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SOLUTION OF DYNAMIC RESPONSE OF FRAMED STRUCTURE USING PIECEWISE BIRKHOFF POLYNOMIAL

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In this paper, the piecewise Birkhoff interpolation polynomials and the modal superposition method were employed for the solution of dynamic response of m.d.o.f. system. The related formulae are derived. Because an exact result can be obtained when each loading can be represented by a piecewise polynomial, the proposed method not only can considerably reduce computational effort compared to the traditional step-by-step integration solution technique, but also can thoroughly avoid the problems of accuracy, convergence and stability encountered in many other numerical procedures.

1. INTRODUCTION

In the analysis of a multi-degree-of-freedom (m.d.o.f.) system subjected to arbitrary dynamic loadings, the step-by-step integration procedure generally provides a powerful solution technique. However, what is worth noting is that it is a very large computational task to evaluate, by the step-by-step integration approach, the response of structures to long-duration loads such as might result from an earthquake. In the case of these long-duration loads, it may be advantageous to use the Duhamel integral and modal superposition method rather than the direct-integration approach.

When compared with the step-by-step integration procedure, another procedure presented by Wilson and Dovey [1], Nigam and Jennings [2], Ly [3] in which an arbitrary dynamic loading is approximated by piecewise linear segments may be preferable. When the dynamic loading is represented by piecewise linear segments, this procedure can provide an exact result. However, when the dynamic loading is varied in the form of an arbitrary curve, the error involved in this procedure can be rather large, especially when the interval between time points is large. In another paper [4], the present author has extended the above described procedure to smoothly varying loading cases by using the piecewise second or third degree Lagrange polynomial for linear s.d.o.f. system. In this paper, in combination with the modal superposition method, the Birkhoff piecewise interpolation functions are employed to approximate arbitrary applied loadings for the solution of the dynamic response of framed structure. Since the Duhamel integral in which the applied loading is replaced by a piecewise Birkhoff polynomial can be exactly integrated, and since the third and fifth degree piecewise polynomials will fit a curve far better than piecewise linear segments, the proposed solution technique has a far better accuracy than the piecewise linear approximation procedure, for comparable time interval and computational effort.

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2. THE EQUATIONS OF MOTION OF m.d.o.f. SYSTEM

The framed structure to be considered in this paper is an m.d.o.f. system subjected to arbitrary dynamic loadings. At any instant of time t, the differential equations of motion are given by

$$[m]\{\ddot{v}\} + [c]\{\dot{v}\} + [k]\{v\} = \{p(t)\}, \tag{1}$$

where $[m]$, $[c]$ and $[k]$ are the mass, damping, and stiffness matrices, respectively, and $\{\ddot{v}\}$, $\{\dot{v}\}$ and $\{v\}$ the nodal acceleration, velocity and displacement vectors, respectively, and $\{p(t)\}$ the applied loading vector. When the exciting force vector is due to ground acceleration, the inertial force vectors may be given by

$$\{p(t)\} = -[m]\{d\}\ddot{v}_g(t).$$

The vector $\{d\}$ extracts the elements of mass matrix acting along the same axis as the base excitation.

From the eigenproblem, the vibration mode-shape matrix $[\Phi]$ and frequency vector $\{\omega\}$ can be determined. The normal co-ordinate transformation can be used to convert the n coupled linear damped equations of motion to a set of n independent equations of motion given by

$$\ddot{Y}_i + 2\xi_i\omega_i\dot{Y}_i + \omega_i^2 Y_i = F_i(t), \tag{2}$$

where

$$\{\phi_i\}^{\mathrm{T}}[m]\{\phi_i\} = 1, \quad \{\phi_i\}^{\mathrm{T}}[k]\{\phi_i\} = \omega_i^2,$$

$$F_i(t) = \{\phi_i\}^{\mathrm{T}}\{p(t)\} = \sum_{j=1}^{n} \phi_{ji}p_j(t).$$

If the initial velocity and displacement are zero, the general response expression given by the Duhamel integral for each mode is

$$Y_i(t) = \frac{1}{\omega_{Di}}\int_0^t F_i(\tau)\mathrm{e}^{-\xi_i\omega_i(t-\tau)}\sin\omega_{Di}(t-\tau)\,\mathrm{d}\tau = \sum_{j=1}^{n}\phi_{ji}Z_{ji}(t) \tag{3}$$

