Journal of Sound and Vibration (2002) **252**(2), 191–221
doi:10.1006/jsvi.2000.3553, available online at http://www.idealibrary.com on IDEAL®

DIFFRACTION BY A HARD HALF-PLANE: USEFUL APPROXIMATIONS TO AN EXACT FORMULATION

D. OUIS

Department of Engineering Acoustics, LTH, Lund University, Box 118, *S*-221 00 *Lund, Sweden. E-mail*: *djamel.ouis@kstr.lth.se*

(*Received* 20 *December* 1999, *and in final form* 18 *December* 2000)

In this paper, the problem of diffraction of a spherical wave by a hard half-plane is considered. The starting point is the Biot–Tolstoy theory of diffraction of a spherical wave by a fluid wedge with hard boundaries. In this theory, the field at a point in the fluid is composed eventually of a geometrical part: i.e., a direct component, one or two components due to the reflections on the sides of the hard wedge, and a diffracted component due exclusively to the presence of the edge of the wedge. The mathematical expression of this latter component has originally been given in an explicit closed form for the case of a unit momentum wave incidence, but Medwin has further developed its expression for the more useful case of a Dirac delta point excitation. The expression of this form is given in the time domain, but it is quite difficult to find exactly its Fourier transform for studying the frequency behaviour of the diffracted field. It is thus the aim of this paper to present various useful approximations of the exact expression. Among the approximations treated, three are most accurate for engineering purposes, and one of them is proposed, for its simplicity, as appropriate for most occurring practical situations.

1. INTRODUCTION

The problem of diffraction of a wave by a half-plane has been the subject of interest of physicists and mathematicians for some centuries. The simple theory of geometrical optics fails to describe certain phenomena occurring when light rays propagate over sharp edges. In fact, an observer moving from the light to the dark regions of space does not really experience a sudden change of light conditions but this transition may occur in a progressive way with the possible appearance of dark and bright bands: the diffraction fringes. Probably, the first mention of this phenomenon dates back to Leonardo da Vinci and a first accurate description of it is due to the Italian Jesuit Professor F. M. Grimaldi in the early 1660s, who used the word "diffractio" to describe the bending of a wave whenever it is obstructed in some way. On the other hand, Newton, considering the corpuscular nature of light, attributed the phenomenon of diffraction to the possible attraction of the light particles by the diffracting edge. This notion was also taken up by Young when in 1807 he presented his own theory of diffraction.

The problems arising in diffraction by infinitely thin surfaces delimited by smooth edges were first solved in a satisfactory manner only after the arrival of Kirchhoff's integral equation. This integral equation is the mathematical formulation of Huygens' principle which stipulates that for a propagating wave, every point on a primary wavefront serves as the source of spherical secondary sources such as the fact that the primary wavefront at some later time is the envelope of these wavelets. Moreover, the secondary wavelets advance in the medium at a speed and frequency equal to those of the primary wave. Huygens'

0022-460X/02/$35.00

principle is inadequate by itself as it fails to account for the process of diffraction. This difficulty, being due to the ignorance of any wavelength consideration, was first solved by Fresnel when in 1818 he considered the nature of the mutual interference between the secondary wavelets. The Huygens–Fresnel principle adds that the amplitude of the field at any point beyond the wavefront is the superposition of all the wavelets. Yet the Huygens–Fresnel principle is purely hypothetical; it gives a satisfactory qualitative description to a limited class of simple diffraction problems [1].

Hence, starting from the Helmholtz equation for the propagation of a monochromatic scalar wave, and using Green's theorem, Kirchhoff succeeded in 1883 to put the Huygens–Fresnel principle on a sound mathematical basis. For a crude approximation of the field diffracted by apertures in thin black screens, Kirchhoff makes two further assumptions to his integral formula: the strength of the field at the opening is that of the field incident on it in the absence of the screen and is null at the shadowed face of the screen. Thus, in respect to the mathematical operations, the integration of Kirchhoff's formula becomes instead limited only to the surface of the aperture, a formulation usually known as that of Fresnel–Kirchhoff. It is worth noting that diffraction problems with complementary surfaces have similar solutions; this is known as the Babinet principle. Unfortunately, Kirchhoff's solution to this last formulation did not satisfy the reasonably assumed boundary conditions. But the fact that his theory agreed quite well with experiment, especially in the so-called Fresnel diffraction, where the line source–observer is not appreciably remote from the diffracting edge, made most authors consider Kirchhoff's solution as a fairly accurate first approximation [2]. Due to the non-exactitude of his two assumptions, Kirchhoff's formula had thus to undergo many refinements and modifications during the years [3]. Among these new improvements, a high-frequency asymptote due to Rubinowicz in 1917 states that in Kirchoff's formula for the aperture, the total field could be decomposed into a geometrical part and a part made of a contour integral along the rim of the aperture. This edge diffraction concept laid afterwards, in 1953, the ground for the Geometrical Theory of Diffraction [4].

On the other hand, in 1896 Sommerfeld succeeded in giving a rigorous solution to the two-dimensional problem of diffraction of a plane wave by a half-plane. The fame of this achievement is due partly to the skill with which the solution was constructed in terms of many valued functions and to that it could exactly and simply be given in terms of the Fresnel integrals which were used in previous approximate theories [5]. Many mathematicians followed Sommerfeld's approach and generalized the particular case of the half-plane to the more general one of the wedge, and from that of the plane wave incidence to that of the line or point sources; see, for instance, reference [6] for more details. In acoustics, the development of theories for solving the more general problem of scattering by wedge-shaped obstacles is often motivated by the urge of having at hand calculation schemes for predicting the performance of simple noise barriers. Furthermore, these theoretical models become even more attractive when they can handle the case of any boundary conditions on the faces of the wedge. In this regard, it is worth mentioning the recent work published by Mechel, in which the author considers the problem of scattering of an incident wave by a wedge with either absorbing [7] or hard flanks [8]. The method of attack is based on modal expansions of the sound field in the space bounded by the wedge, and applications of practical importance include the shielding of urban noise by buildings, or that of vehicles along road traffic lines (regarding this latter, theoretical calculations enable one, for instance, to prove the improvement of the insertion loss of a hard corner [9] or a thin hard barrier [10] when the diffracting edge is covered by an absorbing cylinder).

The use of normal modes as generalized co-ordinates in Hilbert space is common in acoustics to deal with the vibration of enclosures. Almost four decades ago, this method was

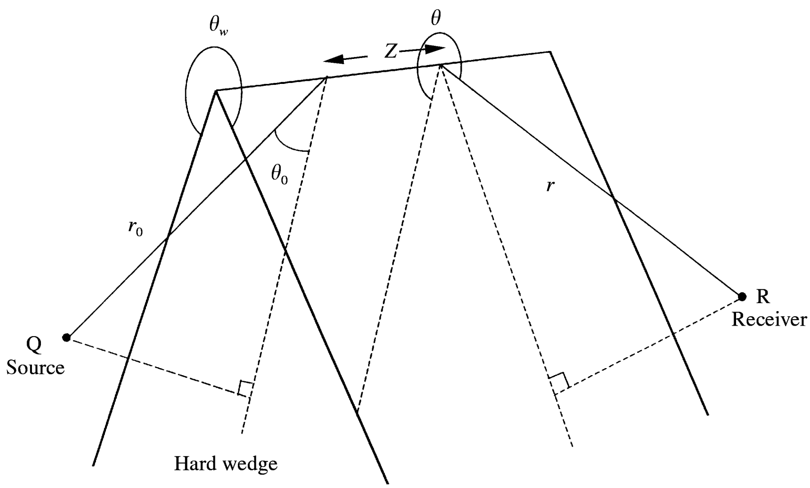

Figure 1. Geometry of the problem of diffraction of a spherical wave by a hard wedge.

extended to deal with unlimited or partially limited mechanical media by first solving the problem for an enclosure and then expanding some or all of its boundaries to infinity [11]. In their early landmark paper, Biot and Tolstoy showed furthermore how to decompose any kind of disturbance into a spectrum of generalized forces. When these two formulations are applied together to the case of diffraction of a spherical explosive pulse in a dissipationless fluid by a hard wedge, an explicit solution of the diffraction problem given in a closed form is obtained which, furthermore, uses only elementary functions. At that time no solution to such an apparently simple problem was presented. The attractiveness of this solution in acoustics lies in its time domain formulation, where, as opposite to the modal representation of the field, the presence of the reverberant decay is more strongly enhanced. In practical applications, using diffraction in its time formulation permits with the help of digital Fourier transform techniques to process important room acoustical descriptors as assessed from the impulse response. Unfortunately, the elegant Biot–Tolstoy theory of diffraction remained a long time in the acoustic literature without a real appreciation of its advantages until 1981 when Medwin showed its usefulness for predicting noise shadowing by finite thin hard barriers [12]. Through simple mathematical manipulations, he was able to express the Biot–Tolstoy solution for the instantly doublet point source: i.e., an infinite compression followed instantly by an infinite rarefaction, to that of a single infinite Dirac-like pulse.

2. THE BIOT–TOLSTOY DIFFRACTION THEORY APPLIED TO THE HARD HALF-PLANE

Figure 1 shows an infinitely rigid wedge subtending a fluid with density ρ, and containing a point source Q and a receiver R.

According to Biot and Tolstoy (B–T) diffraction theory, the acoustical field at the receiver position R due to the point source Q would be made of three components: a direct field, a reflected field and a diffracted field. The possible contribution of the two former components is dictated, respectively, by the fact that the receiver "sees" the source or its image(s) through the side(s) of the wedge. It is consequently decided under pure geometrical considerations. On the other hand, the diffracted component is present everywhere in the space filled by the fluid.

Hence, consider that at Q a point-like delta function of pressure is radiated,

$$u = \frac{\rho S}{4\pi d}\,\delta\left(t - \frac{d}{c}\right), \tag{1}$$

in which S is the strength of the source, i.e., the source volume flow, and c is the speed of propagation of sound. In equation (1) d is the distance from the source and δ the Dirac delta function. The total field is equal to the sum of the geometrical components and the edge diffracted field: that is, $u_{tot} = u_i + u_r + u_d$. The geometrical optics contributions to the total field are u_i and u_r, respectively, the incident, direct wave from the source, and the reflected one, seeming to emanate from the source image of the real source through one, or both faces of the wedge. These two components are expressed as $u_i = \varepsilon_i \rho S \delta(t - d_i/c)/(4\pi d_i)$ and $u_r = \varepsilon_r \rho S \delta(t - d_r/c)/(4\pi d_r)$ for, respectively, the direct and reflected waves, d_i and d_r being the distances to the field point from the real sound source and its image. The coefficient ε_i (ε_r) is equal to one whenever the field point "sees" the real sound source (its image through the face of wedge), and zero otherwise. The diffracted wave due to the tip of the wedge appears at a time τ_0 after the source has emitted its spherically divergent pulse:

$$\tau_0 = [(r + r_0)^2 + z^2]^{1/2}/c. \tag{2}$$

Here τ_0 is the least travel time over the wedge, and is given by the travel time of the wave in its shortest travel path from the point source to the field point via the crest line of the wedge. This quantity appears also in the classical theory of diffraction when in Rubinowicz's representation of Kirchhoff's formula for the aperture in an infinite plane, the surface integral over the area of the aperture is replaced by a line integral over its rim; see, for instance, reference [1] p. 452. A practical consequence of this is the well-known fact in experimental physics that the sharp straight edge of a metallic sheet presents a bright portion when illuminated by a small light source and observed from within the shadowed side. For the wedge, the diffracted field $u_d(t)$ is given by [11, 12]

$$u_d(t) = \frac{-S\rho c}{4\pi\theta_W}\{\beta\}\frac{\exp(-vy)}{rr_0\sinh(y)}, \tag{3}$$

where

$$y = \operatorname{arccosh}\frac{c^2t^2 - (r^2 + r_0^2 + z^2)}{2rr_0} \tag{4}$$

and

$$\{\beta\} = \frac{\sin[v(\pi \pm \theta \pm \theta_0)]}{1 - 2\exp(-vy)\cos[v(\pi \pm \theta \pm \theta_0)] + \exp(-2vy)}, \quad v = \pi/\theta_W. \tag{5}$$

Actually, the curly bracket $\{\beta\}$ is the sum of four terms due to the four possible combinations of the signs in $(\pi \pm \theta \pm \theta_0)$. The quantity v is a wedge index, and has the value 1/2 for the half-plane.

For the half-plane, $\theta_w = 2\pi$ and equation (3) is developed in Appendix A for the important case of the half-plane with $z = 0$. The final result is expressed as

$$u_d(t) = \frac{-S\rho}{4\pi^2 c}\sqrt{\frac{t_+^2 - t_-^2}{t^2 - t_+^2}}\left\{\frac{\cos \pm}{t^2 - t_+^2 + (t_+^2 - t_-^2)\cos \pm\ ^2}\right\}, \tag{6}$$

where $t_{\pm} = (r \pm r_0)/c$ and $\cos\pm = \cos[(\theta \pm \theta_0)/2]$. For writing convenience, the factor $\{\ \}_+$ in equation (6) is the sum of two terms corresponding to the different signs in the argument of the trigonometric function. It may also be noted that in the present case where the source and the receiver lie in the same plane normal to the edge of the half-plane on has $t_+ = \tau_0$.

