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LETTERS TO THE EDITOR

A MULTIPLE FREQUENCY IN THE TWO LOWEST AXISYMMETRIC VIBRATION MODES OF A SHORT CYLINDER

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1. INTRODUCTION

The elementary theory of longitudinal vibrations of slender bars demonstrates that a rod of Young's modulus E, density ρ, length L, whose diameter D is much smaller than L, with free ends and under no external force, is able to vibrate according to normal modes whose natural angular frequencies are $\omega = N\pi(E/\rho)^{1/2}/L$, where $N = 1, 2, 3, \ldots$. If N is odd, the modes corresponding to these frequencies are called symmetric because the vibrations are symmetric with respect to the central cross-section; the vibration amplitude is therefore an odd function of the variable z that denotes the position of a point in the rod. If N is even the corresponding modes are called antisymmetric, the amplitude being an even function of z. All the natural frequencies are integer multiples of the first harmonic. Therefore, when a slender rod vibrates in the lowest natural mode (first symmetric $s1$), its frequency is equal to half that corresponding to the following mode (first antisymmetric $a1$).

A complex function, relating natural frequencies to slenderness L/D, is found when the length of the rod is approximately equal to or smaller than its diameter, even if the rod is vibrating axisymmetrically. The non-dimensional frequency $\Omega = \pi f D\sqrt{\rho/G}$ is often used in order to simplify calculations, where f is the ordinary frequency, measured in Hz, and G the shear modulus. The parameter Ω varies in a complicated manner with both slenderness L/D and the Poisson ratio, and even the natural frequencies of the first two modes become equal for a value of L/D [1]. Therefore, different mode shapes can be obtained for the same natural frequency. The present work is focused on the study of that phenomenon. Preliminary results [2] concerning this problem are also included.

In the study performed, a numerical method has been used because there appears to be no available analytical solution for free vibrations satisfying the boundary conditions. The Ritz method is adequate, between the different methods applicable to the problem, to study vibrations of simple geometric systems. As is well known, this method is based on approximating the solution for the displacement of the points of the system by means of series of functions. Specifically, we are going to follow a methodology already applied to linear elastic cylinders [3]. In the referred work, a combination of power series was used as an approximation function. As our interest is restricted to axisymmetric vibrations, the displacement functions only have radial u and axial w components, which are functions of time t, the distance r' to the revolution axis, and the distance z' to the central cross-section.

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Upon changing ordinary co-ordinates to non-dimensional co-ordinates, $r = 2r'/D$ and $z = z'/L$, the harmonic solution, corresponding to a shape mode, will be

$$u(r, z, t) = U(r, z)\sin(\omega t + \phi) \quad \text{and} \quad w(r, z, t) = W(r, z)\sin(\omega t + \phi), \tag{1}$$

where ϕ is a constant that depends on the initial conditions.

We assumed for the amplitudes U and W polynomial functions:

$$U(r, z) = \sum_{i}^{I}\sum_{j}^{J} A_{ij} r^i z^j \quad \text{and} \quad W(r, z) = \sum_{p}^{P}\sum_{q}^{Q} C_{pq} r^p z^q \tag{2}$$

with $i = 1, 2, 3, \ldots$; $j = 0, 1, 2, \ldots$; $p = 0, 1, 2, \ldots$; $q = 0, 1, 2, \ldots$; where $i = 0$ is not considered to avoid singularities in the stresses and radial displacements at $r = 0$. In symmetric modes, j only takes even values and q odd values and in antisymmetric ones, j takes odd values and q even values.