in which

$$\begin{aligned} Z_{ji}(t) &= \frac{1}{\omega_{Di}}\int_0^t p_j(\tau)\mathrm{e}^{-\xi_i\omega_i(t-\tau)}\sin\omega_{Di}(t-\tau)\,\mathrm{d}\tau \\ &= \frac{\mathrm{e}^{-\xi_i\omega_i t}}{\omega_{Di}}[A_i(t)\sin\omega_{Di}t - B_i(t)\cos\omega_{Di}t], \end{aligned} \tag{4}$$

where

$$\omega_{Di} = \omega_i\sqrt{1-\xi_i^2},$$

$$A(t) = \int_0^t p_j(\tau) e^{\xi_i \omega_i \tau} \cos \omega_{Di} \tau \, d\tau, \tag{5a}$$

$$B(t) = \int_0^t p_j(\tau) e^{\xi_i \omega_i \tau} \sin \omega_{Di} \tau \, d\tau. \tag{5b}$$

The calculation of the Duhamel integral thus requires the evaluation of the integrals $A(t)$ and $B(t)$. In general, integrands $p_j(\tau)e^{\xi_i\omega_i\tau}\cos\omega_{Di}\tau$ and $p_j(\tau)e^{\xi_i\omega_i\tau}\sin\omega_{Di}\tau$ cannot be exactly integrated, unless $p_j(\tau)$ can be expressed as the product of a complex polynomial and a complex exponent. Although numerical integration [5] may be employed in such cases, it cannot provide an exact result. For example, Simpson's rule used in the calculation of the Duhamel integral is equivalent to fitting each set of three consecutive points with a parabola, that is, a second degree polynomial; exact result can be obtained only when both integrands $p_j(\tau)e^{\xi_i\omega_i\tau}\cos\omega_{Di}\tau$ and $p_j(\tau)e^{\xi_i\omega_i\tau}\sin\omega_{Di}\tau$ are polynomials in which the number of degrees is not more than 3. This is implausible, because $e^{\xi_i\omega_i\tau}$, $\sin\omega_{Di}\tau$ or $\cos\omega_{Di}\tau$ can each be expanded in an infinite power-series. Evaluation of the integrals $A(t)$ and $B(t)$ by Simpson's rule is equivalent to approximating the polynomial which is the product of three infinite power-series by a second degree polynomial; consequently, an exact result cannot be obtained by the numerical integration method.

In order to solve the problems mentioned above, one can propose an extension of the procedure in which the dynamic loading is approximated by piecewise linear segments. In the proposed method, one can employ the Birkhoff piecewise interpolation polynomial $p_{jk}^n(\tau)$ for approximating the applied loading $p_j(\tau)$ in the integrals $A(t)$ and $B(t)$, instead of using a piecewise interpolation polynomial for approximating the integrands $p_j(\tau)e^{\xi_i\omega_i\tau}\cos\omega_{Di}\tau$ and $p_j(\tau)e^{\xi_i\omega_i\tau}\sin\omega_{Di}\tau$. Since the integrands $p_{jk}^n(\tau)e^{\xi_i\omega_i\tau}\cos\omega_{Di}\tau$ and $p_{jk}^n(\tau)e^{\xi_i\omega_i\tau}\sin\omega_{Di}\tau$ can be exactly integrated, computational error exists only because of the difference between $p_{jk}^n(\tau)$ and $p_j(\tau)$. Therefore, the error resulting from this is much smaller than that involved in the method in which one treats the entire expression $p_j(\tau)e^{\xi_i\omega_i\tau}\cos\omega_{Di}\tau$ or $p_j(\tau)e^{\xi_i\omega_i\tau}\sin\omega_{Di}\tau$ as one integrand. If the applied loading can be expressed as a piecewise polynomial, no difference exists between $p_{jk}^n(\tau)$ and $p_j(\tau)$, and the proposed method gives an exact result.

