Journal of Sound and Vibration (2002) **252**(2), 385–388
doi:10.1006/jsvi.2001.3946, available online at http://www.idealibrary.com on IDEAL®

LOWER FREQUENCIES OF TRANSVERSE VIBRATIONS OF CIRCULAR MEMBRANES WITH DISTURBED BOUNDARIES

P. A. A. Laura and E. Romanelli

Department of Engineering – CONICET, *Universidad Nacional del Sur*, 8000 *Bahía Blanca*, *Argentina*

(*Received* 26 *July* 2001)

1. INTRODUCTION

In an interesting article, Wang has recently treated the case of vibrating circular membranes with mass density perturbations [1].

It is the goal of the present investigation to analyze the variation of the lower natural frequencies of circular membranes in the case of disturbed boundaries; see Figure 1. It is assumed that the configuration in the z-plane is mapped onto a unit circle in the ξ-plane by means of the analytic function [2]

$$z = \frac{a}{1+m}(1 + m\xi^n)\xi, \tag{1}$$

where a is the radius of the circumscribing circle and $m \leqslant 1/(n+1)$.

The governing Helmholtz equation is transformed using equation (1) and the resulting partial differential equation is solved using polynomial co-ordinate functions in the radial and azimuthal variables. The radial component contains Rayleigh's optimization exponential parameter "p" which allows for minimization of the eigenvalues under investigation.

The problem is also of considerable interest in other fields like wave propagation in electromagnetic and acoustic waveguides, liquid oscillations in a basin, etc., since an ideal circular shape does possess, in some real situations, geometric alterations.

It is important to point out that Lord Rayleigh presented an approximate method for obtaining natural frequencies of vibrating circular membranes whose boundaries deviate slightly from a circular shape [3]. The functional relation (1) was also considered in a previous study [4] but the co-ordinate functions did not take the azimuthal variation into account nor did they contain an optimization parameter.

2. APPROXIMATE ANALYTICAL SOLUTION

The transverse motion of a thin, perfectly flexible membrane is governed by the two-dimensional wave equation,

$$\nabla^2 u(x, y, t) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}(x, y, t), \tag{2}$$

0022-460X/02/$35.00

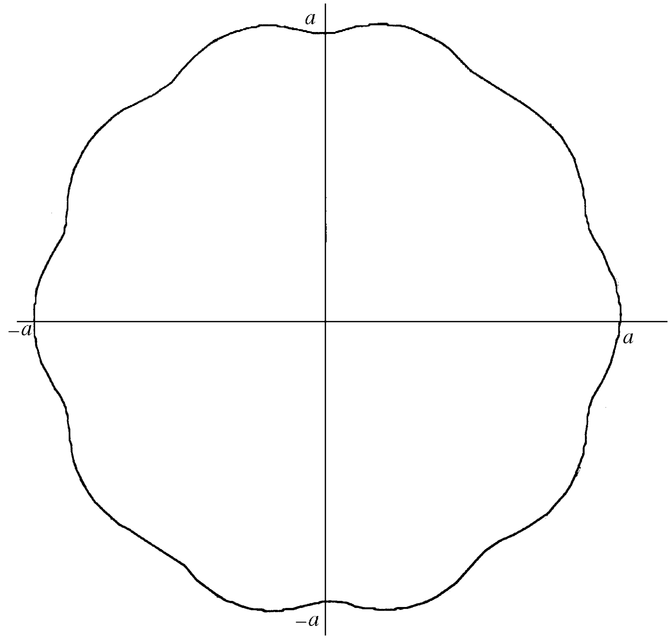

Figure 1. Circular membrane with slightly disturbed boundary ($n = 10$, $m = 0{\cdot}025$).

which for normal modes becomes

$$\nabla^2 U(x, y) + \left(\frac{\omega}{c}\right)^2 U(x, y) = 0, \tag{3}$$

where ω denotes the natural frequency and $U = 0$ on the boundary. If the boundary of the membrane is a curve natural to any of the common co-ordinate systems for which equation (3) separates, the solution can be derived by classical methods, and may be expressed in terms of known transcendental functions. For more "exotic" boundaries it is convenient to use a conformal mapping approach [4].

In the case of membranes of corrugated boundaries which can be mapped onto a unit circle in the complex plane by means of equation (1), the problem can be reduced to the solution of the transformed differential system [2]

$$\nabla^2 U(\xi, \bar{\xi}) + \left(\frac{\omega}{c}\right)^2 |f'(\xi)|^2 U(\xi, \bar{\xi}) = 0, \qquad U(\xi, \bar{\xi})|_{\sqrt{\xi, \bar{\xi}} = 1} = 0, \tag{4a, b}$$

where

$$f'(\xi) = a_1[1 + m(n+1)\xi^n], \quad a_1 = a/(1+m), \quad \text{and} \quad |f'(\xi)|^2 = f'(\xi)\overline{f'(\xi)}. \tag{5a, b}$$

In the manner of reference [4], Galerkin's method will be used.