For an axisymmetric mode, the maximum strain energy functional is

$$V_{max} = 2\pi GL \int_{-1/2}^{1/2}\int_0^1 \left\{ \frac{v}{(1-2v)}\left(\frac{\partial U}{\partial r} + \frac{U}{r} + \frac{R}{L}\frac{\partial W}{\partial z}\right)^2 + \left(\frac{\partial U}{\partial r}\right)^2 + \left(\frac{U}{r}\right)^2 + \left(\frac{R}{L}\frac{\partial W}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{R}{L}\frac{\partial U}{\partial z} + \frac{\partial W}{\partial r}\right)^2 \right\} r\,\mathrm{d}r\,\mathrm{d}z$$

and the maximum kinetic energy is

$$T_{max} = \pi\rho L R^2 \omega^2 \int_{-1/2}^{1/2}\int_0^1 (U^2 + W^2)\, r\,\mathrm{d}r\,\mathrm{d}z.$$

The Hamilton principle for harmonic motion implies that $\partial(V_{max} - T_{max})/\partial A_{ij} = 0$ and $\partial(V_{max} - T_{max})/\partial C_{pq} = 0$, for all values of i, j, p and q. These conditions constitute a homogenous set of linear algebraic equations in A_{ij} and C_{pq}. The eigenvalues of this set are the square of the non-dimensional frequencies (Ω^2) and the eigenvectors are the coefficients of the polynomials. From Ω^2, the ordinary frequency f for a given cylinder may be deduced as well as the vibration shape can be inferred from the coefficients.

The number of terms of an appropriate polynomial depends on the desired precision. After completion of a convergence study, a fifth degree was selected for the polynomials, which were calculated using a PC in a few minutes.

In this paper, the natural frequencies and the mode shapes associated with the lowest modes of a cylinder are obtained by applying the above methodology. The objective is to describe and analyze in considerable detail the two lowest modes of a short cylinder vibrating axisymmetrically. The interest is mainly focused on the fact that such cylinders can vibrate both symmetrically and antisymmetrically at the same frequency, i.e., there exists a multiple frequency. The values of L/D for which the lowest modes vibrate at the same frequency are found for aluminium and steel cylinders. A detailed description of the vibration of such modes is provided. The obtained symmetric and antisymmetric theoretical eigenvectors for the multiple frequency are compared with the displacement measured when the samples are vibrating in conditions close to the multiple frequency.

A solid cylinder has been chosen as an example due to its three-dimensional form. For the axisymmetric vibration modes, the cylinder becomes two dimensional from the mathematical point of view. It also has a multiple frequency. Besides it is easy to be mechanized. In addition, the theoretical and experimental work presented can help to interpret the cylinder vibration spectrum. The more cylinder vibrations are known, the better the experimental results can be used for practical purposes. Thus, cylinder vibrations have been widely treated in the literature. This study could be of interest in determining the dynamic elastic properties of materials.

2. RESULTS OF THE NUMERICAL CALCULATION

Numerical results of the non-dimensional frequencies for both the first symmetric and first antisymmetric modes have been obtained for aluminium samples of the Poisson ratio 0·330 and values for the quotient L/D between 0·1 and 2·0. The results are shown in Figure 1, where circles refer to the symmetric mode and triangles to the antisymmetric one. The crossing point of both curves takes place for a value of the slenderness of 0·764 and for a frequency $\Omega = 2{\cdot}783$. According to the modal theory, this frequency is called multiple with a degree of multiplicity two, because the system can vibrate in two different modes with the same frequency. We have calculated the functions $U(r, z)$ and $W(r, z)$ from the eigenvectors and the amplitude vectors are shown in Figure 2 for the symmetric vibration $s1$ and in Figure 3 for the antisymmetric one $a1$. There is little resemblance between Figure 2 and the fundamental shape of vibration of a slender cylinder. Figure 3 reminds us of the elementary mode of vibration of a circular plate. Actually, none of the two modes seem to match those obtained from the elementary theory for slender rods and plates, which is not applicable to bars of similar length and diameter. Note the difference between the two modes for the same frequency.

The numerical calculation was repeated for a stainless-steel sample of the Poisson ratio 0·298. The frequencies of the first symmetric mode and the first antisymmetric one are equal for a slenderness $L/D = 0{\cdot}786$.