An nth degree piecewise polynomial in the kth time interval may be expressed as

$$p_{jk}^n(\tau) = \sum_{l=0}^{n} a_{kl}\tau^l.$$

By substituting $p_{jk}^n(\tau)$ into equation (5a), one can obtain

$$\begin{aligned} A(t) &= \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} p_j(\tau)e^{\xi_i\omega_i\tau}\cos\omega_{Di}\tau\,d\tau + \int_{t_m}^{t} p_j(\tau)e^{\xi_i\omega_i\tau}\cos\omega_{Di}\tau\,d\tau \\ &\approx \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} p_{jk}^n(\tau)e^{\xi_i\omega_i\tau}\cos\omega_{Di}\tau\,d\tau + \int_{t_m}^{t} p_{j(m+1)}^n(\tau)e^{\xi_i\omega_i\tau}\cos\omega_{Di}\tau\,d\tau \\ &= \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \left(\sum_{l=0}^{n} a_{kl}\tau^l\right) e^{\xi_i\omega_i\tau}\cos\omega_{Di}\tau\,d\tau + \int_{t_m}^{t} \left(\sum_{l=0}^{n} a_{(m+1)l}\tau^l\right) e^{\xi_i\omega_i\tau}\cos\omega_{Di}\tau\,d\tau \\ &= \sum_{k=1}^{m}\sum_{l=0}^{n} a_{kl} I_{cl}\big|_{t_{k-1}}^{t_k} + \sum_{l=0}^{n} a_{(m+1)l} I_{cl}\big|_{t_m}^{t}. \end{aligned} \tag{6a}$$

In the same way, one can obtain

$$B(t) \approx \sum_{k=1}^{m} \sum_{l=0}^{n} a_{kl} I_{sl}|_{t_{k-1}}^{t_k} + \sum_{l=0}^{n} a_{(m+1)l} I_{sl}|_{t_m}^{t} \tag{6b}$$

in which

$$t_0 = 0, \quad t_m \leqslant t \leqslant t_{m+1},$$

$$I_{s0} = \int e^{\xi_i \omega_i \tau} \sin \omega_{Di} \tau \, d\tau = \frac{1}{\omega_i^2} e^{\xi_i \omega_i \tau} (\xi_i \omega_i \sin \omega_{Di} \tau - \omega_{Di} \cos \omega_{Di} \tau)$$

$$I_{c0} = \int e^{\xi_i \omega_i \tau} \cos \omega_{Di} \tau \, d\tau = \frac{1}{\omega_i^2} e^{\xi_i \omega_{i1} \tau} (\xi_i \omega_i \cos \omega_{Di} \tau + \omega_{Di} \sin \omega_{Di} \tau)$$

$$I_{sl} = \int \tau^l e^{\xi_i \omega_i \tau} \sin \omega_{Di} \tau \, d\tau = \tau^l I_{s0} - \frac{1}{\omega_i^2} (\xi_i \omega_i I_{s(l-1)} - \omega_{Di} I_{c(l-1)})$$

$$I_{cl} = \int \tau^l e^{\xi_i \omega_i \tau} \cos \omega_{Di} \tau \, d\tau = \tau^l I_{c0} - \frac{1}{\omega_i^2} (\xi_i \omega_i I_{c(l-1)} + \omega_{Di} I_{s(l-1)}).$$

By substituting equations (6) into equation (4), one can obtain $Z_{ji}(t)$; then one can obtain $\dot{Z}_{ji}(t)$ by substituting $A(t)$, $B(t)$ and $Z_{ji}(t)$ into the equation

$$\dot{Z}_{ji}(t) = -\xi_i \omega_i Z_{ji}(t) + e^{-\xi_i \omega_i t} [A(t) \cos \omega_{Di} t + B(t) \sin \omega_{Di} t]. \tag{7}$$

One can obtain $\ddot{Z}_{ji}(t)$ by substituting $Z_{ji}(t)$ and $\dot{Z}_{ji}(t)$ into the equation

$$\ddot{Z}_{ji}(t) = p_j(t) - 2\xi_i \omega_i \dot{Z}_{jt}(t) - \omega_i^2 Z_{ji}(t). \tag{8}$$

For each mode, modal displacement, velocity and acceleration are given, respectively, by

$$Y_i(t) = \sum_{j=1}^{n} \phi_{ji} Z_{ji}(t), \qquad \dot{Y}_i(t) = \sum_{j=1}^{n} \phi_{ji} \dot{Z}_{ji}(t), \qquad \ddot{Y}_i(t) = \sum_{j=1}^{n} \phi_{ji} \ddot{Z}_{ji}(t). \tag{9–11}$$

The displacement, velocity and acceleration vectors expressed in geometric co-ordinate are given, respectively, by the following normal co-ordinate transformations

$$\{v\} = [\Phi]\{Y\}, \qquad \{\dot{v}\} = [\Phi]\{\dot{Y}\}, \qquad \{\ddot{v}\} = [\Phi]\{\ddot{Y}\}. \tag{12–14}$$

From $\{v\}$, $\{\dot{v}\}$ and $\{\ddot{v}\}$, one can obtain the elemental force vector of framed structure.

3. THE BIRKHOFF INTERPOLATION FUNCTION

Birkhoff interpolation [6] allows the function and a variable number of derivative to be used as data at each point. The three different Birkhoff interpolation functions that will be introduced are the first, third and fifth degree Birkhoff interpolation functions. The piecewise linear segments approximation procedure, which will be restated in the following, was presented, respectively, by Wilson and Dovey [1], Nigam and Jennings [2], Ly [3] in different ways.