The procedure is as follows: $U(\xi, \bar{\xi})$ is approximated by a linear combination of independent co-ordinate functions which identically satisfy the boundary conditions, i.e.,

$$U \cong U_a(\xi, \bar{\xi}) = \sum_1^{k_b} b_j \psi_j(\xi, \bar{\xi}) + \sum_1^{k_c} c_j \varphi_j(\xi, \bar{\xi}), \tag{6}$$

where

$$\psi_j(\xi, \bar{\xi}) = \psi_j(r) \quad (j = 1, 2, \ldots, k_b),$$
$$\varphi_j(\xi, \bar{\xi}) = r\psi_j \cos(n\theta) = \varphi_j(\theta, r) \quad (j = 1, 2, \ldots, k_c) \tag{7a, b}$$

and $\psi_j = 1 - r^{pj}$.

By substituting equation (6) into equation (4a), one obtains the error or residual function $\varepsilon(r, \theta)$ which multiplied by r yields

$$\varepsilon_j r = \sum_1^{k_b} b_j(\psi_j'' r + \psi_j' + \alpha\eta\psi \cdot r) + \sum_1^{k_c} c_j\left(\varphi_j'' r + \varphi_j' + \frac{1}{r}\varphi_{\theta_j}'' + \alpha\eta\varphi_j r\right), \tag{8}$$

where φ_j' and φ_j'' are the first and second partial derivatives of φ_j with respect to r, φ_{θ_j}'' is the second partial derivative of φ_j with respect to θ, $\alpha = [\omega a/c(1 + m)]^2$ and

$$\eta = 1 + 2m(n + 1)r^n \cos(n\theta) + m^2(n + 1)^2 r^{2n}.$$

Multiplying equation (8) by each co-ordinate function and integrating with respect to r and θ, one obtains a linear system of homogeneous equations in the b_j's and c_j's once the orthogonality requirement is applied.

The non-triviality condition leads to a determinantal equation whose lowest root is the fundamental frequency coefficient of the system, $\Omega_j = (\omega_1/c)a$.

3. NUMERICAL RESULTS

Table 1 depicts values of Ω_1 and Ω_2 for $n = 1$ and $m = 0{\cdot}5$. These frequency parameters correspond to quasi-axisymmetric modes and they have been obtained for different values

TABLE 1

Frequency coefficients Ω_1 and Ω_2 corresponding to quasi-axisymmetric modes for $n = 1$, $m = 0{\cdot}5$

k_b, k_c	Ω_1	Ω_2
1, 0	3·2684†	—
	3·2863‡	
1, 1	3·0378†	—
	3·0489‡	
2, 0	3·2604†	7·3935
	3·2608‡	7·6549
2, 1	3·0296†	8·7532
	3·0398‡	9·1816
3, 0	3·2603†	7·2628
	3·2603‡	7·2628
2, 2	3·0220†	7·6369
	3·0227‡	7·7541
3, 1	3·0259†	8·4712
	3·0384‡	8·8599
4, 0	3·2603†	7·2625
	3·2603‡	7·2627

†Optimized value.
‡Eigenvalue determined for $p = 2$.

TABLE 2

Frequency coefficients Ω_1 and Ω_2 for $n = 10$, $m = 0{\cdot}025$

k_b, k_c	Ω_1	Ω_2
1, 0	2·4743	—
1, 1	2·4743	—
2, 0	2·4647	5·8014
2, 1	2·4645	5·7963
3, 0	2·4647	5·6602
2, 2	2·4643	5·7862
3, 1	2·4642	5·6601
4, 0	2·4647	5·6501

of k_b and k_c. For $k_b = k_c = 2$ one obtains the best approximation for Ω_1. This is due to the improvement introduced by taking into account the θ-dependence. On the other hand, the best approximation for Ω_2 is attained for $k_b = 4$ and $k_c = 0$. This is due to the fact that the r-dependence is very strong for the second quasi-axisymmetric mode. The values are in excellent agreement with those obtained in reference [5].

Table 2 shows values of Ω_1 and Ω_2 for $n = 10$ and $m = 0{\cdot}025$ (a slightly disturbed circular shape with 10 axes of symmetry).

The optimum value of Ω_1 is achieved for $k_b = 3$ and $k_c = 1$ while the best value of Ω_2 is obtained for $k_b = 4$ and $k_c = 0$.

One concludes that in the case of a slightly disturbed circular boundary, the fundamental frequency is altered by almost 3% while Ω_2 is increased by 2%.

ACKNOWLEDGMENTS

The present study has been sponsored by CONICET Research and Development Program and by Secretaría General de Ciencia y Tecnología of Universidad Nacional del Sur (Program Director: Professor R. E. Rossi).

REFERENCES

1. C. Y. WANG 1999 *Journal of Sound and Vibration* **227**, 682–684. Fundamental frequency of a wavy non-homogeneous circular membrane.
2. R. SCHINZINGER and P. A. A. LAURA 1991 *Conformal Mapping: Methods and Applications*. Amsterdam: Elsevier.
3. LORD RAYLEIGH 1877 *Theory of Sound* (two volumes). New York: Dover Publications; second edition, 1945 reissue.
4. P. A. A. LAURA and E. ROMANELLI 1974 *Journal of Sound and Vibration* **36**, 69–75. A note on the fundamental frequency of circular membranes with a class of boundary disturbances.
5. P. A. A. LAURA, K. NAGAYA and G. SÁNCHEZ SARMIENTO 1980 *IEEE Transactions on Microwave Theory and Techniques* **MTT-28**, 568–572. Numerical experiments on the determination of cutoff frequencies of waveguides of arbitrary cross section.