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FREE VIBRATION ANALYSIS OF NON-UNIFORM BEAMS WITH AN ARBITRARY NUMBER OF CRACKS AND CONCENTRATED MASSES

Q. S. Li

Department of Building and Construction, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong, People's Republic of China. E-mail: bcqsli@cityu.edu.hk

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An exact approach for free vibration analysis of a non-uniform beam with an arbitrary number of cracks and concentrated masses is proposed. A model of massless rotational spring is adopted to describe the local flexibility induced by cracks in the beam. Using the fundamental solutions and recurrence formulas developed in this paper, the mode shape function of vibration of a non-uniform beam with an arbitrary number of cracks and concentrated masses can be easily determined. The main advantage of the proposed method is that the eigenvalue equation of a non-uniform beam with any kind of two end supports, any finite number of cracks and concentrated masses can be conveniently determined from a second order determinant. As a consequence, the decrease in the determinant order as compared with previously developed procedures leads to significant savings in the computational effort and cost associated with dynamic analysis of non-uniform beams with cracks. Numerical examples are given to illustrate the proposed method and to study the effect of cracks on the natural frequencies and mode shapes of cracked beams.

1. INTRODUCTION

The problem of vibration of cracked beams has been extensively investigated because of its relevance to mechanical and aeronautical engineering. Dimarogonas [1] gave a state-of-the-art review of methods developed to analyze cracked structures. Thus, only some previous works that are directly related to the present study are mentioned below.

Cracks present a serious threat to proper performance of structures and machines. Most early investigations were concentrated on the analysis of the effect of a single crack on the dynamic behavior of simple structures, such as shafts and beams. Several studies introduced cracks into the mathematical model through a simple reduction of the stiffness on a given zone of cracked structures. In order to investigate dynamic characteristics due to the presence of real damage, the evaluation of changes in natural frequencies of a simple cantilever beam due to the presence of one or two cracks was addressed in several papers (e.g., references [2, 3]). Rizos *et al.* [4] developed an approach for vibration analysis of a cracked beam. Their approach leads to a system of $(4n + 4)$ equations for establishing the eigenvalue equation in the case of n cracks in a uniform beam. An improved analytical method for calculating natural frequencies of a uniform beam with an arbitrary number of cracks was proposed by Shifrin and Ruotolo [5]. This procedure was presented based on the use of massless rotational spring to describe the local flexibility induced by cracks and, as a main feature, leads to a system of $(n + 2)$ linear equations for determining the

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eigenvalue equation for a uniform beam with n cracks. It should be noted that these studies mentioned above have been confined to uniform beam with cracks.

In this paper, an attempt is made to present an exact approach for free vibration analysis of a non-uniform beam with an arbitrary number of cracks and concentrated masses. Shifrin and Ruotolo [5] used Dirac's delta function [$\delta(x)$] to express the governing equation for free vibration of a uniform beam with cracks, and the frequency equation of a uniform beam with n cracks can be determined from a system of $(n + 2)$ linear equations. This paper adopts the fundamental solutions and recurrence formulas developed here to express the jump of the slope due to a crack and that of shear force due to a concentrated mass. Based on a model of massless rotational spring to describe the local flexibility induced by cracks in the beam, the frequency equation of a non-uniform beam with n cracks and n concentrated masses can be conveniently established from a second order determinant. As a consequence, the decrease in the determinant order as compared with previously developed procedures (e.g., references [4, 5]) leads to significant savings in the computational effort and cost associated with dynamic analysis of cracked beams. On the other hand, Shifrin and Ruotolo [5] are confined to free vibration of a clamped-free beam with cracks, while the present method can be used to analyze free vibration of a non-uniform beam with any kind of two end supports, any finite number of cracks and concentrated masses. The numerical examples show that the effects of number, depth and location of cracks on the natural frequencies and mode shapes of cracked beams are significant, and the proposed procedure is an exact and efficient method. The comparison between the results calculated by the present method and those obtained by Liang *et al.* [6] is favorable, thus supporting the validity of the proposed method. There are other methods such as finite element method (FEM), which be can also used for vibration analysis of a structure with an arbitrary number of cracks (e.g., reference [7]). However, FEM will also lead to determinants with high order. It should be mentioned that the present analytical method and solutions that can be easily implemented could provide adequate insight into the physics of the problem. Meanwhile, the availability of the exact solutions will help in examining the accuracy of the approximated or numerical solutions.