3.1. THE PIECEWISE FIRST DEGREE BIRKHOFF INTERPOLATION FUNCTION

Let the two sample points be given at the two endpoints of interval t_{k-1} and t_k respectively. The polynomial $p_{jk}^1(\tau)$, which is valid in the kth time interpolation interval $(t_{k-1} \leqslant \tau \leqslant t_k)$, can be expressed using the values $p_j(t_{k-1})$ and $p_j(t_k)$ to represent the coefficients of the first degree polynomial. The function $p_{jk}^1(\tau)$ may be expressed as

$$p_{jk}^1(\tau) = \frac{t_k - \tau}{t_k - t_{k-1}} p_j(t_{k-1}) + \frac{\tau - t_{k-1}}{t_k - t_{k-1}} p_j(t_k) = a_{k0} + a_{k1}\tau$$

in which a_{k0} and a_{k1} may be expressed as

$$a_{k0} = \frac{t_k p_j(t_{k-1}) - t_{k-1} p_j(t_k)}{t_k - t_{k-1}},$$

$$a_{k1} = \frac{p_j(t_k) - p_j(t_{k-1})}{t_k - t_{k-1}}.$$

3.2. THE PIECEWISE THIRD DEGREE BIRKHOFF INTERPOLATION FUNCTION

Let the two sample points be given at the two endpoints of interval t_{k-1} and t_k respectively. The polynomial $p_{jk}^3(\tau)$, which is valid in the kth time interpolation interval $(t_{k-1} \leqslant \tau \leqslant t_k)$, can be expressed using the values $p_j(t_{k-1})$ and $p_j(t_k)$, and first derivatives $p_j'(t_{k-1})$ and $p_j'(t_k)$ to represent the coefficients of the third degree polynomial. The function $p_{jk}^3(\tau)$ may be expressed as

$$\begin{aligned} p_{jk}^3(\tau) = {} & \left[1 - 3\left(\frac{\tau - t_{k-1}}{\Delta t_k}\right)^2 + 2\left(\frac{\tau - t_{k-1}}{\Delta t_k}\right)^3\right] p_j(t_{k-1}) + \left[3\left(\frac{\tau - t_{k-1}}{\Delta t_k}\right)^2\right. \\ & \left. - 2\left(\frac{\tau - t_{k-1}}{\Delta t_k}\right)^3\right] p_j(t_k) + \left[\left(\frac{\tau - t_{k-1}}{\Delta t_k}\right)\left(1 - \frac{\tau - t_{k-1}}{\Delta t_k}\right)^2\right] p_j'(t_{k-1}) \\ & + \left[\left(\frac{\tau - t_{k-1}}{\Delta t_k}\right)^2\left(1 - \frac{\tau - t_{k-1}}{\Delta t_k}\right)\right] p_j'(t_k) \\ = {} & a_{k0} + a_{k1}\tau + a_{k2} + a_{k3}\tau^3 \end{aligned}$$

in which a_{k0}, a_{k1}, a_{k2} and a_{k3} may be obtained by

$$a_{k3} = 2\,\frac{p_j(t_{k-1}) - p_j(t_k)}{\Delta t_k^3} + 2\,\frac{p_j'(t_{k-1}) + p_j'(t_k)}{\Delta t_k^2},$$

$$a_{k2} = 3(t_{k-1} + t_k)\frac{-p_j(t_{k-1}) + p_j(t_k)}{\Delta t_k^3} - \frac{p'_j(t_{k-1})(t_{k-1} + 2t_k) + p'_j(t_k)(2t_{k-1} + t_k)}{\Delta t_k^2},$$

$$a_{k1} = 6t_k t_{k-1}\frac{p_j(t_{k-1}) - p_j(t_k)}{\Delta t_k^3} + \frac{p'_j(t_{k-1})(2t_k^2 + t_{k-1}t_k) + p'_j(t_k)(2t_{k-1}^2 + t_{k-1}t_k)}{\Delta t_k^2},$$

$$a_{k0} = \frac{p_j(t_{k-1})t_k^2(t_k - 3t_{k-1}) + p_j(t_k)t_{k-1}^2(3t_k - t_{k-1})}{\Delta t_k^3} - 2t_{k-1}t_k\frac{p'_j(t_{k-1})t_k + p'_j(t_k)t_{k-1}}{\Delta t_k^2},$$