2.1. MEDWIN'S FIRST APPROXIMATION

Since often the theoretical and experimental results are presented in the frequency domain, the Fourier transform,

$$u_d(f) = FT[u_d(t)] = \int_{\tau_0}^{\infty} u_d(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t \tag{7}$$

of $u_d(t)$ is needed. With this definition of the Fourier transform, the temporal dependence of the fields becomes $\mathrm{e}^{+\mathrm{i}\omega t}$, and this factor will be omitted throughout. Evaluating $u_d(f)$ exactly is very cumbersome, if possible at all, and approximations are therefore necessary. To circumvent this problem, it is informative to note from equation (6) that most of the signal information, i.e., its energy content, lies in proximity of the least time τ_0. Thus, Medwin proposed to decompose the time signal into two parts, one having a simple form with a known exact Fourier transform, and the other component being just the left part of the diffracted field, and the Fourier transform of which is to be made digitally [12, 13]. This latter has, of course, to be truncated somewhere in the time domain depending on the working frequency range, but mostly on the behaviour of the diffracted field, hence depending on the geometry of the problem. Hence, developing $u_d(t)$ in equation (6) for $\tau = t - \tau_0 \ll \tau_0$ gives a first approximation which one may call u_{d1},

$$u_{d1}(\tau) = \frac{-S\rho}{4\pi^2 c}\frac{1}{\sqrt{2t_+(t_+^2 - t_-^2)}}\left\{\frac{1}{\cos\pm}\right\}_+ \frac{1}{\sqrt{\tau}}, \tag{8}$$

i.e.,

$$u_{d1}(\tau) \sim 1/\sqrt{\tau}, \tag{9}$$

a form better suited for calculations. The Fourier transform of this last expression is

$$FT[u_{d1}(\tau)] = u_{d1}(f) = \int_{\tau_0}^{\infty} u_{d1}(t - \tau_0)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t. \tag{10}$$

With the change of variable $\tau = t - \tau_0$ one obtains

$$u_{d1}(f) \sim \mathrm{e}^{\mathrm{i}\omega\tau_0}\int_0^{\infty} u_{d1}(\tau)\,\mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\tau, \tag{11}$$

which, when decomposed into two (cos and sin) Fourier transforms leads to (reference [14], f. 17.33.3, p. 1150)

$$u_{d1}(f) = \frac{-S\rho}{4\pi^2 c}\frac{1}{\sqrt{2t_+(t_+^2 - t_-^2)}}\left\{\frac{1}{\cos\pm}\right\}_+ \mathrm{e}^{\mathrm{i}\omega\tau_0}\frac{1+\mathrm{i}}{2\sqrt{f}}. \tag{12}$$

2.2. AN IMPROVED FIRST APPROXIMATION

With the same calculation strategy in mind, a new approximation of the initial short time range diffracted field is suggested here. One can indeed develop equation (6) as

$$u_d(\tau) = \frac{-S\rho}{4\pi^2 c}\sqrt{t_+^2 - t_-^2}\,\frac{1}{\tau\sqrt{1+2t_+/\tau}}\left\{\frac{\cos\pm}{\tau^2(1+2t_+/\tau)+(t_+^2-t_-^2)\cos\pm^2}\right\}_+ \tag{13}$$

and upon taking into consideration that for comparatively short times after the least time t_+ one can consider that $2t_+/\tau \gg 1$, one then gets a better approach which one can call u_{d2}, and which is given by

$$u_{d2}(\tau) = \frac{-S\rho}{4\pi^2 c}\sqrt{\frac{t_+^2 - t_-^2}{2t_+}}\,\frac{1}{2t_+}\,\frac{1}{\sqrt{\tau}}\left\{\frac{\cos\pm}{\tau+(t_+^2-t_-^2)/2t_+\cos\pm^2}\right\}_+ \tag{14}$$

with all the parameters as defined earlier.

Calculations show that the form (14) is more efficient than form (8) both for the short and the long range time diffracted field. In the frequency domain, Fourier transforming the expression in equation (14) leads to the sum of two terms of the form

$$\mathrm{e}^{\mathrm{i}\omega\tau_0}\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}\omega\tau}}{\sqrt{\tau}(\tau+a)}\,\mathrm{d}\tau, \tag{15}$$

which when performed, and for $\theta \neq \pi \pm \theta_0$, yields (reference [15], f. 2.1.3, p. 16)

$$u_{d2}(f) = \frac{-S\rho}{4\pi^2 c}\sqrt{\frac{t_+^2 - t_-^2}{2t_+}}\,\frac{\pi\mathrm{e}^{\mathrm{i}\omega\tau_0}}{2t_+}\left\{\frac{\cos\pm\ \mathrm{e}^{-\mathrm{i}\omega a\pm}}{\sqrt{a_\pm}}\,\mathrm{erfc}(\sqrt{-\mathrm{i}\omega a_\pm})\right\}_+, \tag{16}$$

where

$$a_\pm = (t_+^2 - t_-^2)\cos\pm^2/(2t_+) \geqslant 0, \tag{17}$$

$\omega = 2\pi f$ and erfc is the complementary error function with a complex argument which is examined in Appendix B. For the argument of the erfc function, the sign of the square root of the complex quantity is taken as positive, i.e., $\sqrt{-\mathrm{i}\omega a_\pm} = +(1-\mathrm{i})\sqrt{\omega a_\pm/2}$. This ensures the correct behaviour of the diffracted field in the vicinity of, and on jumping over the geometrical boundaries: namely, that the amplitude of the diffracted field increases when approaching the boundaries, a property which is not fulfilled for a negative argument. The expression in equation (16) is composed of two terms, one with a_+, which changes sign at the reflection boundary ($\cos(\theta+\theta_0)/2 = 0$ for $\theta = \pi - \theta_0$), and one with a_-, changing sign at the incidence boundary ($\cos(\theta-\theta_0)/2 = 0$ for $\theta = \pi + \theta_0$). The function erfc(z) takes on small values for large amplitudes of the argument z (high frequencies and/or away from the geometrical boundaries), and erfc(z) $\to 1$ for $z \to 0$ and consequently, at each geometrical boundary, the corresponding term becomes more significant than the other term. It may be pointed out that the function $\mathrm{e}^{-z^2}\,\mathrm{erfc}(-\mathrm{i}z)$ is also known as the plasma dispersion function and finds applications in other branches of physics.

2.3. THIRD APPROXIMATION FORMULAE

These two approximations are built from the development of the last term in equation (6). First, if in this latter, the temporal term is neglected in front of the other terms, i.e., for $\tau^2 + 2t_+\tau \ll (t_+^2 - t_-^2)\cos\pm^2$, then

$$u_{d3}(\tau) = \frac{-S\rho}{4\pi^2 c}\frac{1}{\sqrt{t_+^2 - t_-^2}}\frac{1}{\sqrt{\tau(\tau + 2t_+)}}\left\{\frac{1}{\cos\pm}\right\}_+. \tag{18}$$

The frequency form of equation (18) may then be expressed according to (reference [14], f. 1.2.15, p. 14)

$$u_{d3}(f) = \frac{-S\rho}{4\pi^2 c}\frac{1}{\sqrt{t_+^2 - t_-^2}}\left\{\frac{1}{\cos\pm}\right\}_+ \mathrm{K}_0(-\mathrm{i}\omega t_+), \tag{19}$$

in which K_0 is the modified Hankel function of order 0, and which for the case at hand with a pure imaginary argument can be reformulated in terms of the Bessel functions of first and second kinds, J_0 and Y_0, respectively, as (reference [16], f. 9.6.4, p. 375).

$$\mathrm{K}_0(-\mathrm{i}\omega t_+) = -\frac{\pi}{2}[\mathrm{Y}_0(\omega t_+) - \mathrm{i}\mathrm{J}_0(\omega t_+)]. \tag{20}$$

2.4. FOURTH APPROXIMATION FORMULAE

The fourth approximation is obtained by considering instead the square root temporal term in equation (6) and leaving the last term in its original form; i.e., by using $\sqrt{\tau(\tau + 2t_+)} \approx \sqrt{2t_+\tau}$, one obtains

$$u_{d4} = \frac{-S\rho}{4\pi^2 c}\sqrt{\frac{t_+^2 - t_-^2}{2t_+}}\frac{1}{\sqrt{\tau}}\left\{\frac{\cos\pm}{t^2 - t_+^2 + (t_+^2 - t_-^2)\cos\pm^2}\right\}_+. \tag{21}$$

To make it easier to find the Fourier transform of this last expression, the last term is developed into the sum of two terms, namely,

$$\left\{\frac{1}{\tau^2 + 2t_+\tau + (t_+^2 - t_-^2)\cos\pm^2}\right\} = \frac{1}{2\sqrt{\Delta'_\pm}}\left\{\frac{1}{\tau + \tau_1} - \frac{1}{\tau + \tau_2}\right\} = \frac{1}{2\sqrt{\Delta'_\pm}}\left\{\frac{1}{\tau + \tau_{1,2}}\right\}_-, \tag{22}$$

where it is understood from the notation $\{\}_-$ that it is the difference between two terms with respectively τ_1 and τ_2. In equation (22),

$$\Delta'_\pm = t_+^2 - (t_+^2 - t_-^2)\cos\pm^2 = t_+^2\sin\pm^2 + t_-^2\cos\pm^2 \geqslant 0 \tag{23}$$

and

$$\tau_{1,2} = t_+ \mp \sqrt{\Delta'_\pm}. \tag{24}$$

Hence, equation (21) may be written again as

$$u_{d4}(\tau) = \frac{-S\rho}{4\pi^2 c}\sqrt{\frac{t_+^2 - t_-^2}{2t_+}}\frac{1}{2\sqrt{\tau}}\left\{\frac{\cos\pm}{\sqrt{\Delta'_\pm}}\left\{\frac{1}{\tau + \tau_{1,2}}\right\}_-\right\}_+, \tag{25}$$

the frequency form of which is, according to equation (16), given by

$$u_{d4}(f) = \frac{-S\rho}{4\pi^2 c}\sqrt{\frac{t_+^2 - t_-^2}{2t_+}}\frac{\pi e^{i\omega\tau_0}}{2}\left\{\frac{\cos\pm}{\sqrt{\Delta'_\pm}}\left\{\frac{e^{-i\omega\tau_{1,2}}}{\sqrt{\tau_{1,2}}}\,\mathrm{erfc}\,(\sqrt{-i\omega\tau_{1,2}})\right\}_-\right\}_+, \tag{26}$$

where the quantities $\tau_{1,2}$ should be non-negative. This is directly seen for τ_2 in equation (24) and can easily be verified for τ_1.

2.5. FIFTH APPROXIMATION FORMULAE

In the light of the developments in the previous section, one can then write equation (6) in the more suitable form

$$u_{d5}(\tau) = \frac{-S\rho}{4\pi^2 c}\sqrt{t_+^2 - t_-^2}\,\frac{1}{2\sqrt{\tau}\sqrt{\tau + 2t_+}}\left\{\frac{\cos\pm}{\sqrt{\Delta'_\pm}}\left\{\frac{1}{\tau + \tau_{1,2}}\right\}_-\right\}_+, \tag{27}$$

and a new approximation may be found by considering the short time development of the term $1/\sqrt{\tau + 2t_+}$, i.e., (reference [17], f. 12:6:2, p. 94)

$$\frac{1}{\sqrt{\tau + 2t_+}} = \frac{1}{\sqrt{2t_+}}\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(\frac{\tau}{2t_+}\right)^n \quad \text{for } \tau \leqslant 2t_+ \tag{28}$$

and upon considering the first two terms in this last development, $u_{d5}(\tau)$ is then given by

$$u_{d5}(\tau) = u_{d4} + \frac{S\rho}{4\pi^2 c}\sqrt{\frac{t_+^2 - t_-^2}{2t_+}}\frac{1}{8t_+}\left\{\frac{\cos\pm}{\sqrt{\Delta'_\pm}}\left\{\frac{\sqrt{\tau}}{\tau + \tau_{1,2}}\right\}_-\right\}_+, \tag{29}$$

of which the Fourier transform is given by (reference [15], f. 2.1.2, p. 16)

$$u_{d5}(f) = -\frac{S\rho}{4\pi^2 c}\sqrt{\frac{t_+^2 - t_-^2}{2t_+}}\frac{\pi e^{i\omega\tau_0}}{2}$$
$$\times\left\{\frac{\cos\pm}{\sqrt{\Delta'_\pm}}\times\left\{\left[\frac{1}{\sqrt{\tau_{1,2}}} + \frac{\sqrt{\tau_{1,2}}}{4t_+}\right]e^{-i\omega\tau_{1,2}}\,\mathrm{erfc}(\sqrt{-i\omega\tau_{1,2}}) - \frac{1}{4t_+\sqrt{-i\pi\omega}}\right\}_-\right\}_+. \tag{30}$$

This fifth approximation can even be improved by taking the next term in the serial development in equation (28) and one then gets in the frequency domain

$$u_{d5}(f)' = -\frac{S\rho}{4\pi^2 c}\sqrt{\frac{t_+^2 - t_-^2}{2t_+}}\frac{\pi e^{i\omega\tau_0}}{2}$$
$$\times\left\{\frac{\cos\pm}{\sqrt{\Delta'_\pm}}\left\{\begin{array}{l}\left[\dfrac{1}{\sqrt{\tau_{1,2}}} + \dfrac{\sqrt{\tau_{1,2}}}{4t_+}\right]e^{-i\omega\tau_{1,2}}\,\mathrm{erfc}(\sqrt{-i\omega\tau_{1,2}}) - \dfrac{1}{4t_+\sqrt{-i\pi\omega}} \\ + \dfrac{3}{64\pi}\dfrac{\tau_{1,2}^{3/2}}{t_+^2}\left[K_0\left(-\dfrac{i\omega\tau_{1,2}}{2}\right) - K_1\left(-\dfrac{i\omega\tau_{1,2}}{2}\right)\left(1 + \dfrac{1}{i\omega\tau_{1,2}}\right)\right]\end{array}\right\}_-\right\}_+, \tag{31}$$

where the Fourier transform of $\tau^{3/2}/(\tau+\tau_{1,2})$ is given by

$$\int_0^\infty \frac{t^{3/2}}{t+\tau_{1,2}}\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = \frac{1}{2}\,\tau_{1,2}^{3/2}\,\mathrm{e}^{-\mathrm{i}\omega\tau_{1,2}/2}\left[\mathrm{K}_0\left(-\frac{\mathrm{i}\omega\tau_{1,2}}{2}\right) - \mathrm{K}_1\left(-\frac{\mathrm{i}\omega\tau_{1,2}}{2}\right)\left(1+\frac{1}{\mathrm{i}\omega\tau_{1,2}}\right)\right] \quad (32)$$

and is shown in Appendix C.