2. THEORY

A non-uniform beam with an arbitrary number of cracks is shown in Figure 1. It is assumed that the number of cracks is n, and the n cracks are located at sections $x_1, x_2, \ldots,$

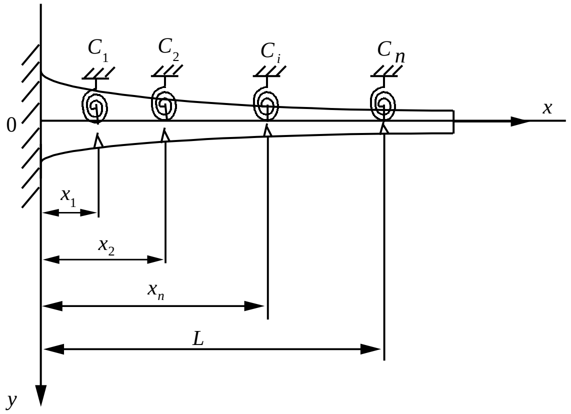

Figure 1. A non-uniform beam with n cracks.

x_n such that $0 < x_1 < x_2 < \cdots < x_n < L$. The beam is divided into $n+1$ segments by the n cracks. A model of massless rotational spring [8] is adopted to describe the local flexibility induced by cracks in the beam, as shown in Figure 1. As is well known, the difference between a beam with a crack at the ith section and the corresponding beam without crack is that the rotation at the ith section has a jump.

The governing differential equation of undamped free flexural vibration for an uncracked beam with variable cross-section can be written as [9]

$$\frac{\partial^2}{\partial x^2}\left[K(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right] + \bar{m}(x)\frac{\partial^2 y(x,t)}{\partial t^2} = 0, \tag{1}$$

in which $K(x)$ is the flexural stiffness, $\bar{m}(x)$ is the mass per unit length and $y(x,t)$ the transverse displacement

Using the method of separation of variables, one obtains the differential equation of the mode shape function of vibration as

$$\frac{d^2}{dx^2}\left[K(x)\frac{d^2 X(x)}{dx^2}\right] - \omega^2\bar{m}(x)X(x) = 0, \tag{2}$$

where $X(x)$ is the mode shape function of vibration. ω is the circular natural frequency.

The general solution of equation (2) can be expressed in the form

$$X(x) = C_1 S_1(x) + C_2 S_2(x) + C_3 S_3(x) + C_4 S_4(x) \tag{3}$$

where $S_i(x)$ and C_i $(i = 1, 2, 3, 4)$ are the linearly independent special solutions and integral constants of equation (2), respectively. Obviously, $S_i(x)$ $(i = 1, 2, 3, 4)$ are dependent on the expressions of $K(x)$ and $\bar{m}(x)$. The exact solutions for free vibration of a non-uniform beam for five types of distributions which cover many structural members are given in Appendix A.

In order to simplify the analysis for the title problem, based on the linearly independent solutions $S_i(x)$ $(i = 1, 2, 3, 4)$ presented in Appendix A, the linearly independent fundamental solutions denoted by $\bar{S}_i(x)$ $(i = 1, 2, 3, 4)$, which satisfy the following normalization condition at the origin of co-ordinate system

$$\begin{bmatrix} \bar{S}_1(0) & \bar{S}'_1(0) & \bar{S}''_1(0) & \bar{S}'''_1(0) \\ \bar{S}_2(0) & \bar{S}'_2(0) & \bar{S}''_2(0) & \bar{S}'''_2(0) \\ \bar{S}_3(0) & \bar{S}'_3(0) & \bar{S}''_3(0) & \bar{S}'''_3(0) \\ \bar{S}_4(0) & \bar{S}'_4(0) & \bar{S}''_4(0) & \bar{S}'''_4(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