where

$$\Delta t_k = t_k - t_{k-1}.$$

3.3. THE PIECEWISE FIFTH DEGREE BIRKHOFF INTERPOLATION FUNCTION

Let the two sample points be given at the two endpoints of interval t_{k-1} and t_k respectively. The polynomial $p_{jk}^5(\tau)$, which is valid in the kth time interpolation interval $(t_{k-1} \leqslant \tau \leqslant t_k)$, can be expressed using the values $p_j(t_{k-1})$ and $p_j(t_k)$, first derivatives $p'_j(t_{k-1})$ and $p'_j(t_k)$, and second derivatives $p''_j(t_{k-1})$ and $p''_j(t_k)$ to represent the coefficients of the fifth degree polynomial. The function $p_{jk}^5(\tau)$ may be expressed as

$$p_{jk}^5(\tau) = \sum_{l=0}^{5} a_{kl}\tau^l = a_{k0} + a_{k1}\tau + a_{k2}\tau^2 + a_{k3}\tau^3 + a_{k4}\tau^4 + a_{k5}\tau^5$$

in which a_{k0}, a_{k1}, a_{k2}, a_{k3}, a_{k4} and a_{k5} may be obtained by

$$\begin{bmatrix} 1 & t_{k-1} & t_{k-1}^2 & t_{k-1}^3 & t_{k-1}^4 & t_{k-1}^5 \\ 0 & 1 & 2t_{k-1} & 3t_{k-1}^2 & 4t_{k-1}^3 & 5t_{k-1}^4 \\ 0 & 0 & 2 & 6t_{k-1} & 12t_{k-1}^2 & 20t_{k-1}^3 \\ 1 & t_k & t_k^2 & t_k^3 & t_k^4 & t_k^5 \\ 0 & 1 & 2t_k & 3t_k^2 & 4t_k^3 & 5t_k^4 \\ 0 & 0 & 2 & 6t_k & 12t_k^2 & 20t_k^3 \end{bmatrix} \begin{Bmatrix} a_{k0} \\ a_{k1} \\ a_{k2} \\ a_{k3} \\ a_{k4} \\ a_{k5} \end{Bmatrix} = \begin{Bmatrix} p_j(t_{k-1}) \\ p'_j(t_{k-1}) \\ p''_j(t_{k-1}) \\ p_j(t_k) \\ p'_j(t_k) \\ p''_j(t_k) \end{Bmatrix}.$$

4. NUMERICAL EXAMPLE

For the sake of investigating and demonstrating the efficiency of the proposed method, let us take a four-storey building as an illustration.

The building shown in Figure 1 is treated as a lump–mass framed structure. The masses, which are concentrated at each floor level from the first to the fourth storey, are 160, 160, 160 and 80 metric tons respectively. The Young's modulus is $E = 3 \times 10^7\,\mathrm{kN/m^2}$. The moment of inertia of all the columns is $I_c = 7{\cdot}2 \times 10^{-3}\,\mathrm{m^4}$; that of all the beams is $I_b = 1{\cdot}28 \times 10^{-2}\,\mathrm{m^4}$. The area of all columns is $A_c = 0{\cdot}024\mathrm{m^2}$; that of all beams is $A_b = 0{\cdot}024\,\mathrm{m^2}$. The storey heights from the first to the fourth storey are 5, 4, 4 and 4 m respectively. The spans from left to right are 6, 6 and 6 m respectively. The damping ratio for all modes is assumed to be 0. Determine the response of the framed structure subjected to the following load cases:

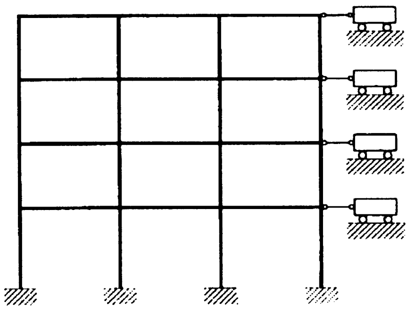

Figure 1. Framed structure considered in example.

(1) when the frame is subjected to a suddenly applied constant acceleration $a_g = -0{\cdot}2g$ at its base;
(2) when the frame is subjected to an acceleration $a_g = -0{\cdot}2g\sin(\pi t)$ at its base.