3. EXPERIMENTAL RESULTS

In order to verify the results of the aforesaid numerical calculation, a series of experiments have been carried out. The procedure used to generate the vibration of the sample and the

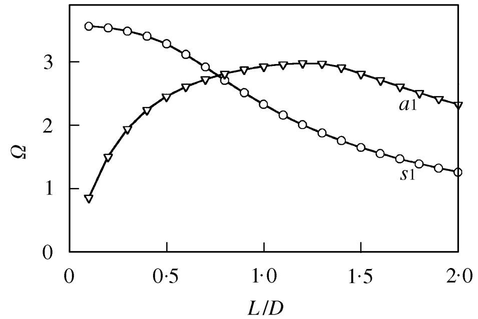

Figure 1. Calculated values of the non-dimensional frequency parameter $\Omega = \pi f D\sqrt{\rho/G}$, versus the slenderness L/D of an aluminium cylinder of the Poisson ratio 0·330 for the first symmetric and antisymmetric modes.

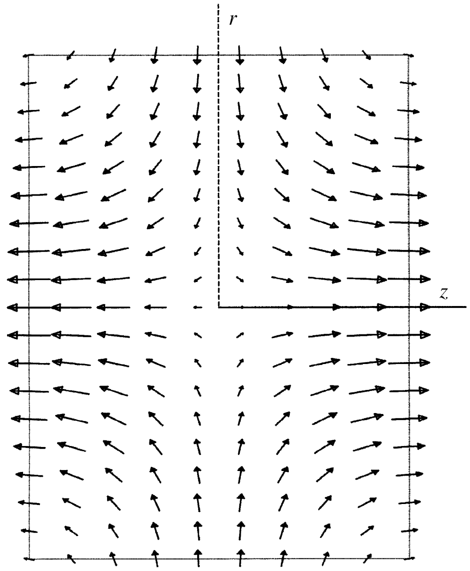

Figure 2. Vibration amplitude for the first symmetric mode. Z is the axis of revolution. The vibration shapes are obtained by numerical calculation.

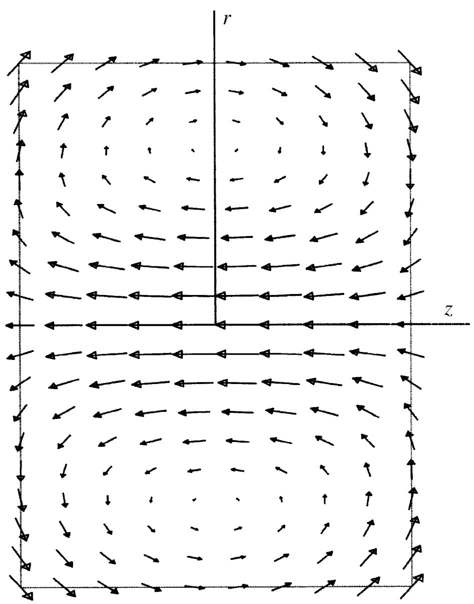

Figure 3. Calculated amplitude of vibration for the first antisymmetric mode.

posterior detection has been described previously [4]. The test piece is located so that it can vibrate almost freely. An axial impact is applied at the centre of the cylinder base using a small steel sphere. This kind of excitation allows the sample to vibrate freely after the impact.

A laser interferometer OP-35 I/O from Ultra Optec Inc. has been used to measure the vibration of the central point on the opposite base to that where the impact was applied.

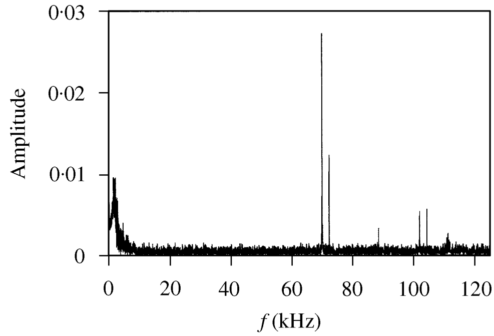

Figure 4. Spectrum of the free out-of-plane vibration measured for the aluminium cylinder of slenderness $L/D = 0{\cdot}800$.