can be easily constructed by

$$\begin{bmatrix} \bar{S}_1(x) \\ \bar{S}_2(x) \\ \bar{S}_3(x) \\ \bar{S}_4(x) \end{bmatrix} = \begin{bmatrix} S_1(0) & S'_1(0) & S''_1(0) & S'''_1(0) \\ S_2(0) & S'_2(0) & S''_2(0) & S'''_2(0) \\ S_3(0) & S'_3(0) & S''_3(0) & S'''_3(0) \\ S_4(0) & S'_4(0) & S''_4(0) & S'''_4(0) \end{bmatrix}^{-1} \begin{bmatrix} S_1(x) \\ S_2(x) \\ S_3(x) \\ S_4(x) \end{bmatrix} \tag{5}$$

The primes in equations (4) and (5) indicate differentiation with respect to the co-ordinate variable x.

Using the fundamental solutions, $\bar{S}_i(x)$, the mode shape function for the first interval $[0, x_1)$ can be expressed as

$$X_1(x) = X(0)\bar{S}_1(x) + X'(0)\bar{S}_2(x) - \frac{M(0)}{K(0)}\bar{S}_3(x) - \frac{1}{K(0)}[Q(0) - \mu(0)M(0)]\bar{S}_4(x), \quad x \in [0, x_1) \quad (6)$$

$$\mu(0) = \frac{K'(0)}{K(0)} \quad (7)$$

where $X(0)$, $X'(0)$, $M(0)$ and $Q(0)$ are the displacement, slope, bending moment and shear force of this beam at $x = 0$ respectively. They are called the initial parameters in this paper, only two of them are unknown for any kind of support configuration at $x = 0$. It is evident that $K'(0) = 0$ for a uniform beam.

The displacement, bending moment and shear force at all the boundaries of two neighboring segments are required to be continuous:

$$\begin{aligned} X_{i+1}(x_i) &= X_i(x_i), \\ M_{i+1}(x_i) &= M_i(x_i), \\ Q_{i+1}(x_i) &= Q_i(x_i). \end{aligned} \quad (8)$$

As introduced above, a model of massless rotational spring is adopted in this paper to describe the local flexibility induced by cracks in non-uniform beams. If a crack is located at the section $x = x_i$, the slope has a jump

$$X'_{i+1}(x_i) = X'_i(x_i) + C_i X''_i(x_i), \quad (9)$$

where C_i is the flexibility of the rotational spring which is a function of the crack depth and beam height. For one sided crack, C_i can be expressed as [8]

$$C_i = 5{\cdot}346\, h_i\, f(\xi_i) \quad (10a)$$

where h_i is the height of the cross-section of the beam at $x = x_i$,

$$\xi_i = a_i/h_i \quad (10b)$$

in which a_i is the depth of the ith crack. $f(\xi_i)$ is called the flexibility function expressed as [8]

$$\begin{aligned} f(\xi_i) = {} & 1{\cdot}862\xi_i^2 - 3{\cdot}95\xi_i^3 + 16{\cdot}375\xi_i^4 - 37{\cdot}226\xi_i^5 + 76{\cdot}81\xi_i^6 - 126\xi_i^7 \\ & + 172\xi_i^8 - 143{\cdot}97\xi_i^9 + 66{\cdot}56\xi_i^{10} \end{aligned} \quad (10c)$$

The case for two-sided cracks can be considered similarly [5].

Considering the continuous conditions of displacement, bending moment and shear force as well as the jump of slope at the boundary of the ith segment and the $(i+1)$th segment, we have

$$X_{i+1}(x) = X_i(x) + C_i X''_i(x_i)\bar{S}_2(x - x_i)H(x - x_i) \quad (11)$$

where $X_i(x)$ is the mode shape function of the ith segment (Figure 1), $x \in [x_{i-1}, x_i)$. The second term represents the jump of the slope at the boundary of the two neighboring segments.

Equation (11) is a recurrence formula of mode shape functions. Using $X_1(x)$, the mode shape function of the first segment which is given in equation (6), and equation (11) for $i = 2, 3, \ldots, n$, leads to

$$X_{n+1}(x) = X_1(x) + \sum_{i=1}^{n} C_i X''_i(x_i)\bar{S}_2(x - x_i)H(x - x_i). \quad (12)$$

The frequency equation can be easily established by using equations (6), (12) and the boundary conditions. The frequency equations for seven types of boundary conditions are derived and given in Appendix B.