The adopted means of reducing the number of degrees of freedom is by static condensation and by kinematic constraints which express the displacements of many degrees of freedom in terms of a much smaller set of primary displacement variables. For example, consider the four-storey building frame shown in Figure 1, which includes 48 d.o.f. (one horizontal translation, one vertical translation and one rotation displacement per joint). For plane building frames, one can assume that the beams are inextensible. After considering kinematic constraints, a total of 36 d.o.f. are left (one horizontal translation, four vertical translation and four rotation displacements per storey). Static condensation can reduce the dynamic degrees of freedom to only the horizontal translation displacements. Thus, the final result of this reduction is a total of four dynamic degrees of freedom, only about 8% of the 48 included in the original finite element model. The corresponding ratio concerning the global stiffness matrix is $(4\times 4)/(48\times 48) = 0{\cdot}7\%$. Consequently, there is no considerable difference between the step-by-step method and the proposed method with respect to storage space.

In the following tables, the numbers in parentheses are percentage errors; disp. 1, disp. 2, disp. 3 and disp. 4 denote the first, second, third and fourth storey horizontal translation displacements respectively.

When the structure is subjected to the first load case, some of the results are as shown in Table 1. The time step used in solution A by the Wilson-θ method is $\Delta t = 0{\cdot}02$ s; that in solution B is $\Delta t = 0{\cdot}01$ s; that in solution C is $\Delta t = 0{\cdot}002$ s. The time step used in solution C by the Wilson-θ method is kept extremely small so as to improve the precision of the results. The interpolation functions adopted in the proposed method could be any one of the first [1–3], third or fifth degree piecewise Birkhoff interpolation polynomial, the results obtained in each case are exact results.

When the structure is subjected to the second load case, some of the results are as shown in Table 2. Because exciting forces are sinusoidal, that is

$$p_j(\tau) = p_{0j}\sin\theta_j\tau$$

and because the damping ratios are zero, by substituting the above equation into equation (4), exact results can be obtained as

$$Z_{ji}(t) = \frac{p_{0j}}{\omega_i^2 - \theta_j^2}\left(\sin\theta_j\tau - \frac{\theta_j}{\omega_i}\sin\omega_{Di}t\right).$$

TABLE 1

Horizontal displacement of structure subjected to the first load case

Time (s)	Disp.	Wilson-θ method Solution A ($\Delta t = 0{\cdot}02$ s)	Solution B ($\Delta t = 0{\cdot}01$ s)	Solution C ($\Delta t = 0{\cdot}002$ s)	Proposed method
0·30	Disp.1	0·0292749(0·683%)	0·0294192(0·194%)	0·0294739(0·008%)	0·0294763
	Disp.2	0·0488780(0·476%)	0·0490511(0·123%)	0·0491092(0·005%)	0·0491117
	Disp.3	0·0616596(0·054%)	0·0616941(0·002%)	0·0616929(0·000%)	0·0616927
	Disp.4	0·0669765(0·030%)	0·0669801(0·024%)	0·0669955(0·001%)	0·0669963
2·00	Disp.1	0·0342836(2·125%)	0·0338400(0·804%)	0·0335811(0·033%)	0·0335701
	Disp.2	0·0571403(1·336%)	0·0564454(0·104%)	0·0563866(0·000%)	0·0563868
	Disp.3	0·0717079(0·121%)	0·0716285(0·231%)	0·0717893(0·008%)	0·0717947
	Disp.4	0·0776529(1·162%)	0·0787225(0·199%)	0·0785734(0·096%)	0·0785659
11·5	Disp.1	0·0326974(6·311%)	0·0351768(0·793%)	0·0349162(0·046%)	0·0349000
	Disp.2	0·0587430(2·795%)	0·0575974(0·790%)	0·0572200(0·130%)	0·0571458
	Disp.3	0·0660612(7·104%)	0·0706930(0·591%)	0·0710839(0·041%)	0·0711134
	Disp.4	0·0786908(1·868%)	0·0758713(1·782%)	0·0771375(0·014%)	0·0772477
33·0	Disp.1	8·13198	0·0349742(5·308%)	0·0332308(0·059%)	0·0332113
	Disp.2	−14·3929	0·0579661(4·412%)	0·0553747(0·026%)	0·0555165
	Disp.3	15·1294	0·0707632(0·796%)	0·0702596(0·079%)	0·0702041
	Disp.4	−13·5707	0·0761942(0·537%)	0·0769791(0·049%)	0·0766055
69·5	Disp.1	579·857	0·0346989(1·179%)	0·0341079(0·544%)	0·0342944
	Disp.2	−3546·11	0·0563789(1·152%)	0·0567095(0·572%)	0·0570360
	Disp.3	−2445·17	0·0706052(1·578%)	0·0717508(0·019%)	0·0717372
	Disp.4	−7897·96	0·0744816(4·636%)	0·0787479(0·083%)	0·0781021

In order to obtain more accurate results, the time interval of the base acceleration used by the Wilson-θ method is also 0·002 s. For the first, third and fifth degree piecewise interpolation polynomials, interpolation with equal time interval, denoted by Δt, is adopted. With identical time intervals, the accuracy obtained with the third and fifth degree polynomials is far better than that obtained with a linear polynomial. For evaluating the dynamic response at $t = 50$ s, one needs only 101 calculations when using the proposed fifth degree polynomial approximation procedure, but more than 25 001 calculations when using the Wilson-θ method, for comparable accuracy.