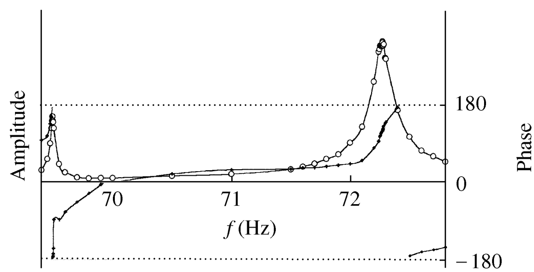

Figure 5. Experimental out-of-plane displacement amplitude and phase at the central point of a base of the aluminium cylinder, submitted to a periodic longitudinal force. —○—, amplitude; —●—, phase.

The out-of-plane displacement component was detected. The obtained signal is proportional to the instantaneous displacement of the detection point and it is digitized with an oscilloscope. The sampling frequency was $f_s = 250$ kHz. The maximum amplitudes of the spectrum, obtained from the fast Fourier transform of the signal, correspond to the natural frequencies of the induced vibration.

The first sample tested was an aluminium cylinder, $D = 39{\cdot}00$ mm in diameter and $L = 31{\cdot}20$ mm in length, i.e., $L/D = 0{\cdot}800$. Figure 4 shows the vibration spectrum of the sample obtained applying the previous procedure. The maxima corresponding to the lowest natural frequencies are at $f_{s1} = 69\,975$ and $f_{a1} = 72\,300$ Hz. These values agree with the numerical calculations. The sample experimentally tested does not have a ratio $L/D = 0{\cdot}764$ although its slenderness is close to the theoretical ratio. A reduction in the complexity of the experiments referred to below is then expected since the experimental layout and the interpretation of the results become simpler. Consequently, the natural frequencies corresponding to $s1$ and $a1$ are not exactly equal.

In view of the previous results, a second experiment was carried out with the same aluminium cylinder, but now using a harmonic exciter. The purpose of such an experiment was to evaluate the behaviour of the cylinder plus a piezoelectric element system. The piezoelectric disc was added to a base of the cylinder and the out-of-plane component of the displacements was detected at the central point of the opposite base. Figure 5 shows the amplitude–frequency results yielded when a sweeping of the exciting frequencies in the range of interest is performed. Variation of oscillation amplitude and phase shift between the displacement and the applied force can be observed. Note that the peaks are very sharp although they do not seem so, due to the horizontal scale used. The resonance frequency (maximum amplitude) for the mode $s1$ is 69 495 Hz and for $a1$ it is 72 273 Hz, which do not agree completely with the previously measured frequencies for the free cylinder. The resonant frequencies do not agree totally with the obtained ones for the previously analyzed free vibration of the cylinder because the sample was overloaded due to the added transducer. The variation of natural frequency due to the mass added to the bar can be roughly estimated [4] by means of the formula $\tan kL = -m_a kL/m$. The exciter used is a commercial transducer of diameter 1·5 cm and mass 0·3 g. The rod mass is $m = 105{\cdot}4$ g, and so the product of the wave number and the length of the bar is $kL = 3{\cdot}1327$. Without the added mass, it would be $k'L = \pi$. The relative variation of frequency is then $(f - f')/f = (k - k')/k = -0{\cdot}0028$. The aforesaid experimental natural free vibration frequencies $f_{s1} = 69\,975$ and $f_{a1} = 72\,300$ Hz for the non-loaded sample become, after the estimated shift for the loaded sample, $f_{s1} = 69\,779$ and $f_{a1} = 72\,098$ Hz. Thus, the estimated and measured frequencies are in close agreement.