In equation (32), the modified Bessel function K_1 for the negative pure imaginary argument $-\mathrm{i}\omega t_+$, may be expressed, similar to K_0, in terms of the Bessel functions J_1 and Y_1 for a positive real argument, i.e. (reference [16], f. 9.6.4, p. 375),

$$\mathrm{K}_1(-\mathrm{i}\omega t_+) = -\frac{\pi}{2}\,[\mathrm{J}_1(\omega t_+) + \mathrm{Y}_1(\omega t_+)]. \quad (33)$$

This process of considering more terms may be pursued for higher terms by using the general integration formula (reference [18], f. 2.3.6.15, p. 325)

$$\int_0^\infty \frac{t^{n-1/2}}{t+\alpha}\,\mathrm{e}^{-pt}\,\mathrm{d}t = (-1)^n\pi\alpha^{n-1/2}\mathrm{e}^{p\alpha}\,\mathrm{erfc}(\sqrt{p\alpha})$$

$$+\,2^{1-n}\sqrt{\pi}p^{1/2-n}\sum_{m=0}^{n-1}(2n-2m-3)!!(2p\alpha)^m, \quad (34)$$

and which, by the way gives a generalization of

$$u_{d5g}(f) = -\frac{S\rho}{4\pi^2 c}\sqrt{\frac{t_+^2-t_-^2}{2t_+}}\,\frac{\pi\mathrm{e}^{\mathrm{i}\omega\tau_0}}{2}$$

$$\times\left\{\frac{\cos\pm}{\sqrt{\Delta'_\pm}}\left\{\begin{array}{l}\dfrac{\mathrm{e}^{-\mathrm{i}\omega\tau_{1,2}}}{\sqrt{\tau_{1,2}}}\,\mathrm{erfc}(\sqrt{-\mathrm{i}\omega\tau_{1,2}}) + \displaystyle\sum_{n=1}^{\infty}\binom{-1/2}{n}\frac{1}{(2t_+)^n} \\ \left[\begin{array}{l}(-1)^n\tau_{1,2}^{n-1/2}\mathrm{e}^{-\mathrm{i}\omega\tau_{1,2}}\,\mathrm{erfc}(\sqrt{-\mathrm{i}\omega\tau_{1,2}}) + 2^{1-n} \\ \sqrt{\dfrac{2}{\pi}}(-2\mathrm{i}\omega)^{1/2-n}\displaystyle\sum_{m=0}^{n-1}(2n-2m-3)!!(-2\mathrm{i}\omega\tau_{1,2})^m\end{array}\right]\end{array}\right\}_-\right\}_+. \quad (35)$$

2.6. SIXTH APPROXIMATION FORMULAE

This form is obtained on considering the first two terms of the serial development of $1/(\tau+\tau_{1,2})$, or

$$\frac{1}{\tau+\tau_{1,2}} = \frac{1}{\tau_{1,2}}\sum_{n=0}^{\infty}\left(-\frac{\tau}{\tau_{1,2}}\right)^n \quad \text{for } \tau<\tau_{1,2} \quad (36)$$

and the new time domain approximation for the diffracted field becomes

$$u_{d6}(\tau) = \frac{-S\rho}{4\pi^2 c}\sqrt{t_+^2-t_-^2}\,\frac{1}{2}\left\{\frac{\cos\pm}{\sqrt{\Delta'_\pm}}\left\{\frac{1}{\tau_{1,2}}\left(\frac{1}{\sqrt{\tau}\sqrt{\tau+2t_+}} + \frac{1}{\tau_{1,2}}\frac{\sqrt{\tau}}{\sqrt{\tau+2t_+}}\right)\right\}_-\right\}_+ \quad (37)$$

and by using for the first term in the bracket the result of equation (19) and for the second term (reference [18], f. 2.3.6.11, p. 325)

$$\int_0^\infty \frac{\sqrt{t}}{\sqrt{t+a}} \mathrm{e}^{-pt}\, \mathrm{d}t = \frac{a}{2} \mathrm{e}^{pa/2} \left[\mathrm{K}_1\left(\frac{pa}{2}\right) - \mathrm{K}_0\left(\frac{pa}{2}\right)\right], \tag{38}$$

one gets a transform in the frequency domain which appears as follows:

$$u_{d6}(\tau) = \frac{-S\rho}{4\pi^2 c} \sqrt{t_+^2 - t_-^2}\, \frac{1}{2}$$
$$\times \left\{ \frac{\cos\pm}{\sqrt{\Delta'_\pm}} \left\{ \frac{1}{\tau_{1,2}} \left[\mathrm{K}_0(-\mathrm{i}\omega t_+) \left(1 + \frac{t_+}{\tau_{1,2}}\right) - \frac{t_+}{\tau_{1,2}} \mathrm{K}_1(-\mathrm{i}\omega t_+) \right] \right\}_- \right\}_+ . \tag{39}$$

Similar to the fifth approximation form, equation (39) may be improved by taking one more term in the development (36). In this case, one needs an expression for the Fourier transform of $\tau^{3/2}/\sqrt{\tau + 2t_+}$ which is given by

$$\int_0^\infty \frac{t^{3/2}}{\sqrt{t + 2t_+}} \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}t = t_+^2\, \mathrm{e}^{-\mathrm{i}\omega t_+} \left[2\mathrm{K}_0(-\mathrm{i}\omega t_+) - \mathrm{K}_1(-\mathrm{i}\omega t_+) \left(2 + \frac{1}{\mathrm{i}\omega t_+}\right)\right] \tag{40}$$

and is detailed in Appendix D.

Hence, the new approximation $u'_{d6}(f)$ is expressed by

$$u'_{d6}(\tau) = \frac{-S\rho}{4\pi^2 c} \sqrt{t_+^2 - t_-^2}\, \frac{1}{2}$$
$$\times \left\{ \frac{\cos\pm}{\sqrt{\Delta'_\pm}} \left\{ \frac{1}{\tau_{1,2}} \left[\begin{array}{c} \mathrm{K}_0(-\mathrm{i}\omega t_+)\left(1 + \dfrac{t_+}{\tau_{1,2}}\right) - \dfrac{t_+}{\tau_{1,2}} [\mathrm{K}_1(-\mathrm{i}\omega t_+) - \mathrm{K}_0(-\mathrm{i}\omega t_+)] \\ -\left(\dfrac{t_+}{\tau_{1,2}}\right)^2 \left[\mathrm{K}_1(-\mathrm{i}\omega t_+)\left(2 + \dfrac{1}{\mathrm{i}\omega t_+}\right) - 2\mathrm{K}_0(-\mathrm{i}\omega t_+)\right] \end{array} \right] \right\}_- \right\}_+ . \tag{41}$$

However, for all the terms in the series of equation (36) it would be more suitable to consider the general integration formula (reference [18] f. 2.3.6.9, p. 324)

$$\int_0^\infty \frac{t^{\nu-1}}{(t+\alpha)^\lambda} \mathrm{e}^{-pt}\, \mathrm{d}t = \Gamma(\nu) \alpha^{\nu-\lambda} U(\nu, \nu + 1 - \lambda; p\alpha), \tag{42}$$

which leads to the generalized sixth approximation:

$$u_{d6}(\tau) = \frac{-S\rho}{4\pi^2 c} \sqrt{t_+^2 - t_-^2}\, \frac{1}{2} \mathrm{e}^{\mathrm{i}\omega\tau_0} \left\{ \frac{\cos\pm}{\sqrt{\Delta'_\pm}} \left\{ \frac{1}{\tau_{1,2}} \sum_{n=0}^{\infty} \left(-\frac{2t_+}{\tau_{1,2}}\right)^n \Gamma\left(n + \frac{1}{2}\right)\right.\right.$$
$$\left.\left. \times U\left(n + \frac{1}{2}, n+1; -\mathrm{i}\omega 2t_+\right)\right\}_- \right\}_+ . \tag{43}$$

The function Γ is the Gamma function, and $\Gamma(n+1/2)$ satisfies a simple relation:

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{1{\cdot}3{\cdot}5\cdots(2n-1)}{2^n}\,\Gamma\left(\frac{1}{2}\right)=\frac{(2n-1)!!}{2^n}\sqrt{\pi}, \tag{44}$$

where $\Gamma(1/2)=\sqrt{\pi}$ (reference [16], ff. 6.1.12 and 6.1.8, p. 255).

Note that equation (42) is obtained by using, respectively, equations (C.1) and (C.2) in Appendix C. The function $\mathrm{U}(n+1/2, n+1;\ z)$ is the Tricomi function (also called the confluent hypergeometric function and sometimes also denoted as $\Psi(a,b,z)$ or $z_2^{-a}F_0(a, 1+a-b, -1/z)$) which may be derived by recurrence from $\mathrm{U}(1/2, 1; z)$, this latter being a simple expression of the modified Bessel function K_0 introduced in equation (19). This is explained in detail in Appendix E (from this same appendix one can see that equations (39) and (41) may be recovered by considering the expressions of $\mathrm{U}(1/2, 1; -\mathrm{i}\omega 2t_+)$ and $\mathrm{U}(3/2, 2; -\mathrm{i}\omega 2t_+)$ in terms of $\mathrm{K}_0(\omega t_+)$ and $\mathrm{K}_1(\omega t_+)$. It would be worth noting also that equation (34) may be processed from the more general formula in equation (42) by using the property (reference [18], Vol. III, f. 7.11.4.4, p. 584)

$$\mathrm{U}\left(n+\frac{1}{2}, n+\frac{1}{2}; z\right)=\mathrm{e}^z\,\Gamma\left(\frac{1}{2}-n, z\right) \tag{45}$$

and the recurrence relation for the incomplete gamma function Γ in equation (G.9) in Appendix G.

2.7. SEVENTH APPROXIMATION FORMULAE

From the considerations which led to the fifth and sixth approximations, one can make instead a serial development of the bracketed rational expression in equation (6), and deduce therefrom a new approximation. Hence, the inverse quadratic expression $1/(ax^2+bx+c)$ may be developed as (reference [17], f. 16:6:1, p. 126)

$$\frac{1}{ax^2+bx+c}=\frac{1}{c}\left\{1-\frac{bx}{c}+\frac{(b^2-ac)x^2}{c^2}-\frac{(b^3-2abc)x^3}{c^3}+\cdots\right\}=\frac{1}{c}\sum_{n=0}^{\infty}\left[\frac{-x(ax+b)}{c}\right]^n, \tag{46}$$

which in combination with the binomial development formula

$$(ax+b)^n=\sum_{m=0}^{n}\binom{n}{m}(ax)^m b^{n-m} \tag{47}$$

may be written as

$$\frac{1}{ax^2+bx+c}=\frac{1}{c}\left[1+\sum_{n=1}^{\infty}\sum_{m=0}^{E(n/2)+1}\binom{n-m}{m}\left(\frac{a}{b}\right)^m\left(-\frac{b}{c}\right)^{n-m}x^n\right], \tag{48}$$

where the symbol $E(x)$ means the entire part of x.

The development in equation (45) or in equation (47) is valid only for small values of x, or more precisely for x satisfying $|ax^2+bx|/|c|<1$. In respect of equation (6), this implies that τ be between the roots τ_s and τ_l of the equation $\tau^2+2t_+\tau-(t_+^2-t_-^2)\cos{\pm}^2=0$.