The computational time depends principally on the variation of loads. In this paper, the computation is done with Pentium III 667 (128RAM). For the second load case, the total computational time at $t = 50$ s consumed by the Wilson-θ method is 101·28 s. Those consumed by the proposed method are 0·28 s (in solution A by first degree), 0·37 s (in solution B by first degree), 0·32 s (in solution A by third degree), 0·46 s (in solution B by third degree), and 0·48 s (by fifth degree) respectively. These results demonstrate that the proposed method is computationally more efficient than the traditional method for long-duration excitings.

Although low damping ratios which are typical of most practical structures remove energy from the dynamically responding system and reduce the transient response, they almost do not influence the accuracy for the structure of the numerical example; however, they may affect computational effort. In the numerical example, zero damping ratios are assumed. In such cases, the direct solution avoids the potentially very large cost of the modal co-ordinate evaluation, because the damping matrix is generally computed from the modal matrix together with the specified damping ratios when they are non-zero. For

TABLE 2

Horizontal displacement of structure subjected to the second load case

Time (s)	Disp.	Wilson-θ method ($\Delta t = 0{\cdot}002$ s)	First degree Solution A($\Delta t = 0{\cdot}5$ s)	First degree Solution B($\Delta t = 0{\cdot}25$ s)	Third degree Solution A($\Delta t = 0{\cdot}5$ s)	Third degree Solution B($\Delta t = 0{\cdot}25$ s)	Fifth degree ($\Delta t = 0{\cdot}5$ s)	Exact result
0·50	Disp. 1	0·0268030(0·001%)	0·0209822(21·72%)	0·0254759(4·952%)	0·0265372(0·992%)	0·0267894(0·051%)	0·0267982(0·019%)	0·0268032
	Disp. 2	0·0439445(0·000%)	0·0341263(22·34%)	0·0417323(5·035%)	0·0434943(1·025%)	0·0439215(0·052%)	0·0439362(0·019%)	0·0439447
	Disp. 3	0·0546682(0·002%)	0·0422057(22·79%)	0·0518807(5·101%)	0·0540958(1·049%)	0·0546398(0·054%)	0·0546583(0·020%)	0·0546691
	Disp. 4	0·0592363(0·002%)	0·0456174(22·99%)	0·0561905(5·144%)	0·0586105(1·059%)	0·0592058(0·054%)	0·0592259(0·020%)	0·0592377
3·50	Disp. 1	−0·0253645(0·007%)	−0·0191896(24·33%)	−0·0241725(4·693%)	−0·0250727(1·144%)	−0·0253508(0·047%)	−0·0253573(0·022%)	−0·0253628
	Disp. 2	−0·0415681(0·014%)	−0·0310626(25·26%)	−0·0394980(4·966%)	−0·0410697(1·185%)	−0·0415407(0·051%)	−0·0415528(0·022%)	−0·0415621
	Disp. 3	−0·0517548(0·020%)	−0·0383135(25·96%)	−0·0490428(5·221%)	−0·0511161(1·214%)	−0·0517157(0·055%)	−0·0517324(0·023%)	−0·0517443
	Disp. 4	−0·0561286(0·021%)	−0·0413846(26·25%)	−0·0531095(5·359%)	−0·0554288(1·226%)	−0·0560847(0·057%)	−0·0561038(0·023%)	−0·0561168
13·50	Disp. 1	−0·0208039(0·070%)	−0·0132495(36·36%)	−0·0198442(4·680%)	−0·0204457(1·790%)	−0·0208088(0·046%)	−0·0208112(0·035%)	−0·0208184
	Disp. 2	−0·0338899(0·084%)	−0·0210279(38·00%)	−0·0321860(5·107%)	−0·0332857(1·865%)	−0·0339004(0·053%)	−0·0339061(0·036%)	−0·0339183
	Disp. 3	−0·0420192(0·099%)	−0·0255418(39·27%)	−0·0397360(5·527%)	−0·0412537(1·919%)	−0·0420360(0·059%)	−0·0420454(0·037%)	−0·0420609
	Disp. 4	−0·0454984(0·123%)	−0·0274204(39·81%)	−0·0429345(5·751%)	−0·0446702(1·941%)	−0·0455260(0·063%)	−0·0455376(0·037%)	−0·0455545
32·50	Disp. 1	0·0248316(0·220%)	0·0184683(25·46%)	0·0236431(4·577%)	0·0244777(1·208%)	0·0247659(0·045%)	0·0247713(0·023%)	0·0247771
	Disp. 2	0·0407272(0·193%)	0·0298955(26·45%)	0·0386503(4·917%)	0·0401402(1·251%)	0·0406282(0·051%)	0·0406391(0·024%)	0·0406488
	Disp. 3	0·0507085(0·170%)	0·0368163(27·27%)	0·0479540(5·272%)	0·0499732(1·283%)	0·0505943(0·056%)	0·0506103(0·024%)	0·0506226
	Disp. 4	0·0549675(0·134%)	0·0397159(27·65%)	0·0518888(5·474%)	0·0541819(1·297%)	0·0548615(0·059%)	0·0548803(0·024%)	0·0548937
49·50	Disp. 1	−0·0280515(0·187%)	−0·0227360(18·80%)	−0·0267524(4·453%)	−0·0277597(0·855%)	−0·0279869(0·043%)	−0·0279946(0·016%)	−0·0279991
	Disp. 2	−0·0461703(0·122%)	−0·0371105(19·52%)	−0·0439081(4·784%)	−0·0457069(0·883%)	−0·0460916(0·049%)	−0·0461066(0·016%)	−0·0461142
	Disp. 3	−0·0576567(0·097%)	−0·0459981(20·14%)	−0·0546451(5·131%)	−0·0570798(0·905%)	−0·0575695(0·054%)	−0·0575912(0·017%)	−0·0576008
	Disp. 4	−0·0626074(0·117%)	−0·0497511(20·44%)	−0·0592003(5·332%)	−0·0619628(0·914%)	−0·0624987(0·057%)	−0·0625240(0·017%)	−0·0625344