In a third experiment, two transducers were glued to the sample in order to induce forced vibrations in the sample. Each one was adhered to a base of the cylinder being connected in parallel to a sinusoidal signal generator. If symmetrically applied forces act on the bases for a certain polarity in the connection, the excitation will be symmetric. The antisymmetric excitation will be reached with the opposite polarization. We assume that the variation of the resonant frequency of the system will be double because a mass is added to each base. The resulting forced vibrations have been detected for both the symmetric and antisymmetric excitations. The measurement of the amplitude and phase of the displacement at a point of the cylinder generatrix should be taken when the exciting frequency is close to the resonance. Therefore, the displacement is expected to be easily detected. However, if the exciting frequencies are too close to the resonance, any parasitic variation of frequency brings about sharp changes of amplitude and phase. For that reason, close but non-equal frequencies to those of resonance have been applied, when carrying out a scanning along a generatrix.

The results of the in-plane displacements corresponding to the symmetric mode $s1$ are shown in Figure 6, which shows the variations of amplitude and phase for a frequency of 69 130 Hz. This figure represents in the vertical axis seven approximately equidistant points $P1$, $P2, \ldots, P7$ placed along the generatrix of the test specimen. Point $P1$ is placed at 1·14 mm from a base. The in-plane displacements for each point are shown during the first 50 μs. The amplitude at $P1$ is about 5 nm and it decreases to zero at the central point $P4$. Note the agreement with the theoretical results shown in Figure 2, in amplitude as well as in phase. Figure 7 represents for mode $s1$ the amplitude vectors from their measured in-plane and out-of-plane displacement components of the seven points along the generatrix. We

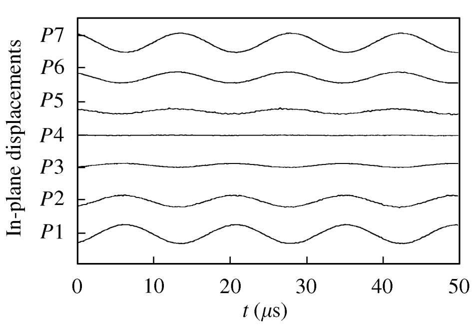

Figure 6. Measured in-plane displacements as a function of time at seven equidistant points at the bottom generatrix of the aluminium cylinder in the case of resonant symmetric excitation. The vibration amplitude at 1·14 mm from a base, point $P1$, is about 5 nm.

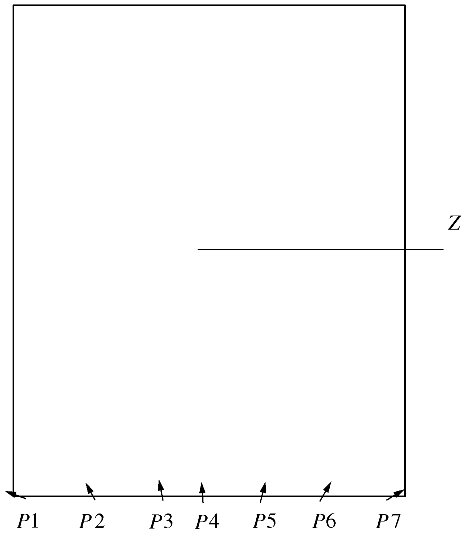

Figure 7. Graph of the in-plane and out-of-plane experimental amplitudes and resultant displacements at seven points along a generatrix of the cylinder for the first symmetric mode.

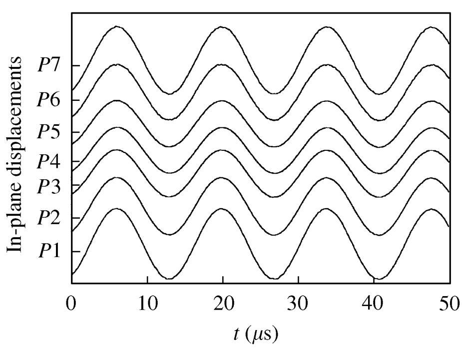

Figure 8. Experimental in-plane displacement versus time at seven points along a generatrix for the first antisymmetric mode. Note the phase concordance for this component of the displacement.

noted a very satisfactory coincidence between these experimental results and those shown at the bottom of Figure 2 obtained by the aforementioned numerical calculation.