These roots are given by $\tau_{s,l} = -t_+ \mp \sqrt{\Delta'}$ with $\Delta' = t_+^2(1+\cos\pm^2) + t_-^2\cos\pm^2 > t_+^2$. Hence, $\tau_s < 0$ and $\tau_l > t_+ > 0$. Thus, equation (6) becomes

$$u_d(\tau) = \frac{-S\rho}{4\pi^2 c}\frac{1}{\sqrt{t_+^2 - t_-^2}}\left\{\frac{1}{\cos\pm}\left\{1 + \sum_{n=1}^{\infty}\left[\sum_{m=0}^{E(n/2)+1}\binom{n-m}{m}\left(\frac{1}{2t_+}\right)^m \right.\right.\right.$$

$$\left.\left.\left.\times\left(\frac{-2t_+}{(t_+^2 - t_-^2)\cos\pm^2}\right)^{n-m}\right]\tau^n\right\}\frac{1}{\sqrt{\tau}\sqrt{\tau + 2t_+}}\right\}_+ . \tag{49}$$

The Fourier transform of this latter then reads as

$$u_d(f) = \frac{-S\rho}{4\pi^2 c}\frac{1}{\sqrt{t_+^2 - t_-^2}}\mathrm{e}^{\mathrm{i}\omega\tau_0}$$

$$\times\left\{\frac{1}{\cos\pm}\left\{\begin{array}{l}\mathrm{e}^{-\mathrm{i}\omega t_+}\mathrm{K}_0(-\mathrm{i}\omega t_+) + \displaystyle\sum_{n=1}^{\infty}\left[\sum_{m=0}^{E(n/2)+1}\binom{n-m}{m}\left(\frac{1}{2t_+}\right)^m\left(\frac{-2t_+}{(t_+^2 - t_-^2)\cos\pm^2}\right)^{n-m}\right] \\ \Gamma\left(n + \dfrac{1}{2}\right)(2t_+)^n\mathrm{U}\left(n + \dfrac{1}{2}, n + \dfrac{3}{2}; -\mathrm{i}2t_+\omega\right)\end{array}\right\}\right\}_+ , \tag{50}$$

where again use has been made of equation (42) for $\nu = n + 1/2$ and $\lambda = 1/2$. One finds again the Tricomi function $\mathrm{U}(n + 1/2, n + 3/2; z)$, which this time for the case where its parameters differ by just unity reduces to a simple inverse power function (see Appendix F),

$$\mathrm{U}\left(n + \frac{1}{2}, n + \frac{3}{2}; z\right) = z^{-(n+1/2)} \tag{51}$$

and which when inserted in equation (50) gives

$$u_{d7}(f) = \frac{-S\rho}{4\pi^2 c}\frac{1}{\sqrt{t_+^2 - t_-^2}}\mathrm{e}^{\mathrm{i}\omega\tau_0}$$

$$\times\left\{\frac{1}{\cos\pm}\left\{\begin{array}{l}\mathrm{e}^{-\mathrm{i}\omega t_+}\mathrm{K}_0(-\mathrm{i}\omega t_+) + \displaystyle\sum_{n=1}^{\infty}\left[\sum_{m=0}^{E(n/2)+1}\binom{n-m}{m}\frac{(-2t_+)^{n-2m}}{[(t_+^2 - t_-^2)\cos\pm^2]^{n-m}}\right] \\ \Gamma\left(n + \dfrac{1}{2}\right)(-\mathrm{i}\omega)^{-n-1/2}\end{array}\right\}\right\}_+ . \tag{52}$$

Alternatively, if the quadratic polynomial $ax^2 + bx + c$ has two roots x_1 and x_2 then

$$\frac{1}{ax^2 + bx + c} = \frac{1}{a}\frac{1}{x - x_1}\frac{1}{x - x_2} \tag{53}$$

with $x_{1,2} = (-b \pm \sqrt{\Delta})/2a$ and $\Delta = b^2 - 4ac$. Hence, for small values of x one can use the serial development in equation (36) in equation (53) and get

$$\frac{1}{ax^2 + bx + c} \approx \frac{1}{a}\frac{1}{x_1 x_2}\left[\sum_{i=0}^{\infty}\left(-\frac{x}{x_1}\right)^i\right]\left[\sum_{j=0}^{\infty}\left(-\frac{x}{x_2}\right)^j\right], \tag{54}$$

which is valid for $|x| < \min(|x_1|, |x_2|)$. The product of the roots x_1 and x_2 satisfies the relationship $x_1 x_2 = c/a$ and upon rearranging the product of the sums, equation (54) is given again by

$$\frac{1}{ax^2 + bx + c} \approx \frac{1}{c} \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} \frac{1}{x_1^m x_2^{n-m}} \right] (-x)^n. \tag{55}$$

Using these results with equation (42) in the expression for the diffracted field one finds

$$u'_{d7}(f) = \frac{-S\rho}{4\pi^2 c} \frac{1}{\sqrt{t_+^2 - t_-^2}} \mathrm{e}^{\mathrm{i}\omega\tau_0} \sum_{n=0}^{\infty} (-2t_+)^n \left\{ \frac{1}{\cos \pm} \left[\sum_{m=0}^{n} \frac{1}{\tau_1^m \tau_2^{n-m}} \right] \right\}_+ \Gamma\left(n + \frac{1}{2}\right)$$

$$\times \mathrm{U}\left(n + \frac{1}{2}, n + 1; -\mathrm{i}\omega 2t_+\right). \tag{56}$$

One recognizes already in this last expression the third approximation as given by equation (19). This can be shown by taking $n = 0$ and using equation (E.2) in Appendix E.

2.8. EIGHTH APPROXIMATION FORMULA

This last approximation is achieved by making a double serial expansion of the product $[(1/\sqrt{(\tau + 2t_+)}][1/(\tau + \tau_{1,2})]$ and choosing a suitable series development for the range of values taken by the variable τ. First, from equation (28),

$$\frac{1}{\sqrt{\tau + 2t_+}} = \frac{1}{\sqrt{2t_+}} \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(\frac{\tau}{2t_+}\right)^n \quad \text{for } \tau \leqslant 2t_+$$

and (reference [17], f. 12.6.4, p. 94)

$$\frac{1}{\sqrt{\tau + 2t_+}} = \frac{1}{\sqrt{\tau}} \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(\frac{2t_+}{\tau}\right)^n \quad \text{for } \tau \geqslant 2t_+. \tag{57}$$

Hence, calling u_{d8} the new development of u_d one gets

$$u_{d8}(f) = -\frac{S\rho}{4\pi^2 c} \sqrt{t_+^2 - t_-^2}\, \mathrm{e}^{\mathrm{i}\omega\tau_0}$$
$$\times \left\{ \frac{\cos \pm}{2\sqrt{\Delta'_\pm}} \left\{ \begin{matrix} \dfrac{1}{\sqrt{2t_+}} \displaystyle\sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{1}{(2t_+)^n} \int_0^{2t_+} \frac{\tau^{n-1/2}}{\tau + \tau_{1,2}} \mathrm{e}^{\mathrm{i}\omega\tau} \mathrm{d}\tau \\ + \displaystyle\sum_{n=0}^{\infty} \binom{-1/2}{n} (2t_+)^n \int_{2t_+}^{\infty} \frac{\tau^{-n-1}}{\tau + \tau_{1,2}} \mathrm{e}^{\mathrm{i}\omega\tau} \mathrm{d}\tau \end{matrix} \right\}_- \right\}_+, \tag{58}$$

which one may write in a more compact form as

$$u_{d8}(f) = -\frac{S\rho}{4\pi^2 c} \sqrt{t_+^2 - t_-^2}\, \mathrm{e}^{\mathrm{i}\omega\tau_0} \left\{ \frac{\cos \pm}{2\sqrt{\Delta'_\pm}} \left\{ \sum_{n=0}^{\infty} \binom{-1/2}{n} [(2t_+)^{-n-1/2} I_{1n} + (2t_+)^n I_{2n}] \right\}_- \right\}_+ \tag{59}$$

with

$$I_{1n} = \int_0^{2t_+} \frac{\tau^{n-1/2}}{\tau + \tau_{1,2}} \, \mathrm{e}^{\mathrm{i}\omega\tau} \, \mathrm{d}\tau \tag{60}$$

and

$$I_{2n} = \int_{2t_+}^{\infty} \frac{\tau^{-n-1}}{\tau + \tau_{1,2}} \, \mathrm{e}^{\mathrm{i}\omega\tau} \, \mathrm{d}\tau. \tag{61}$$

From equation (24) it is seen that $0 < \tau_1 < \tau_2 < 2t_+$ and this implies that another pair of serial development is needed for I_{1n}: i.e., by using (reference [17], ff. 7:6:1 and 7:6:2, p. 55)

$$\frac{1}{\tau + \tau_{1,2}} = \frac{1}{\tau_{1,2}} \frac{1}{1 + \tau/\tau_{1,2}} = \frac{1}{\tau_{1,2}} \sum_{m=0}^{\infty} \left(- \frac{\tau}{\tau_{1,2}} \right)^m \quad \text{for } \tau < \tau_{1,2} \tag{62a}$$

and

$$\frac{1}{\tau + \tau_{1,2}} = \frac{1}{\tau} \frac{1}{1 + \tau_{1,2}/\tau} = \frac{1}{\tau} \sum_{m=0}^{\infty} \left(- \frac{\tau_{1,2}}{\tau} \right)^m \quad \text{for } \tau > \tau_{1,2}. \tag{62b}$$

Equation (59) may again be expressed as

$$u_{d8}(f) = - \frac{S\rho}{4\pi^2 c} \sqrt{t_+^2 - t_-^2} \, \mathrm{e}^{\mathrm{i}\omega\tau_0}$$
$$\times \left\{ \frac{\cos \pm}{2\sqrt{\Delta'_\pm}} \left\{ \sum_{n=0}^{\infty} \binom{-1/2}{n} \left\{ \begin{aligned} &(2t_+)^{-n-1/2} \left[\sum_{m=0}^{\infty} \left(- \frac{I_{1nm1}}{(-\tau_{1,2})^{m+1}} + (-\tau_{1,2})^m I_{1mn2} \right) \right] \\ &+ (2t_+)^n \left[\sum_{m=0}^{\infty} (-\tau_{1,2})^m I_{2nm} \right] \end{aligned} \right\} \right\} \right\}_-, \tag{63}$$

where the integrals to be evaluated are

$$I_{1nm1} = \int_0^{\tau_{1,2}} \tau^{n+m-1/2} \, \mathrm{e}^{\mathrm{i}\omega\tau} \, \mathrm{d}\tau, \quad I_{1nm2} = \int_{\tau_{1,2}}^{2t_+} \tau^{n-m-1/2} \, \mathrm{e}^{\mathrm{i}\omega\tau} \, \mathrm{d}\tau, \quad I_{2nm} = \int_{2t_+}^{\infty} \tau^{n-m-2} \, \mathrm{e}^{\mathrm{i}\omega\tau} \, \mathrm{d}\tau. \tag{64a–c}$$

These are shown in Appendix G.

The final result for the n-term in the innermost bracket of equation (63) reads then as

$$(2\omega t_+)^{-n-1/2} \left[\sum_{m=0}^{\infty} \left(\begin{aligned} &- \frac{\omega \gamma(n + m + 1/2, -\mathrm{i}\omega\tau_{1,2})}{(-\omega\tau_{1,2})^{m+1} (-\mathrm{i})^{n+m+1/2}} \\ &+ (-\omega\tau_{1,2})^m \left\{ \begin{aligned} &\frac{\gamma(n - m + 1/2, -\mathrm{i}\omega\tau)|_{\tau_{1,2}}^{2t_+}}{(-\mathrm{i})^{n-m+1/2}} \\ &- \mathrm{e}^{\mathrm{i}(n-m+1/2)\pi/2} \Gamma(n - m + 1/2, -\mathrm{i}\omega\tau)|_{\tau_{1,2}}^{2t_+} \end{aligned} \right. \end{aligned} \right) \right]$$
$$+ (2\omega t_+)^n \left[\sum_{m=0}^{\infty} (-\mathrm{i})^{n+m-1} \Gamma(-n - m - 1, -\mathrm{i}\omega 2t_+ \right], \tag{65}$$

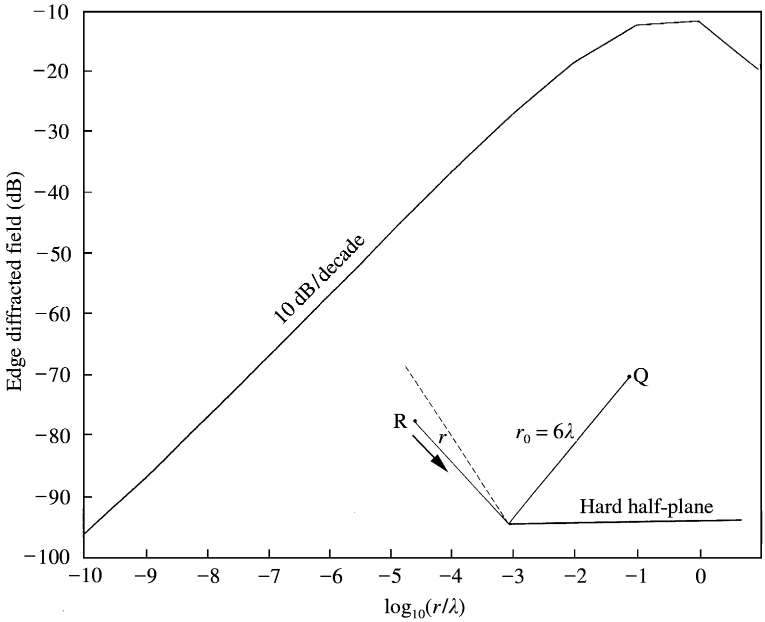

Figure 2. Variation of the amplitude of the diffracted field at approaching the edge of a hard half-plane.

where the factor of $(-\,\omega\tau_{1,2})^m$ is either the upper or lower term depending on whether $n - m + 1/2$ is positive or negative. $\gamma(\alpha, z)$ and $\Gamma(\alpha, z)$ are the incomplete gamma functions and are evaluated by a recurrence process as detailed in Appendix G.

3. NUMERICAL EXAMPLES

Before making numerical comparisons of the different approximations, it would be interesting first to make some general remarks concerning the behaviour of the diffracted field on approaching the edge of the hard half-plane. This is illustrated in Figure 2.

In accordance with the principle of conservation of energy, the edge of the wedge should neither absorb, nor radiate energy and this sets the condition on the diffracted field that on approaching the edge, its amplitude should vary at most as the square root of the distance to it [19]. This is clearly seen in the slope of the curve in Figure 2.

Figure 3 illustrates the behaviour of the edge diffracted field around the edge of the hard half-plane at a constant distance from it.

In Figure 3, the curves of the approximations given by u_{d2}, u_{d4} and u_{d5} are indiscernible from that of u_d which is given by a continuous line. The departures between the curves amount to less than 0·5 dB at the geometrical boundaries. The approximations u_{d1}, u_{d3} and u_{d6} fail when approaching these boundaries, and the angular extension of these failure zones diminishes with higher frequency. This is illustrated in Figure 4 for a frequency equal to 3400 Hz corresponding to $r_0 = 10\lambda$. This feature reminds one somehow of Keller's Geometrical Theory of Diffraction [4]. In this latter, and depending on the geometry of the problem and the frequency, the failure zones for a plane incident wave are delimited by parabolae whose axes are the geometrical boundary lines, and whose foci are centred at the edge of the half-plane (in the case of a line source parallel to the edge of the half-plane, the failure zones are delimited by hypebolae) [20].