3. NUMERICAL EXAMPLE

Example 1. A uniform beam with cracks.

In order to illustrate the application of the proposed method, this numerical example will show how to determine the natural frequencies of a simply supported steel beam with uniform cross-section and with cracks, which was considered in reference [6], as shown in Figure 2.

(1) *Determination of structural parameters*: The structural parameters of the beam, which were found from reference [6], are: length $L = 800\,\text{mm}$, width $= 10\,\text{mm}$, height $= 60\,\text{mm}$, Young's modulus $E = 2{\cdot}0 \times 10^{11}$ Pa, mass per unit volume $\rho = 7{\cdot}8 \times 10^{3}\text{kg/m}^{3}$.

(2) *Determination of the fundamental solutions*: The exact solutions, $S_i(x)$, for this example are given in equation (A2) listed in Appendix A. Using $S_i(x)$ and equation (5) one obtains the fundamental solutions as follows:

$$\left.\begin{aligned}
\bar{S}_1(x) &= \tfrac{1}{2}(\cosh(kx) + \cos(kx)),\\
\bar{S}_2(x) &= \frac{1}{2k}(\sinh(kx) + \sin(kx)),\\
\bar{S}_3(x) &= \frac{1}{2k^2}(\cosh(kx) - \cos(kx)),\\
\bar{S}_4(x) &= \frac{1}{2k^3}(\sinh(kx) - \sin(kx)).
\end{aligned}\right\} \tag{13}$$

(3) *Determination of the mode shape functions*: The mode shape function for the first interval $[0, x_1)$ is given in equation (A29), i.e.,

$$X_1(x) = X'(0)\bar{S}_2(x) - \frac{Q(0)}{K}\bar{S}_4(x), \quad [0, x_1), \tag{14}$$

where x_1 denotes the crack location.

The mode shape function of the second segment can be determined from equation (11) as

$$\begin{aligned}
X_2(x) &= X'(0)\bar{S}_2(x) - \frac{Q(0)}{K}\bar{S}_4(x)\\
&\quad + C_1\left[X'(0)\bar{S}_2''(x_1) - \frac{Q(0)}{K}\bar{S}_4''(x_1)\right]\bar{S}_2(x - x_1), \quad x \in [x_1, L)
\end{aligned} \tag{15}$$

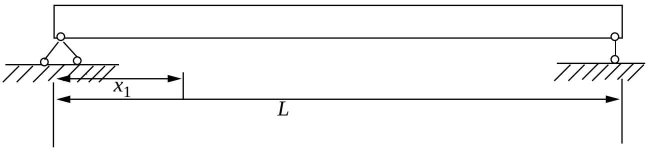

Figure 2. A simply supported beam with a crack.

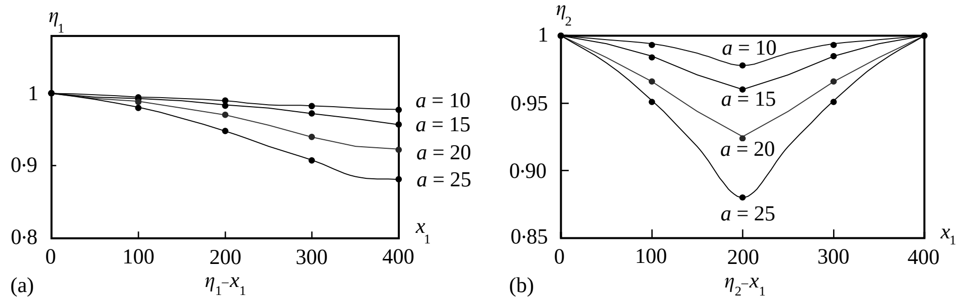

Figure 3. Crack-induced eigenfrequency changes for various crack locations and crack depths. (Notes: ——, present results; •, reference [6]; η_1, Mode 1; η_2 Mode 2, a, depth of crack (mm)).

or

$$X_2(x) = X'(0)\bar{S}_{2c}(x) - \frac{Q(0)}{K}\bar{S}_{4c}(x), \quad x \in [x_1, L), \tag{16}$$

where

$$\begin{aligned} \bar{S}_{2c}(x) &= \bar{S}_2(x) + C_1\bar{S}''_2(x_1)\bar{S}_2(x - x_1), \\ \bar{S}_{4c}(x) &= \bar{S}_4(x) + C_1\bar{S}''_4(x_1)\bar{S}_2(x - x_1), \end{aligned} \tag{17}$$

in which C_1 can be determined from equation (10).