Figure 8 shows the measured in-plane displacements versus time for seven points vibrating in the first antisymmetric mode $a1$. All the points have the same phase for that component. Figure 9 represents the amplitude (eigenvectors) for the same seven points along the generatrix. The vibration shape shown differs from that expected from the elementary theory for slender bars. The experimental results are in complete agreement with the theoretical ones given in Figure 3.

A new experiment was carried out using a stainless-steel sample that theoretically has a multiple frequency. The steel test piece used in the laboratory has a length of 31·35 mm, a diameter of 39·90 mm, then its slenderness is 0·7857, which agrees with the theoretical value of the crossing point of Ω–L/D curves for $a1$ and $s1$. Figure 10 shows in detail the obtained spectrum for the free vibration of the sample, which is left to vibrate on its own after an initial axial impulse. There is only one resonance frequency at 67 875 Hz as it is

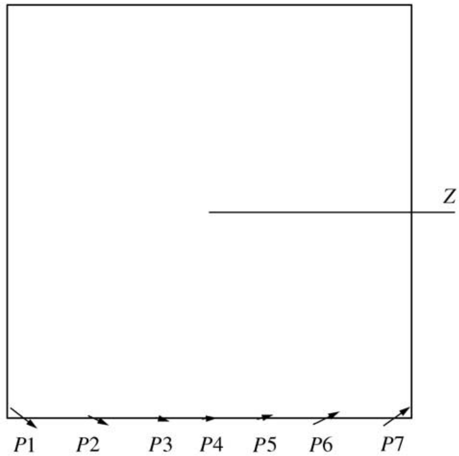

Figure 9. Experimental eigenvectors at the seven points of the generatrix for the first antisymmetric mode. They differ from the elementary mode shape.

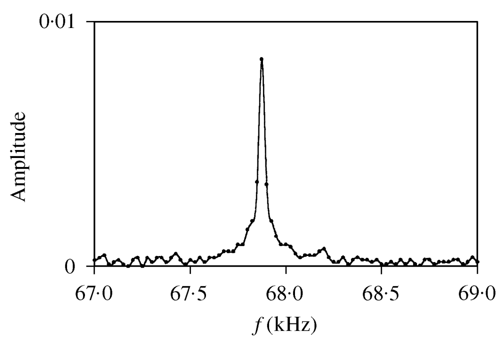

Figure 10. Detailed spectrum of the free vibration of a steel cylinder that presents only one frequency for symmetric and antisymmetric modes $s1$ and $a1$.

expected. This demonstrates the good precision achieved in numerical calculation and experiments as well as the consistency of the results.

The only aforesaid maximum is spread out in two, if the steel sample is put under harmonic oscillation with piezoelectric pieces placed on both bases of the cylinder. On analyzing the in-plane component at the centre and ends of the generatrix, it is deduced that one value corresponds to the symmetric oscillation and the other to the antisymmetric one.

4. CONCLUSION

This paper describes the application of the Ritz method to calculate the eigenfrequencies for axisymmetric vibration modes of aluminium and stainless-steel short cylinders. In the study presented, the problem of multiple frequency for such cylinders is investigated. A numerical solution for the frequency of axisymmetric vibration modes of a short cylinder foresees a double frequency for the lowest symmetric and antisymmetric modes. From the numerical results, it is concluded that the cross-over of frequencies versus slenderness curves for the lowest antisymmetric and symmetric modes should be at $L/D = 0{\cdot}764$ and 0·786 for aluminium and steel samples respectively. The analysis of the eigenvectors at the crossing point shows the vibration shapes of such modes for the same frequency. The accuracy of the numerical calculation is experimentally verified. Each test sample is put under free oscillation in the laboratory in order to determine the natural frequencies. The detected out-of-plane displacement, using a laser interferometer, proves the existence of the multiple frequency. Under forced oscillations, the out-of-plane and in-plane displacements are detected. The shapes obtained experimentally for the lowest antisymmetric and symmetric modes near the crossing point are in good agreement with the numerical ones and demonstrate that the experimental mode shapes and the theoretical ones are very close.

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