the systems with viscous damping, the proposed method can offer even more computational advantage than the step-by-step integration method, for the reasons that the latter method also requires the calculation of the modal matrix and that the response analysis for the individual modal equations requires very little computational effort with the former method.

It is shown that the proposed method can evaluate with much higher accuracy the dynamic response of a multi-storey framed structure to long-duration loads which each varies in the form of an arbitrary curve when compared with the piecewise linear approximation method or the step-by-step integration method, and that for comparable accuracy the proposed method requires less computational effort than the latter two.

5. CONCLUSIONS

The response of a multi-storey framed structure is determined by means of the modal superposition method and the Duhamel integral. Firstly, one can approximate every applied loading $p_j(t)$ with a piecewise Birkhoff interpolation polynomial. Secondly, one can integrate precisely the Duhamel integrals for each mode and for each loading. Thirdly, one can calculate the displacements, velocities and accelerations expressed in geometric co-ordinate by the normal co-ordinate transformation. Finally, one can obtain the elemental forces. The equations associated with the first, third and fifth degree piecewise Birkhoff interpolation polynomials are presented. For the applied loadings which each can be represented by a piecewise polynomial in which the number of degrees is not more than five, an exact result is obtained from the proposed method; for the applied loadings that do not satisfy the conditions mentioned above, an exact result cannot be obtained from this method, but the error is rather small. The proposed method has a higher accuracy than the traditional step-by-step integration procedure and the piecewise linear approximation procedure in the case when each loading varies in the form of a curve. For long-duration loads, the proposed method requires much less computational effort as compared to the effort involved in the step-by-step integration method. For example, the computational time consumed by the proposed method is less than 0·5% of the time consumed by the step-by-step method for comparable accuracy and storage space, for the framed structure in numerical example. Unlike the step-by-step integration solution technique, the error brought by the proposed method does not remarkably increase in the order of magnitude as the time of response goes on, even when the structure is subjected to an extremely long duration loading. The proposed method not only has very high accuracy at a very large interpolation interval, it also converges to the exact result rapidly as the interpolation interval is reduced. The proposed method can not only provide benefits in the systems that have a few degrees of freedom, but also offers a very great advantage in that an adequate estimate of the dynamic response can often be obtained by considering only a few modes of vibration, even in the systems that may have dozens or hundreds of degrees of freedom; thus the computational effort may be reduced significantly.

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