(4) *Determination of the frequency equation*: Using equation (16) and the following boundary condition of the beam at $x = L$, $X_2(L) = X''_2(L) = 0$, leads to

$$\begin{aligned} X'(0)\bar{S}_{2c}(L) - \frac{Q(0)}{K}\bar{S}_{4c}(L) &= 0 \\ X'(0)\bar{S}''_{2c}(L) - \frac{Q(0)}{K}\bar{S}''_{4c}(L) &= 0. \end{aligned} \tag{18}$$

Because $X'(0) \neq 0$, $Q(0) \neq 0$, the frequency equation is

$$\begin{vmatrix} \bar{S}_{2c}(L) & -\bar{S}_{4c}(L) \\ \bar{S}''_{2c}(L) & -\bar{S}''_{4c}(L) \end{vmatrix} = 0 \tag{19}$$

i.e.,

$$\bar{S}_{2c}(L)\bar{S}''_{4c}(L) - \bar{S}_{4c}(L)\bar{S}''_{2c}(L) = 0. \tag{20}$$

Solving equation (20) a set of ω_{cj} can be obtained, which are expressed as

$$\omega_{cj} = \eta_j\omega_j,$$

where ω_{cj} and ω_j are the jth natural frequency of the cracked and uncracked beams, respectively, η_j is the reduced coefficient of the jth natural frequency. η_1 and η_2 are shown in Figure 3. The results obtained by Liang *et al.* [6] are also presented in Figure 3. The comparison between the results calculated by the present method and those obtained by Liang *et al.* [6] is favorable, thus supporting the validity of the proposed method.

Example 2. A non-uniform beam with cracks is shown in Figure 4. The sectional width of the beam is a constant, only the height of section varies along the length of the beam. The procedure for determining the natural frequencies and mode shapes of this cracked beam is as follows.

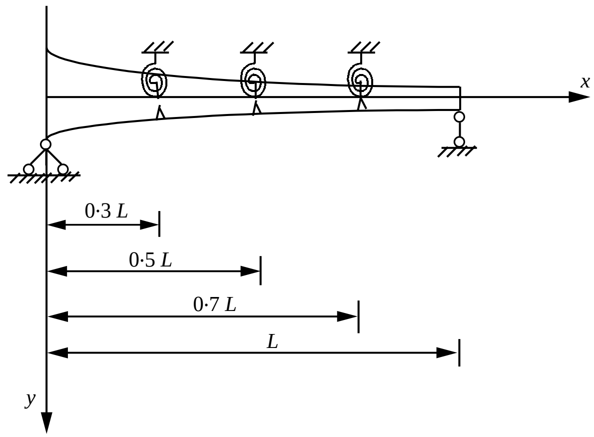

Figure 4. A hinged–hinged non-uniform beam with three cracks.

(1) *Distributions of flexural stiffness and mass intensity*. The distributions of flexural stiffness and mass of the beam are continuous, which can be expressed as

$$\bar{m}(x) = a(1 + \beta x),$$
$$K(x) = \alpha(1 + \beta x)^3, \tag{21}$$

where a and α are the mass intensity and flexural stiffness at $x = 0$, respectively, β is dependent on the variation of mass intensity.

These parameters are found as

$$a = \bar{m}(0) = 60\ \mathrm{kg/m}, \quad \bar{m}(L) = 52{\cdot}7\ \mathrm{kg/m}, \quad L = 4.5\ \mathrm{m},$$
$$\alpha = K(0) = 4{\cdot}635 \times 10^6\ \ \mathrm{N/m^2},$$
$$\beta = \left[\frac{\bar{m}(L)}{\bar{m}(0)} - 1\right] \Big/ L = -\,0{\cdot}05.$$

The heights of section are found as 0·2831, 0·2663 and 0·2494 m at $x = 0{\cdot}25L$, $0{\cdot}5L$ and $0{\cdot}75L$, respectively.