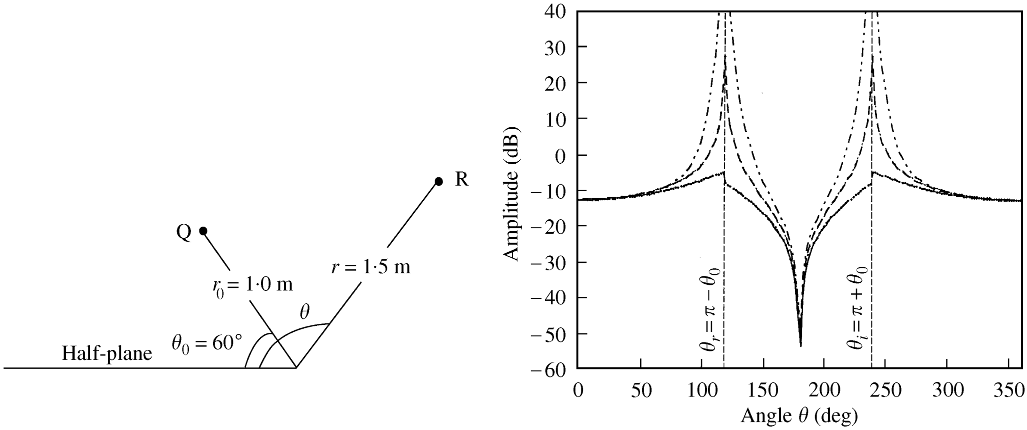

Figure 3. Diffraction of a spherical wave by a hard half-plane. Left, geometry, and right, amplitude of the edge diffracted field normalized to the free field at $r + r_0$. Frequency $f = 340$ Hz corresponding to $r_0 = \lambda$. ------, u_{d1}; ······, u_{d3}; -·-·-·-·-·-·, u_{d6}.

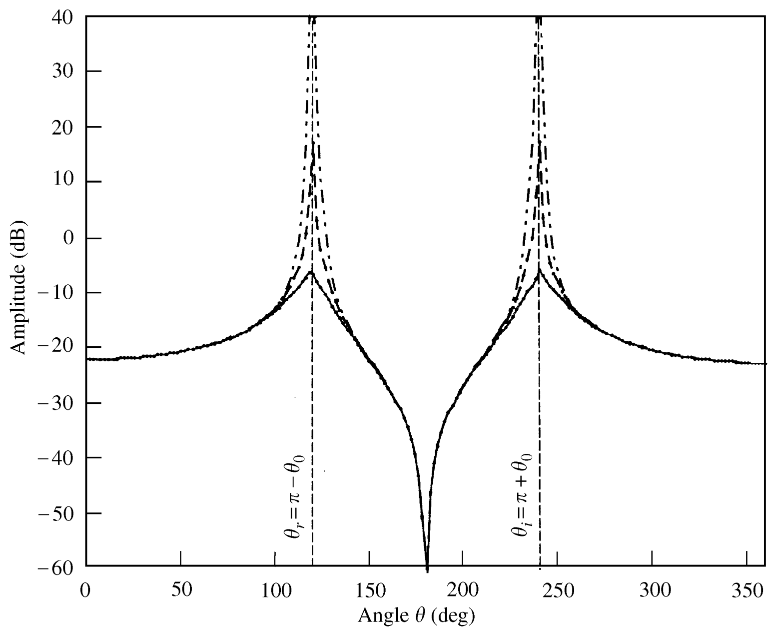

Figure 4. As Figure 3 but for $f = 3400$ Hz corresponding to $r_0 = 10\ \lambda$.

The same calculations were carried out for the approximations u_{d6}, u_{d7} and u_{d8} for which the same observations apply. The curves for these approximations are plotted in Figure 5 for $f = 340$ Hz and in Figure 7 for $f = 3400$ Hz.

Similar calculations have been made for fixed positions of the source and the receiver but with the frequency as the variable quantity. The results are plotted in the curves of Figures 7 and 8.

These latter calculations have been performed for emphasizing the performance of the various approximations for the engineering purposes of estimating the sound reduction of

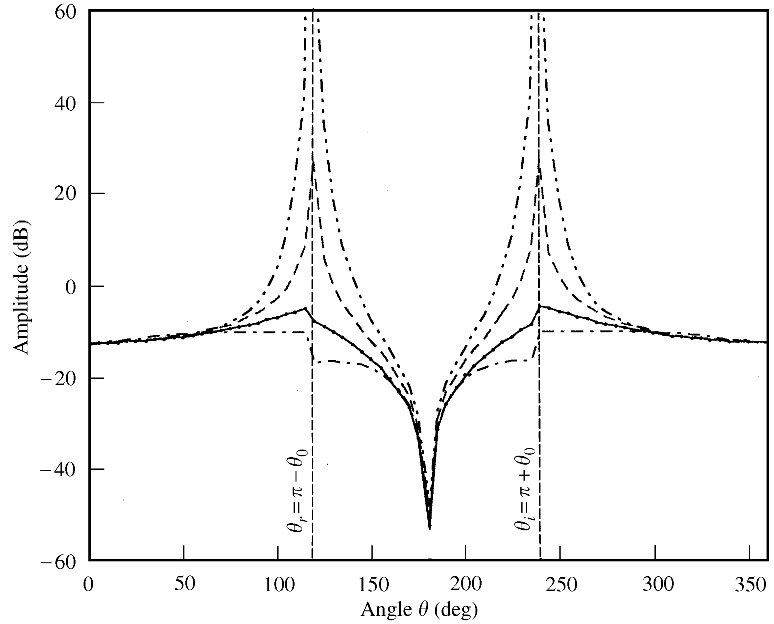

Figure 5. As Figure 3, $f = 340$ Hz, but with ·············, u_{d6}; - - - - - -, u_{d7}; -·-·-·-·-·-·, u_{d8} (2·2 terms).

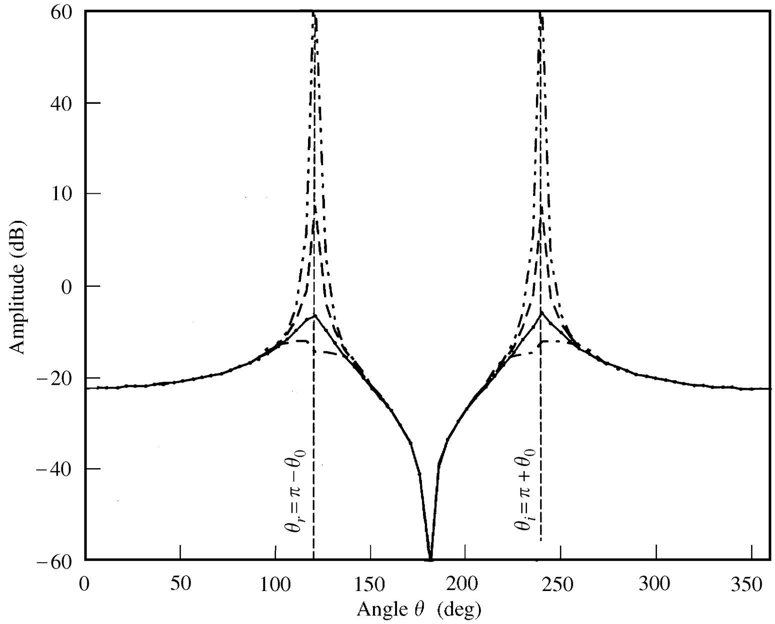

Figure 6. As Figure 5 but for $f = 3400$ Hz.

simple noise barriers. As may be seen, the different approximations are relatively quite accurate for frequencies above the frequency corresponding to the distance source–edge of the barrier equal to a wavelength. It should, however, be pointed out that some approximations, namely u_{d1}, u_{d3}, u_{d6} and u_{d7} give an overestimation of the diffracted field at

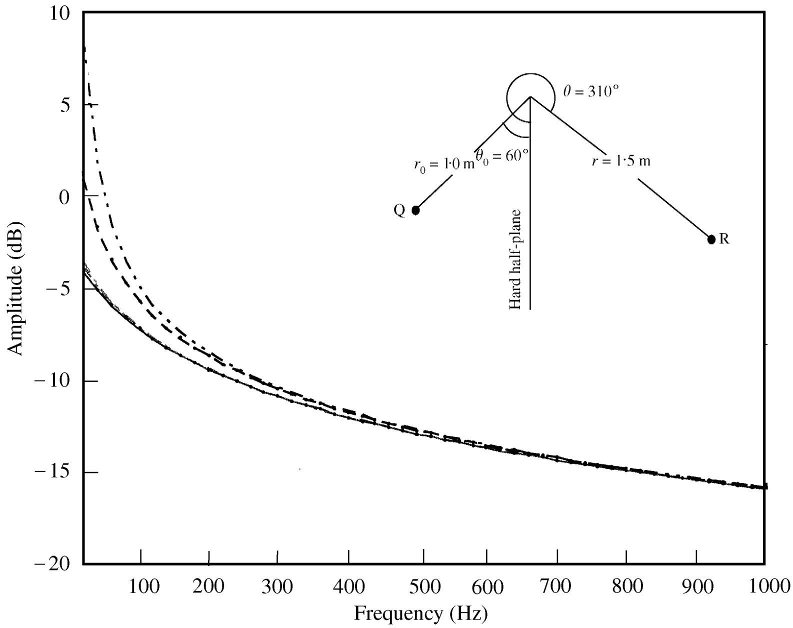

Figure 7. Diffracted field in the shadow region of a hard half-plane normalized to free field at $r + r_0$. ——, u_{d2}; - - - - - - -, u_{d3}; - ·· - ·· - ··-, u_{d6}.

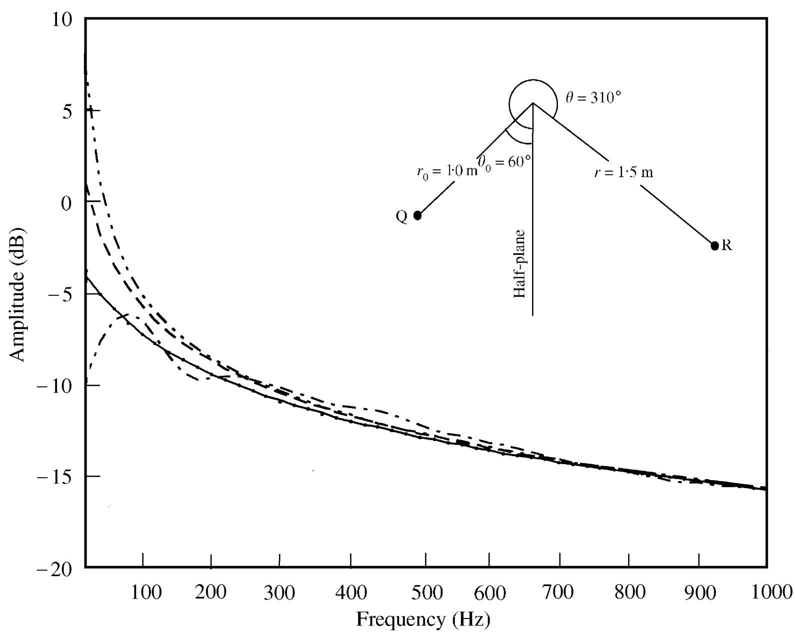

Figure 8. As Figure 7 but ——, u_{d2}; ·· - ·· - ··, u_{d6}; - - - - - -, u_{d7}; - · - · - ·, u_{d8}.

very low frequencies. A plausible explanation is that the failure zones discussed earlier at such frequencies extend to an angular opening with such width that the receiver position falls in these very zones.

4. DISCUSSION AND CONCLUSIONS

In this paper, the problem of diffraction of a spherical wave by a hard half-plane has been considered as a special case of the Biot–Tolstoy theory of diffraction by a hard wedge. Several authors have contributed to the understanding of the general phenomenon of wave propagation around wedges in the presence of different sound sources. One can cite some of the latest interesting contributions to this subject such as Ambaud and Bergassoli's [21] (equation (8) of this latter reference ought to be the expression of the Fourier transform of equation (3) in the present text) and Hadden and Pierce's [22]. Friedlander's work [23] also contains valuable information and illustrative examples on the effect of obstacles on the propagation of sound pulses of various shapes (Friedlander devotes chapter 5 of his book to the problem of diffraction of an instantaneous spherical wave in the presence of a wedge. The solution expressed in terms of Green functions is rather complicated but Hadden and Pierce give useful approximations to the case of the monochromatic point source).

To return to the problem of the hard half-plane, a more tractable form is presented in this work for the exact expression given by Biot and Tolstoy for the diffracted field. From this time-domain expression of the diffracted field several approximations are derived, mostly for the short time range after the arrival of the front of the diffracted field. These approximations are inspired from the approximation produced by Medwin which has been considered as the first in the series of the present approximations. In the frequency domain, comparisons are made between the numerical integration of the exact expression of the diffracted field and the Fourier transform of its approximations. The numerical Fourier transform of the exact expression of the diffracted field uses Gauss and Konrod point rules and is of the automatic and adaptive type [24]. As the diffracted field resulting from an instantaneous pulse is zero at times earlier than that corresponding to the fastest path source–receiver via the edge of the half-plane, then a change of variable permits one to bring the Fourier transform to a one-sided transform, or equivalently to a Laplace transform. At exactly the arrival time of the front of the diffracted field, this latter tends towards infinity and special routines may be used depending on the behaviour of the function at such singularities.

