{26} - c\frac{D_{26}}{R_2}, \quad c_{2222} = A_{22} + \frac{B_{22}}{R_2},$$

$$c_{3311} = 2\left(\frac{B_{11}}{R_1} + \frac{B_{12}}{R_2}\right), \quad c_{3312} = 4\left(\frac{B_{16}}{R_1} + \frac{B_{26}}{R_2}\right), \quad c_{3322} = 2\left(\frac{B_{12}}{R_1} + \frac{B_{22}}{R_2}\right),$$

$$d_{13111} = -B_{11}, \quad d_{13112} = -(3B_{16} + cD_{16}), \quad d_{13122} = -B_{11},$$

$$d_{13222} = -B_{26} + cD_{26}, \quad d_{23111} = -B_{16} + cD_{16},$$

$$d_{23112} = -B_{12} - 2B_{66} + 2cD_{66}, \quad d_{23122} = -(3B_{26} + cD_{26}), \quad d_{23222} = -B_{22},$$

$$d_{31111} = B_{11} + \frac{D_{11}}{R_1}, \quad d_{31112} = 3B_{16} + 3\frac{D_{16}}{R_1} + cD_{16},$$

$$d_{31122} = B_{12} + \frac{D_{12}}{R_1} + 2B_{66} + \frac{2D_{66}}{R_1} + 2cD_{66}, \quad d_{31222} = B_{26} + \frac{D_{26}}{R_1} + cD_{26},$$

$$d_{32111} = B_{16} + \frac{D_{16}}{R_2} - cD_{16}, \quad d_{32112} = B_{12} + \frac{D_{12}}{R_2} + 2B_{66} + \frac{2D_{66}}{R_1} - 2cD_{66},$$

$$d_{32122} = 3B_{26} + 3\frac{D_{26}}{R_1} - cD_{26}, \quad d_{32222} = B_{22} + \frac{D_{22}}{R_2}, \quad e_{331111} = -D_{11},$$

$$e_{331112} = -4D_{16}, \quad e_{331122} = -2D_{12} - 4D_{66}, \quad e_{331222} = -4D_{26}, \quad e_{332222} = -D_{22}, \tag{C1}$$

where A_{ij}, B_{ij}, and D_{ij} $(i, j = 1, 2, 6)$ are extensional, coupling, and bending rigidities, respectively, while A_{ij} $(i, j = 4, 5)$ denote transverse shear rigidities. The constant, c in equations (C1) above, represents a correction factor to the conventional classical shallow shell theory due to Donnell, which is given by

$$c = \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right). \tag{C2}$$