(2) *Determination of the fundamental solutions*: The distributions of flexural stiffness and mass intensity of this beam belong to Case 2 discussed in Appendix A. For this example, we have

$$\left.\begin{aligned} S_1(x) &= (1 + \beta x)^{-1/2} J_1[\lambda(1 + \beta x)^{1/2}] \\ S_2(x) &= (1 + \beta x)^{-1/2} Y_1[\lambda(1 + \beta x)^{1/2}] \\ S_3(x) &= (1 + \beta x)^{-1/2} I_1[\lambda(1 + \beta x)^{1/2}] \\ S_4(x) &= (1 + \beta x)^{-1/2} K_1[\lambda(1 + \beta x)^{1/2}] \end{aligned}\right\}. \tag{22}$$

The fundamental solutions $\bar{S}_i(x)$ $(i = 1, 2, 3, 4)$ can be determined from equations (22) and (5).

(3) *Evaluation of natural frequencies and mode shapes*: The mode shape of the first segment, $x \in [0, x_1)$, is given in equation (A29). The frequency equation can be established by use of equation (A30), in which C_i can be determined from equation (10).

If only one crack occurs at $x = 0{\cdot}25L$, $0{\cdot}5L$, respectively, the depth of the crack is $0{\cdot}2\,h$, then the first and second circular natural frequencies are found for the two cases, respectively, as

$$\omega_1 = 13{\cdot}0132,\ 12{\cdot}6509\ \mathrm{rad/s}; \qquad \omega_2 = 52{\cdot}0561,\ 50{\cdot}6132\ \mathrm{rad/s}.$$

TABLE 1

The first and second mode shapes

x/L	0	0·1	0·2	0·25	0·3	0·4	0·5	0·6	0·7	0·75	0·8	0·9	1·0
$X_1(x)$	0	0·3164	0·6086	0·7608	0·8109	0·9567	1·1008	0·9579	0·8286	0·7711	0·6069	0·3180	0
$X_2(x)$	0	0·5806	0·9527	1·1089	0·9536	0·5209	− 0·1420	− 0·5879	− 0·9574	− 1·1198	− 0·9620	− 0·5897	0
$X_{u1}(x)$	0	0·3085	0·5874	0·7069	0·8088	0·9509	0·9991	0·9512	0·8091	0·7098	0·5886	0·3091	0
$X_{u2}(x)$	0	0·5879	0·9503	0·9946	0·9509	0·5193	− 0·0140	− 0·5881	− 0·9519	− 0·9956	− 0·9603	− 0·5884	0

Note: $X_i(x)$—cracked beam, $X_{ui}(x)$—uncracked beam, $i = 1, 2$.

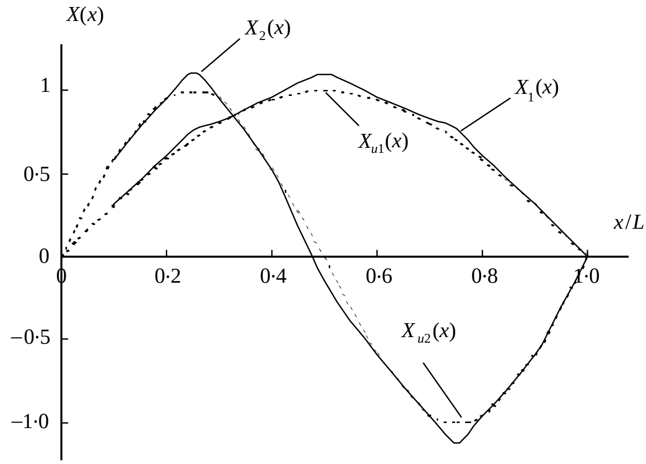

Figure 5. The first and second mode shapes of the cracked and uncracked beams (Note: $X_i(x)$, cracked beam; $X_{ui}(x)$, uncracked beam, $i = 1, 2$.