As to the results in the calculations, one can first make the general observation that the amplitude of the diffracted field as given by all the approximations becomes smaller and smaller as the frequency takes higher values. This ascertains the fact that the diffraction of sound waves by sharp edges is a low-frequency phenomenon. Among the different approximations to the expression of the diffracted field presented in this work, three approximations seem to be accurate enough for most practical engineering purposes. These are the ones labelled u_{d2}, u_{d4} and u_{d5}, this latter with its generalization, as given, respectively, by equations (16), (26), (30) and (35). These approximations have almost comparable performances except at proximity to the geometrical boundaries where very slight differences may be noticed. Some calculations are illustrated in Figure 9 for the receiver position near the reflection boundary of the configuration of Figure 4.

As expected, the diffracted field exhibits its strongest amplitude around the geometrical transition regions. This is the well-known Fresnel diffraction phenomenon. Another observation is the peculiar behaviour of the diffracted field on traversing these transitional regions. In fact, the discontinuity of this latter with the eventual combination of that of the geometrical incident and/or reflected fields ensure the smooth transition of the total field through all space. If one takes the exact time domain expression as given in equation (3), at exactly the geometrical boundaries, that is at $\theta = \pi \pm \theta_0$, the diffracted field suffers sudden amplitude and phase changes leading to compensating effects in the total field due to the abrupt presence or disappearance of the ray acoustics fields. To take an example, one can consider the behaviour of the diffracted field as approximated by u_{d2} in equation (16). Near

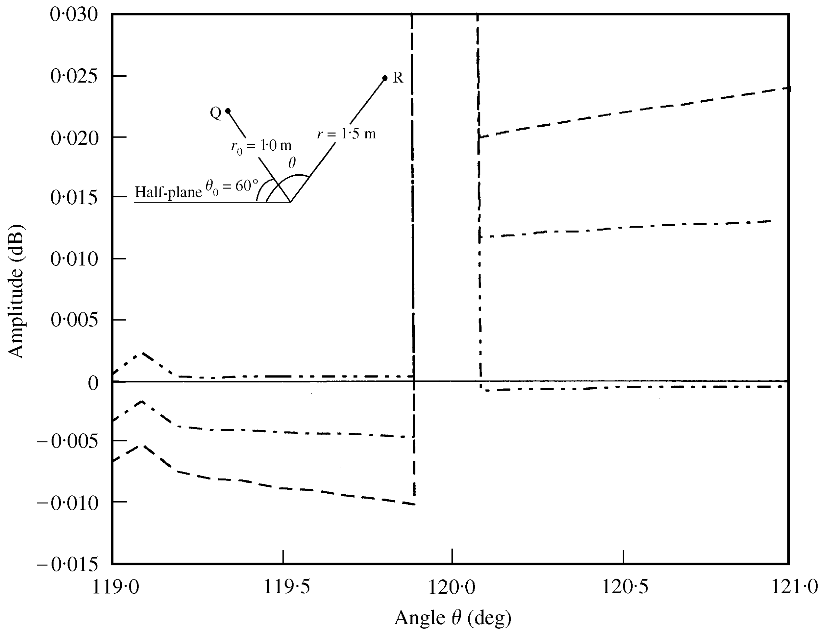

Figure 9. Close look at the behaviour of u_{d2}, u_{d4} and u_{d5} at transiting across the geometrical reflection boundary. $f = 340$ Hz corresponding to $r_0 = \lambda$. - - - - -, $u_{d2} - u_d$; - · - · - · -, $u_{d4} - u_d$; - ·· - ·· - ··, $u_{d5} - u_d$.

the incidence boundary, that is at $\theta = \pi + \theta_0$, a_+ in equation (17) takes relatively large values and the corresponding contribution to the total diffracted field varies slowly. On the other hand, a_- takes small values and, at two symmetrically lying positions about the geometrical boundary, the corresponding contributions to the diffracted field become larger than those due to a_+. In the neighbourhood of the geometrical boundary, the amplitude of a_- component is almost constant, whereas the phases at two symmetrical positions are opposite. Hence, if one denotes the corresponding part of the diffracted field by

$$u_{d-} = \frac{-S\rho}{4\pi^2 c}\frac{\pi e^{i\omega\tau_0}}{2t_+}\,e^{-i\omega\sin^2\varepsilon(t_+^2-t_-^2)/2t}\,\mathrm{erfc}\left(\sqrt{-i\omega\sin^2\varepsilon(t_+^2-t_-^2)/2t_+}\right), \tag{66}$$

then on crossing the geometrical boundary the diffracted field suffers a jump which in the limit is equal to

$$\Delta u_d = u_d(\pi+\theta_0+\varepsilon) - u_d(\pi+\theta_0-\varepsilon) = 2u_{d-} \xrightarrow[\varepsilon\to 0]{} \frac{-S\rho}{4\pi^2 c}\frac{\pi e^{i\omega\tau_0}}{t_+}, \tag{67}$$

which is exactly the incident field. In equation (67), use has been made of (see reference [17], p. 399)

$$\lim_{z\to 0} e^z\,\mathrm{erfc}(\sqrt{z}) = 1. \tag{68}$$

The same reasonsing applies to a_+ at the reflection geometrical boundary and the reflected field.

It would also be interesting to note two further important features of the diffracted field. The first is its vanishing amplitude at points lying in the extension plane of the diffracting half-plane. This feature can be predicted by using the principle of reciprocity and the fact that for a point source situated on the extension of the half-plane, the total field anywhere in space is only that due to the direct field and that the scattered field is zero everywhere as ascertained in earlier works (see reference [6], p. 335). In the time domain, this can easily be verified by assigning the value π to the angle θ in the exact expression of the diffracted field as given by equation (6) (note that in this latter expression, due to the parity of the cosine function the diffracted field obviously satisfies the reciprocity principle). The second remark is the symmetrical behaviour of the amplitude of the diffracted field about this plane. This characteristic can also be predicted theoretically by setting the value of θ in equation (6) equal to $2\pi - \theta$ and upon using the relationship between the cosines of two supplementary angles, i.e. $\cos(\pi - \theta) = -\cos(\theta)$. Hence, at two positions symmetrical about the half-plane, the diffracted field has the same amplitude, whereas, as may be verified, the phases are opposite. Again, this is in full agreement with earlier results that the sum of the total fields at two opposite positions about the half-plane is equal to the sum of the incident field and its image through the half-plane (see reference [6], p. 334). As a direct consequence, the diffracted field suffers, upon traversing the half-plane, a discontinuity, the exact value of which may be approximated by

$$u_{d,\theta=0} - u_{d,\theta=2\pi} = -\frac{S\rho}{4\pi^2 c}\frac{1}{2t_+}\,\mathrm{e}^{\mathrm{i}\omega\tau_0}\,\pi\exp\left(-\mathrm{i}\omega\alpha\cos^2\frac{\theta_0}{2}\right)4\,\mathrm{erfc}\left(\sqrt{-\mathrm{i}\omega\alpha}\cos\frac{\theta_0}{2}\right), \quad (69)$$

where $\alpha = (t_+^2 - t_-^2)/2t_+$ and use has been made of the parity relation

$$\mathrm{erfc}(-z) = 2 - \mathrm{erfc}(z). \quad (70)$$

For a plane incident wave, $r_0 \to \infty$ and $\alpha \to 2r/c$ which makes the expression given in equation (69) tend in the limit to the exact form (see reference [20], p. 556). In reference (6) an approximate expression is given for the value of the total field at a point on the half-plane and for a source remote from it. This value is equivalent to the approximation u_{d3} as given by equation (19). It is worth noting that as expressed by equation (6), the value of the diffracted field at $\theta = 0$ is different from that at $\theta = 2\pi$. Although in the real physical space these two positions coincide they are distinguished from each other upon the introduction of the fictive infinitely thin half-plane. The halving of its angular parameter makes the diffracted field have a 4π period, and this reminds one of Sommerfeld's approach to the half-plane diffraction problem.

Some calculations made on the series of u_{d7} and u_{d8} show that these approximations have a poor convergence, which becomes worse for lower frequencies.

In conclusion, among the approximations presented here for the diffracted field caused by the presence of a hard half-plane in the acoustical field of a spherical wave, u_{d2}, u_{d4} and u_{d5} are most adequate for engineering purposes. For not too low frequencies, and for receiver positions that are not too near the geometrical boundaries of the incident and reflected fields, u_{d2} is accurate enough despite the simplicity of its form over that of u_{d4} and u_{d5}.

ACKNOWLEDGMENTS

The author is much indebted to Professor Sven Lindblad for following and helping in the development of this work. This project has been financially supported by a grant from the Technical Bridge Foundation in Lund, and this is gratefully acknowledged.

REFERENCES

1. M. Born and E. Wolf 1986 *Principles of Optics*. Oxford: Pergamon Press; sixth edition.
2. B. B. Baker and E. T. Copson 1950 *The Mathematical Theory of Huygens' Principle*. Oxford: Oxford University Press; second edition.
3. C. J. Bouwkamp 1954 *Reports on Progress in Physics* **17**, 35–100. Diffraction theory.
4. J. B. Keller 1962 *Journal of the Optical Society of America* **52**, 116–130. The geometrical theory of diffraction.
5. A. Sommerfeld 1954 *Lectures in Theoretical Physics*, Vol. IV, *Optics*. New York: Academic Press.
6. J. J. Bowman, T. B. A. Senior and P. L. E. Uslenghi 1987 *Electromagnetic and Acoustic Scattering by Simple Shapes*. New York: Hemisphere Publishing Corporation; second edition.
7. F. P. Mechel 1998 *Journal of Sound and Vibration* **219**, 581–601. The scattering at a corner with absorbing flanks and an absorbing cylinder.
8. E. P. Mechel 1998 *Journal of Sound and Vibration* **219**, 105–132. Scattering at rigid building corners.
9. E. P. Mechel 1998 *Journal of Sound and Vibration* **219**, 559–579. Improvement of corner shielding by an absorbing cylinder.
10. M. Möser 1995 *Acustica* **81**, 565–585. Die Wirkung von Zylindrischen Aufsäten an Schallschirmen.
11. M. A. Biot and I. Tolstoy 1957 *Journal of the Acoustical Society of America* **29**, 381–391. Formulation of wave propagation in infinite media by normal coordinates with an application to diffraction.
12. H. Medwin 1981 *Journal of the Acoustical Society of America* **69**, 1060–1064. Shadowing by finite noise barriers.
13. W. A. Kinney, C. S. Clay and G. A. Sandness 1983 *Journal of the Acoustical Society of America* **73**, 183–194. Scattering from a corrugated surface: comparison between experiment, Helmholtz–Kirchhoff theory and the facet ensemble method.
14. I. S. Gradshteyn and I. M. Ryzhik 1980 *Table of Integrals, Series and Products*. Orlando: Academic Press.
15. G. E. Roberts and H. Kaufman 1966 *Table of Laplace Transforms*. London: W. B. Saunders Company.
16. M. Abramovitz and I. Stegun 1972 *Handbook of Mathematical Functions*. New York: Dover Publications.
17. J. Spanier and K. Oldham 1987 *An Atlas of Functions*. Berlin: Springer-Verlag.
18. A. P. Prudnikov, Y. A. Brychkov and O. I. Marichev 1986 *Integrals and Series*, Vols. I–III, New York: Gordon and Breach Science Publishers.
19. H. Hönl, A. W. Maue and K. Westphal 1961 *Handbuch der Physik*, Vol. 25 "*Theori der Beugeung*". Berlin: Springer-Verlag.
20. D. S. Jones 1985 *Acoustic and Electromagnetic Waves*, 554. Oxford: Clarendon Press.
21. P. Ambaud and A. Bergassoli 1972 *Acoustica* **27**, 291–298. Le Problème du dièdre en acoustique.
22. W. J. Hadden and A. D. Pierce 1981 *Journal of the Acoustical Society of America* **69**, 1266–1276 and Erratum. 1982 **71**, 1290. Sound diffraction around screens and wedges of arbitrary point source locations.
23. F. G. Friedlander 1958 *Sound Pulses*. Cambridge: Cambridge University Press.
24. NAG 1985 The Numerical Algorithms Group, Mark 15, Oxford.

APPENDIX A: EXPRESSION OF EQUATION (3) FOR THE CASE OF A HALF-PLANE AND $z = 0$

This case is of the most occurrence in experimentation and a simplification of the complicated expression is often desired. With $\theta_W = 2\pi$ equation (3) becomes

$$u_d(t) = \frac{-S\rho c}{8\pi^2}\{\beta\}\frac{1}{rr_0 \sinh y}\,\mathrm{e}^{-y/2}, \tag{A.1}$$

where β is now

$$\{\beta\} = \frac{\sin[(\pi \pm \theta \pm \theta_0)/2]}{1 - 2\mathrm{e}^{-y/2}\cos[(\pi \pm \theta \pm \theta_0)/2] + \mathrm{e}^{-y}} = \frac{\sin[(\pi \pm \theta \pm \theta_0)/2]}{2\mathrm{e}^{-y/2}\{\cosh(y/2) - \cos[(\pi \pm \theta \pm \theta_0)/2]\}} \tag{A.2}$$

with (see reference [14], f. 1.622.6, p. 47]

$$y = \operatorname{arccosh} a = \log[a + \sqrt{a^2 - 1}], \quad a = \frac{c^2t^2 - (r^2 + r_0^2)}{2rr_0}, \tag{A.3}$$

$$\cosh(y/2) = \sqrt{(\cosh y + 1)/2}, \tag{A.4}$$

$$\sinh y = \frac{\mathrm{e}^y - \mathrm{e}^{-y}}{2} = \sqrt{\cosh^2 y - 1} = \sqrt{a^2 - 1}. \tag{A.5}$$

For the different sign combinations in the argument of the sin and cos of $(\pi \pm \theta \pm \theta_0)/2$ one gets

	$+\,+$	$+\,-$	$-\,+$	$-\,-$
$\sin[(\pi \pm \theta \pm \theta_0)/2]$	$\cos +$	$\cos -$	$\cos -$	$\cos -$
$\cos[(\pi \pm \theta \pm \theta_0)/2]$	$-\sin +$	$-\sin -$	$\sin -$	$\sin +$

with

$$\begin{matrix}\cos\\ \sin\end{matrix} \pm = \begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{\theta}{2} \pm \frac{\theta_0}{2}\right).$$

Hence,

$$\{\beta\} \sim \frac{1}{2}\left[\frac{\cos +}{x + \sin +} + \frac{\cos -}{x + \sin -} + \frac{\cos -}{x - \sin -} + \frac{\cos +}{x - \sin +}\right], \tag{A.6}$$

where x stands for $x = \cosh(y/2) = \sqrt{(a+1)/2}$ β then becomes

$$\{\beta\} \sim x\left[\frac{\cos +}{x^2 - \sin +^2} + \frac{\cos -}{x^2 - \sin -^2}\right] \tag{A.7}$$

and $u_d(t)$ then takes the form

$$u_d(t) = \frac{-S\rho c}{8\pi^2}\, x\left[\frac{\cos +}{x^2 - \sin +^2} + \frac{\cos -}{x^2 - \sin -^2}\right]\frac{1}{rr_0\sqrt{a^2 - 1}}. \tag{A.8}$$

Next,

$$rr_0\sqrt{a^2 - 1} = rr_0\left\{\left[\frac{c^2t^2 - (r^2 + r_0^2)}{2rr_0}\right]^2 - 1\right\}^{1/2} = \frac{c^2}{2}\sqrt{(t^2 - t_+^2)(t^2 - t_-^2)} \tag{A.9}$$

with

$$t_\pm = (r \pm r_0)/c \tag{A.10}$$

and because

$$rr_0 = c^2(t_+^2 - t_-^2)/4, \tag{A.11}$$

x may also be expressed in terms of t_+ and t_-, namely,

$$x = \cosh(y/2) = \sqrt{(a+1)/2} = \frac{c}{2}\sqrt{(t^2 - t_-^2)/rr_0} = \sqrt{\frac{t^2 - t_-^2}{t_+^2 - t_-^2}}. \tag{A.12}$$

Then, inserting equation (A.9) and (A.11) in equation (A.8) one gets

$$u_d(t) = \frac{-S\rho}{4\pi^2 c}\sqrt{\frac{t_+^2 - t_-^2}{t^2 - t_+^2}}\left\{\frac{\cos[(\theta+\theta_0)/2]}{t^2 - t_-^2 - (t_+^2 - t_-^2)\sin^2[(\theta+\theta_0)/2]}\right.$$
$$\left. + \frac{\cos[(\theta-\theta_0)/2]}{t^2 - t_-^2 - (t_+^2 - t_-^2)\sin^2[(\theta-\theta_0)/2]}\right\}, \tag{A.13}$$

which when using $\sin^2 = 1 - \cos^2$ gives equation (6).

The expression for the short time range after the least time $\tau_0 = t_+ = (r + r_0)/c$, is given as an expression of $\tau = t - \tau_0$, which when introduced in equation (6) with the necessary transformations leads to equation (8).

APPENDIX B: EXPRESSION OF $\operatorname{erfc}(\sqrt{-\mathrm{i}\omega a})$ IN TERMS OF THE FRESNEL INTEGRALS FOR A REAL POSITIVE ARGUMENT

For the complex argument $\sqrt{-\mathrm{i}\omega a} = (1-\mathrm{i})\sqrt{\pi f a}$ ($f = \omega/2\pi$ is the frequency), the function $\operatorname{erfc}(z)$ defined by

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty \mathrm{e}^{-u^2}\,\mathrm{d}u \tag{B.1}$$

is sometimes not available in calculation softwares and a reformulation in available functions is desirable. With, respectively, the following artifices (see reference [16], f. 7.1.3, p. 299, and f. 7.3.22, p. 301]

$$\operatorname{erfc}(-\mathrm{i}z) = \mathrm{e}^{z^2} w(z), \tag{B.2}$$

$$\mathrm{C}(x) + \mathrm{iS}(x) = \frac{1+\mathrm{i}}{2}\left\{1 - \mathrm{e}^{\mathrm{i}\pi x^2/2}\, w\left[\frac{\sqrt{\pi}}{2}(1+\mathrm{i})x\right]\right\}, \tag{B.3}$$

where the Fresnel cosine and sine integrals are defined by

$$\mathrm{C}, \mathrm{S}(x) = \int_0^x \cos, \sin\left(\frac{\pi}{2}u^2\right)\mathrm{d}u. \tag{B.4}$$

Then by setting $z = (1+\mathrm{i})\sqrt{\pi f a}$ in equation (B.1), and rearranging equation (B.3) one gets finally

$$\operatorname{erfc}(\sqrt{-\mathrm{i}\omega a}) = \operatorname{erfc}[\sqrt{-\mathrm{i}2\pi f a}] = 1 - \frac{2}{1+\mathrm{i}}[\mathrm{C}(x) + \mathrm{iS}(x)] \tag{B.5}$$

or

$$\operatorname{erfc}(\sqrt{-\mathrm{i}\omega a}) = \operatorname{erfc}[\sqrt{-\mathrm{i}2\pi f a}] = 1 - [\mathrm{C}(x) + \mathrm{S}(x)] + \mathrm{i}[\mathrm{C}(x) - \mathrm{S}(x)] \tag{B.6}$$

with $x = 2\sqrt{fa}$.

APPENDIX C: EVALUATION OF THE FOURIER TRANSFORM OF $t^{3/2}/(t + \tau_{1,2})$

From the general formula (see reference [15], f. 1.2.10, p. 13),

$$\int_0^\infty t^\mu (t+a)^\nu \mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = \Gamma(\mu+1)a^{(\mu+\nu)/2}(-\mathrm{i}\omega)^{-(\mu+\nu+2)/2}\mathrm{e}^{-(\mathrm{i}\omega a/2)}\mathrm{W}_{(\nu-\mu)/2,(\mu+\nu+1)/2}(-\mathrm{i}\omega a), \tag{C.1}$$

where Γ is the Gamma function and $W_{\alpha,\beta}$ is the Whittaker function, then one gets for the case where $\mu = 3/2$, $\nu = -1$ and $a = \tau_{1,2}$,

$$\int_0^\infty \frac{t^{3/2}}{t+\tau_{1,2}}\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = \Gamma\left(\frac{5}{2}\right)\tau_{1,2}^{1/4}(-\mathrm{i}\omega)^{-5/4}\mathrm{e}^{-\mathrm{i}\omega\tau_{1,2}/2}\mathrm{W}_{-5/4,3/4}(-\mathrm{i}\omega\tau_{1,2}). \tag{C.2}$$

Next, the Whittaker function may be expressed in terms of the Tricomi function U (see reference [16], f. 13.1.33, p. 505) as

$$\mathrm{W}_{\kappa,\mu}(z) = \mathrm{e}^{-z/2}z^{1/2+\mu}\mathrm{U}\left(\frac{1}{2}+\mu-\kappa, 1+2\mu; z\right) \tag{C.3}$$

and for the present case

$$\mathrm{W}_{-5/4,3/4}(-\mathrm{i}\omega\tau_{1,2}) = \mathrm{e}^{\mathrm{i}\omega\tau_{1,2}/2}(-\mathrm{i}\omega\tau_{1,2})^{5/4}\mathrm{U}\left(\frac{5}{2},\frac{5}{2};-\mathrm{i}\omega\tau_{1,2}\right). \tag{C.4}$$

For the nth derivative, the Tricomi functions satisfy the recurrence relation increasing the orders of both parameters of the function (see reference [16], f. 13.4.22, p. 507)

$$\mathrm{U}^{(n)}(a,b;z) = (-1)^n(a)_n\mathrm{U}(a+n,b+n;z), \tag{C.5}$$

where $(a)_n$ denotes the Pochammer polynomial. For U in equation (C.4) one then gets

$$\mathrm{U}\left(\frac{5}{2},\frac{5}{2};-\mathrm{i}\omega\tau_{1,2}\right) = \frac{1}{(1/2)_2}\mathrm{U}^{(2)}\left(\frac{1}{2},\frac{1}{2};-\mathrm{i}\omega\tau_{1,2}\right) \tag{C.6}$$

and $\mathrm{U}(1/2, 1/2, -\mathrm{i}\omega\tau_{1,2})$ may be expressed in terms of the incomplete gamma function Γ (see reference [18], Vol. III, p. 584)

$$\mathrm{U}\left(\frac{1}{2},\frac{1}{2};-\mathrm{i}\omega\tau_{1,2}\right) = \mathrm{e}^{-\mathrm{i}\omega\tau_{1,2}}\Gamma\left(\frac{1}{2},-\mathrm{i}\omega\tau_{1,2}\right). \tag{C.7}$$

Furthermore, for the differentiation operation, the function $\Gamma(\nu, z)$ satisfies (see reference [18], Vol. II, p. 726)

$$\Gamma'(\nu,z) = -z^{\nu-1}\mathrm{e}^{-z}. \tag{C.8}$$

Hence, using this property for equation (C.5) one gets

$$\mathrm{U}\left(\frac{5}{2},\frac{5}{2};-\mathrm{i}\omega\tau_{1,2}\right)=\frac{4}{3}\left[\mathrm{e}^{z}\Gamma\left(\frac{1}{2},-\mathrm{i}\omega\tau_{1,2}\right)-\frac{1}{\sqrt{-\mathrm{i}\omega\tau_{1,2}}}\left(1+\frac{1}{2\mathrm{i}\omega\tau_{1,2}}\right)\right] \tag{C.9}$$

and with (see reference [18], Vol. II, p. 726)

$$\Gamma\left(\frac{1}{2},z\right)=\sqrt{\pi}\,\mathrm{erfc}(\sqrt{z}) \tag{C.10}$$

one sums up for the integral in equation (C.2)

$$\int_0^{\infty}\frac{t^{3/2}}{t+\tau_{1,2}}\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t=\sqrt{\pi}\tau_{1,2}^{3/2}\left[\sqrt{\pi}\mathrm{e}^{-\mathrm{i}\omega\tau_{1,2}}\,\mathrm{erfc}(\sqrt{-\mathrm{i}\omega\tau_{1,2}})-\frac{1}{\sqrt{-\mathrm{i}\omega\tau_{1,2}}}\left(1+\frac{1}{\mathrm{i}\omega\tau_{1,2}}\right)\right], \tag{C.11}$$

where use has also been, respectively, made of (see reference [16], formulae 6.1.12, 6.1.22, and pp. 255, 256).

$$\Gamma\left(\frac{5}{2}\right)=\frac{3\sqrt{\pi}}{4}\quad\text{and of}\left(\frac{1}{2}\right)_2=\frac{3}{4}. \tag{C.12}$$

APPENDIX D: EVALUATION OF THE FOURIER TRANSFORM OF $t^{3/2}/\sqrt{(t+2t_+)}$

Using equation (C.1) one gets

$$\int_0^{\infty}\frac{t^{3/2}}{\sqrt{t+2t_+}}\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t=\Gamma\left(\frac{5}{2}\right)(2t_+)^{1/2}(-\mathrm{i}\omega)^{-3/2}\mathrm{e}^{-\mathrm{i}\omega t_+}\mathrm{W}_{-1,1}(-\mathrm{i}\omega 2t_+). \tag{D.1}$$

Then, with the help of equation (C.3)

$$\mathrm{W}_{-1,1}(-\mathrm{i}\omega 2t_+)=\mathrm{e}^{\mathrm{i}\omega t_+}(-\mathrm{i}\omega 2t_+)^{3/2}\mathrm{U}\left(\frac{5}{2},3;-\mathrm{i}\omega 2t_+\right). \tag{D.2}$$

Then using equation (C.5) leads to

$$\mathrm{U}\left(\frac{5}{2},3;-\mathrm{i}\omega 2t_+\right)=\frac{1}{(1/2)_2}\mathrm{U}^{(2)}\left(\frac{1}{2},1;-\mathrm{i}\omega 2t_+\right). \tag{D.3}$$

$\mathrm{U}(1/2, 1; z)$ may be expressed in terms of the modified Bessel function K_0 (see reference [16], f. 13.6.21, p. 510),

$$\mathrm{U}\left(\frac{1}{2},1;z\right)=\frac{1}{\sqrt{\pi}}\mathrm{e}^{z/2}\mathrm{K}_0(z/2) \tag{D.4}$$

and using the differentiation properties of K_0 and K_1, namely that (see reference [16], f. 9.6.27, p. 376)

$$\mathrm{K}_0'(z)=-\mathrm{K}_1(z)\Rightarrow\mathrm{K}_0'(z/2)=-\frac{\mathrm{K}_1(z/2)}{2} \tag{D.5}$$

and (see reference [16], f. 9.6.28, p. 376)

$$\mathrm{K}_1'(z) = -\mathrm{K}_0(z) - \frac{\mathrm{K}_1(z)}{z} \Rightarrow \mathrm{K}_1'(z/2) = -\frac{1}{2}\left[\mathrm{K}_0(z/2) + \frac{\mathrm{K}_1(z/2)}{z/2}\right] \tag{D.6}$$

in equation (D.3) gives the final result

$$\int_0^{\infty} \frac{t^{3/2}}{\sqrt{t+2t_+}} \mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = \frac{1}{2}(2t_+)^2 \mathrm{e}^{-\mathrm{i}\omega t_+}\left[\mathrm{K}_0(-\mathrm{i}\omega t_+) - \mathrm{K}_1(-\mathrm{i}\omega t_+)\left(1 + \frac{1}{\mathrm{i}\omega 2t_+}\right)\right]. \tag{D.7}$$

APPENDIX E: EXPRESSION OF $\mathrm{U}\left(n+\frac{1}{2}, n+1; z\right)$ IN TERMS OF $\mathrm{U}\left(\frac{1}{2}, 1; z\right)$

In order to express $\mathrm{U}(n+1/2, n+1; z)$ in terms of $\mathrm{U}(1/2, 1; z)$ which one can call, for brevity, respectively, U_n and U_0 one needs the recurrence relation (see reference [16], f. 13.4.21, p. 507)

$$\mathrm{U}'(a, b; z) = -a\mathrm{U}(a+1, b+1; z) \tag{E.1}$$

and an expression for the first element in the series of $\mathrm{U}(n+1/2, n+1; z)$ which is given by (see reference [16], f. 13.6.21, p. 510)

$$\mathrm{U}\left(\frac{1}{2}, 1; z\right) = \frac{1}{\sqrt{\pi}} \mathrm{e}^{z/2}\mathrm{K}_0(z/2), \tag{E.2}$$

z stands here for $z = -\mathrm{i}2\omega t_+$.