If the crack occurs at 0·5L and its depth is 0·3 h, then

$$\omega_1 = 11{\cdot}1449 \text{ rad/s}, \qquad \omega_2 = 44{\cdot}5789 \text{ rad/s}.$$

If the beam has three cracks at $x = 0{\cdot}25L$, 0·5L and 0·75L, and the depth of the three cracks is 0·3 h, then the first and second circular natural frequencies are found, respectively, as

$$\omega_1 = 10{\cdot}5927 \text{ rad/s}, \qquad \omega_2 = 43{\cdot}3711 \text{ rad/s}.$$

The first and second circular natural frequencies of the corresponding uncracked beam are found, respectively, as

$$\omega_1 = 13{\cdot}4793 \text{ rad/s}, \qquad \omega_2 = 53{\cdot}9177 \text{ rad/s}.$$

It can be seen from the above results that the effects of the number, depth and location of cracks on the natural frequencies of a cracked beam are significant.

After ω_1 and ω_2 are determined, using equations (A29), (A30) and (12) one obtains the first and second mode shapes for the case of three cracks located at $x = 0{\cdot}25L$, 0·5L and 0·75L, and the depth of the cracks is 0·3 h,

In order to compare the first and second mode shapes of the cracked beam with those of the uncracked beam, both are presented in Table 1 and Figure 5.

Because the slope of the section where a crack occurs has a jump, the effect of the cracks on the mode shapes is obvious. Based on the aforementioned procedure, the higher natural frequencies and mode shapes can also be determined.

4. CONCLUSIONS

An exact approach for determining the natural frequencies and mode shapes of a non-uniform beam with an arbitrary number of cracks and concentrated masses is presented in this paper. Exact solutions for free vibration of a non-uniform beam are derived for five different distributions of flexural stiffness and mass. A model of massless rotational spring is adopted to describe the local flexibility induced by cracks. The procedure for determining the fundamental solutions with a unit matrix property is presented. Using the fundamental solutions and recurrence formulas developed in this paper, the mode shape functions of vibration of a non-uniform beam with an arbitrary number of cracks and concentrated masses can be easily determined. The main advantage of the proposed method is that the eigenvalue equation of a non-uniform beam with any kind of two end supports, any finite number of cracks and concentrated masses can be conveniently determined from a second order determinant. As a consequence, due to the decrease in the determinant order as compared with previously developed procedures (e.g., references [4, 5]), the computational time required by the present method for solving the title problem can be reduced significantly. Because the slope of cracked section has a jump when the flexural deformation of the cracked beam occurs, any crack located at any section of the beam will decrease the value of natural frequency and change the shapes of vibration modes. The numerical examples demonstrate that the results determined by the proposed method are in good agreement with those obtained by Liang *et al.* [6], thus supporting the validity of the proposed method. It is also shown through the numerical examples that the effects of number, depth and location of cracks on natural frequencies and mode shapes of a non-uniform beam are significant and the proposed procedure is an exact and efficient method.

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APPENDIX A: EXACT SOLUTIONS FOR FIVE TYPES OF DISTRIBUTIONS

Case 1.

$$K(x) = K, \qquad \bar{m}(x) = \bar{m}, \tag{A1}$$

where K and $\bar{m}$ are constants. This case corresponds to a uniform beam. The solutions, $S_i(x)$, for this case are given by

$$\begin{aligned} &S_1(x) = \mathrm{e}^{kx}, \qquad S_2(x) = \mathrm{e}^{-kx}, \\ &S_3(x) = \sin kx, \quad S_4(x) = \cos kx \\ &k^4 = \frac{\bar{m}\omega^2}{K}. \end{aligned} \tag{A2}$$

Case 2.

$$K(x) = \alpha(1 + \beta x)^{\gamma+2}, \qquad \bar{m}(x) = a(1 + \beta x)^{\gamma}, \tag{A3}$$

where α, β, γ, a are constants which can be determined from the distributions of $K(x)$ and $\bar{m}(x)$ [10–14].