Also one has relations (see reference [16], f. 9.6.27, p. 376)

$$\mathrm{K}_0'(z) = -\mathrm{K}_1(z) \Rightarrow \mathrm{K}_0'(z/2) = -\frac{\mathrm{K}_1(z/2)}{2} \tag{E.3}$$

and (see reference [16], f. 9.6.28, p. 376)

$$\mathrm{K}_1'(z) = -\mathrm{K}_0(z) - \frac{\mathrm{K}_1(z)}{z} \Rightarrow \mathrm{K}_1'(z/2) = -\frac{1}{2}\left[\mathrm{K}_0(z/2) + \frac{\mathrm{K}_1(z/2)}{z/2}\right]. \tag{E.4}$$

These last two relations are important inasmuch as they help to express the U_n of any order just in terms of K_0 and K_1. The first few U_n's are then given by

$$\mathrm{U}_1 = -\frac{1}{\sqrt{\pi}} \mathrm{e}^{z/2}[\mathrm{K}_0(z/2) - \mathrm{K}_1(z/2)],$$

$$\mathrm{U}_2 = \frac{1}{3}\frac{1}{\sqrt{\pi}} \mathrm{e}^{z/2}\left[2\mathrm{K}_0(z/2) - \mathrm{K}_1(z/2)\left(2 + \frac{1}{z/2}\right)\right],$$

$$\mathrm{U}_3 = -\frac{1}{3{\cdot}5}\frac{1}{\sqrt{\pi}} \mathrm{e}^{z/2}\left[\mathrm{K}_0(z/2)\left(4 + \frac{1}{z/2}\right) + \mathrm{K}_1(z/2)\left(-4 - \frac{3}{z/2}\right)\right]. \tag{E.5}$$

If one supposes for $n > 4$ that

$$\mathrm{U}_{n-1} = \frac{(-1)^{n-1}}{1\cdot 3\cdot 5\cdots(2n-3)}\frac{1}{\sqrt{\pi}}\,\mathrm{e}^{z/2}\left[\mathrm{K}_0(z/2)\sum_{i=1}^{n-2}\frac{\alpha_{n-1,i}}{(z/2)^{i-1}} + \mathrm{K}_1(z/2)\sum_{i=1}^{n-1}\frac{\beta_{n-1,i}}{(z/2)^{i-1}}\right] \tag{E.6}$$

or, which when using the property (see reference [16], f. 6.1.12 and 6.1.8, p. 255)

$$(2n-3)!! = 1\cdot 3\cdot 5\cdots(2n-3) = 2^{n-1}\frac{\Gamma(n-1/2)}{\sqrt{\pi}}, \tag{E.7}$$

may be rewritten as

$$\mathrm{U}_{n-1} = \frac{(-1)^{n-1}}{2^{n-1}\Gamma(n-1/2)}\,\mathrm{e}^{z/2}\left[\mathrm{K}_0(z/2)\sum_{i=1}^{n-2}\frac{\alpha_{n-1,i}}{(z/2)^{i-1}} + \mathrm{K}_1(z/2)\sum_{i=1}^{n-1}\frac{\beta_{n-1,i}}{(z/2)^{i-1}}\right], \tag{E.8}$$

then using equations (E.1), (E.3) and (E.4) one can show that

$$\mathrm{U}_n = \frac{(-1)^n}{2^n\Gamma(n+1/2)}$$

$$\mathrm{e}^{z/2}\left\{\begin{aligned}&\mathrm{K}_0(z/2)\left[\sum_{i=1}^{n-2}\left(\frac{\alpha_{n-1,i}-\beta_{n-1,i}}{(z/2)^{i-1}} - \frac{(\mathrm{i}-1)\alpha_{n-1,i}(\mathrm{i}-1)}{(z/2)^i}\right) - \frac{\beta_{n-1,n-1}}{(z/2)^{n-2}}\right]\\ &+\mathrm{K}_1(z/2)\left[\sum_{i=1}^{n-2}\left(\frac{\beta_{n-1,i}-\alpha_{n-1,i}}{(z/2)^{i-1}} - \frac{\beta_{n-1,i}(\mathrm{i}-2)}{(z/2)^i} + \frac{\beta_{n-1,n-1}}{(z/2)^{n-2}} - \frac{\beta_{n-1,n-1}(n-3)}{(z/2)^{n-1}}\right]\right]\end{aligned}\right\} \tag{E.9}$$

and identifying with

$$\mathrm{U}_n = \frac{(-1)^n}{2^n\Gamma(n+1/2)}\,\mathrm{e}^{z/2}\left[\mathrm{K}_0(z/2)\sum_{i=1}^{n-1}\frac{\alpha_{n,i}}{(z/2)^{i-1}} + \mathrm{K}_1(z/2)\sum_{i=1}^{n}\frac{\beta_{n,i}}{(z/2)^{i-1}}\right] \tag{E.10}$$

one gets for the expressions of the coefficients in U_n:

for $i = 1$

$$\alpha_{n,1} = \alpha_{n-1,1} - \beta_{n-1,i}, \qquad \beta_{n,1} = \beta_{n-1,1} - \alpha_{n-1,i}$$

for $i = 2, 3, \ldots, n-2$

$$\alpha_{n,i} = \alpha_{n-1,i} - (i-2)\alpha_{n-1,i-1} - \beta_{n-1,i}, \qquad \beta_{n,i} = \beta_{n-1,i} - (i-1))\beta_{n-1,i-1} - \alpha_{n-1,i},$$

for $i = n-1$

$$\alpha_{n,n-1} = -(n-3)\alpha_{n-1,n-2} - \beta_{n-1,n-1}, \qquad \beta_{n,n-2} = -n\beta_{n-1,n-2} + \beta_{n-1,n-1},$$

and for $i = n$

$$\beta_{n,n} = -(n-3)\beta_{n-1,n-1}.$$

APPENDIX F: EVALUATION OF U$(a, a+1; z)$

From the important transformation formula relating two Tricomi functions of the same argument (see reference [16], f. 13.1.29, p. 505)

$$\mathrm{U}(a, b; z) = z^{1-b}\mathrm{U}(1 + a - b, 2 - b; z), \tag{F.1}$$

one gets for $b = a + 1$

$$\mathrm{U}(a, a + 1; z) = z^{-a}\mathrm{U}(0, 1 - a; z) \tag{F.2}$$

and setting $n = 0$ and $b = 1 - a$ in (see reference [18], Vol. III, f. 7.11.4.12, p. 584)

$$\mathrm{U}(-n, b; z) = (-1)^n n!\,\mathrm{L}_n^{b-1}(z), \tag{F.3}$$

one gets

$$\mathrm{U}(0, 1 - a; z) = \mathrm{L}_0^{-a}(z) \tag{F.4}$$

and the generalized Laguerre polynomial $\mathrm{L}_0^a(z)$ satisfies (see reference [18], Vol. II, p. 733)

$$\mathrm{L}_0^{\lambda}(z) = 1. \tag{F.5}$$

So, to sum up,

$$\mathrm{U}(a, a + 1; z) = z^{-a}. \tag{F.6}$$

APPENDIX G: EVALUATION OF THE INTEGRALS I_{1nm1}, I_{1nm2} AND I_{nm2} AS GIVEN BY EQUATION (63a–c)

(a) Evaluation of $I_{1nm1} = \displaystyle\int_0^{\tau_{1,2}} \tau^{n+m-1/2}\,\mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\tau$

One has (see reference [18], f. 13.2.3, p. 137)

$$\int_0^t t^{\lambda-1}\mathrm{e}^{-\alpha t}\,\mathrm{d}t = \frac{1}{\alpha^{\lambda}}\gamma(\lambda, \alpha t), \tag{G.1}$$

where γ is the incomplete gamma function, so one gets

$$I_{1nm1} = \frac{1}{(-\mathrm{i}\omega)^{n+m+1/2}}\gamma\left(n + m + \frac{1}{2}, -\mathrm{i}\omega\tau_{1,2}\right). \tag{G.2}$$

The function γ satisfies also the recurrence relation (see reference [18], p. 726)

$$\gamma(v + 1, z) = v\gamma(v, z) - z^{v}\mathrm{e}^{-z} \tag{G.3}$$

with

$$\gamma\left(\frac{1}{2}, z\right) = \sqrt{\pi}\,\mathrm{erf}(\sqrt{z}). \tag{G.4}$$

With the help of equation (B.5) this gives for $z = -\,\mathrm{i}\omega\tau_{1,2}$

$$\operatorname{erf}(\sqrt{-\,\mathrm{i}\omega\tau_{1,2}}) = 1 - \operatorname{erfc}\sqrt{-\,\mathrm{i}\omega\tau_{1,2}} = \mathrm{C}(x) + \mathrm{S}(x) - \mathrm{i}[\mathrm{C}(x) - \mathrm{S}(x)] \tag{G.5}$$

with $x = 2\sqrt{\omega\tau_{1,2}/2\pi}$ and C and S, respectively, the cosine and the sine Fresnel integrals as defined in equation (B.4).

(b) Evaluation of $I_{1nm2} = \int_{\tau_{1,2}}^{2t_+} \tau^{n-m-1/2}\,\mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\tau$

For $n - m + 1/2 > 0$ one uses the results in (a) for I_{1nm1}. For $n - m + 1/2 \leqslant 0$,

$$I_{1nm2} = \int_0^{2t_+} - \int_0^{\tau_{1,2}} \tau^{n-m-1/2}(\cos + \mathrm{i}\sin)(\omega\tau)\,\mathrm{d}\tau. \tag{G.6}$$

Using (see reference [14], ff. 2.632.2 and 2.632.4, p. 183)

$$\int t^{\lambda-1}\begin{matrix}\cos\\ \sin\end{matrix}(\omega t)\,\mathrm{d}t = -\frac{1}{2\omega^\lambda}\left\{\begin{matrix}\mathrm{e}^{\mathrm{i}\lambda\pi/2}\\ \mathrm{e}^{\mathrm{i}(\lambda-1)\pi/2}\end{matrix}\,\Gamma(\lambda, -\,\mathrm{i}\omega t) + \begin{matrix}\mathrm{e}^{-\,\mathrm{i}\lambda\pi/2}\\ \mathrm{e}^{\mathrm{i}(1-\lambda)\pi/2}\end{matrix}\,\Gamma(\lambda, \mathrm{i}\omega t)\right\} \tag{G.7}$$

for $\lambda = n - m + \frac{1}{2}$ one gets

$$I_{1nm2} = -\frac{\mathrm{e}^{\mathrm{i}(n-m+1/2)\pi/2}}{\omega^{n-m+1/2}}\,\Gamma\left(n - m + \frac{1}{2}, -\,\mathrm{i}\omega\tau\right)\Bigg|_{\tau_{1,2}}^{2t_+}, \tag{G.8}$$

The incomplete gamma function $\Gamma(v, z)$ satisfies also a recurrence relation, namely (see reference [18], p. 726)

$$\Gamma(v + 1, z) = v\Gamma(v, z) + z^v \mathrm{e}^{-z} \tag{G.9}$$

with (see reference [18], p. 726)

$$\Gamma\left(\frac{1}{2}, z\right) = \sqrt{\pi}\operatorname{erfc}(\sqrt{z}) \tag{G.10}$$

and where $\operatorname{erfc}(\sqrt{z})$ for $z = \mathrm{i}\omega\tau$ may be evaluated according to equation (B.5).

(c) Evaluation of $I_{2nm} = \int_{2t_+}^{\infty} \tau^{n-m-2}\,\mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\tau$

This integral may be evaluated by using (see reference [18], f. 1.3.2.4, p. 137)

$$I_{2nm} = (-\,\mathrm{i}\omega)^{n+m+1}\Gamma(-\,n - m - 1, -\,\mathrm{i}\omega 2t_+) \tag{G.11}$$

and the recurrence relation (G.9) with now for an integer order of the incomplete gamma function (see reference [18], p. 726]

$$\Gamma(0, z) = -\,\mathrm{Ei}(-\,z). \tag{G.12}$$

Ei is the exponential integral and for a pure imaginary argument, it may be expressed in terms of the cosine and sine integrals of a real argument according to (see reference [17], f. 37:11:6, p. 358)

$$\mathrm{Ei}(iy) = \mathrm{Ci}(y) + \mathrm{i}\left[\mathrm{Si}(y) - \frac{\pi}{2}\mathrm{sgn}(y)\right] \tag{G.13}$$

with

$$\mathrm{Ci}(y) = -\int_y^\infty \frac{\cos(x)}{x}\,\mathrm{d}x \quad \text{and} \quad \mathrm{Si}(y) = \int_0^y \frac{\sin(x)}{x}\,\mathrm{d}x. \tag{G.14}$$

(In equation (G.13) $y = 2\omega t_+$ is positive.)