The solutions, $S_i(x)$, for Case 2 are as follows:

$$\begin{aligned} &S_1(x) = \left(\frac{\varsigma}{\lambda}\right)^{-\gamma} J_\gamma(\varsigma), \qquad S_2(x) = \left(\frac{\varsigma}{\lambda}\right)^{-\gamma} Y_\gamma(\varsigma), \\ &S_3(x) = \left(\frac{\varsigma}{\lambda}\right)^{-\gamma} I_\gamma(\varsigma), \qquad S_4(x) = \left(\frac{\varsigma}{\lambda}\right)^{-\gamma} K_\gamma(\varsigma), \\ &\varsigma = \lambda(1 + \beta x)^{1/2}, \qquad \lambda = \frac{2}{|\beta|}\left(\frac{a\omega^2}{\alpha}\right)^{1/4}, \end{aligned} \tag{A4}$$

where $|\beta|$ represents the absolute value of β, $J_\gamma(\varsigma)$, $Y_\gamma(\varsigma)$, $I_\gamma(\varsigma)$ and $K_\gamma(\varsigma)$ are Bessel functions of the first, second, third and fourth kinds, of order γ respectively.

Case 3.

$$K(x) = \alpha(1 + \beta x)^{\gamma+4}, \qquad \bar{m}(x) = a(1 + \beta x)^{\gamma}. \tag{A5}$$

The solutions, $S_i(x)$, for this case are as follows:

$$\begin{aligned} &S_i(x) = \mathrm{e}^{-\lambda_i \varsigma}, \qquad i = 1, 2, 3, 4, \\ &\varsigma = \ln(1 + \beta x), \\ &\lambda_{1,2,3,4} = -\tfrac{1}{2}\left(\gamma + 1 \pm \sqrt{\gamma^2 - 3 \pm 4k^2}\right), \\ &k^2 = \left(\frac{\gamma + 2}{2}\right)^2 + \frac{a\omega^2}{\alpha}. \end{aligned} \tag{A6}$$

Case 4.

$$K(x) = \alpha(1 + \beta x)^{\gamma+4}, \qquad \bar{m}(x) = a(1 + \beta x)^4. \tag{A7}$$

The solutions, $S_i(x)$, for Case 4 are as

$$S_1(x) = (1 + \beta x)^{-2} e^{cx}, \qquad S_2(x) = (1 + \beta x)^{-2} e^{-cx}$$

$$S_3(x) = (1 + \beta x)^{-2} \sin cx, \quad S_4(x) = (1 + \beta x)^{-2} \cos cx \tag{A8}$$

$$c^4 = \frac{a\omega^2}{\alpha}.$$

Case 5.

$$K(x) = \alpha e^{-\beta x}, \qquad \bar{m}(x) = a e^{-\beta x}. \tag{A9}$$

The solutions, $S_i(x)$, are:

$$S_i(x) = e^{-\lambda_i \varsigma}, \qquad \mathrm{i} = 1, 2, 3, 4,$$

$$\varsigma = \ln x,$$

$$\lambda_{1,2,3,4} = \tfrac{1}{2}(\beta \pm \sqrt{\beta^2 \pm 4d^2}),$$

$$d^4 = \frac{a\omega^2}{\alpha}. \tag{A10}$$

APPENDIX B: FREQUENCY EQUATIONS FOR SEVEN TYPES OF BOUNDARY CONDITIONS

B.1. FIXED-FREE BEAM WITH n CRACKS (Figure 1)

If a flexural beam with n cracks is fixed at the left end, $x = 0$, and the right is free, then the boundary condition can be written as

$$X_1(0) = X_1'(0) = 0, \tag{B1}$$

$$M(L) = 0, \quad \text{i.e.,} \quad X_{n+1}''(L) = 0, \tag{B2}$$

$$Q(L) = 0, \quad \text{i.e.,} \quad X_{n+1}'''(L) + \mu(L) X_{n+1}''(L) = 0 \quad \text{or} \quad X_{n+1}'''(L) = 0, \tag{B3}$$

where

$$\mu_L = K'(L)/K(L). \tag{B4}$$

Because the amplitude of a mode shape is indeterminate, $X_1(x)$ can also be written as

$$X_1(x) = M(0)\bar{S}_3(x) + [Q(0) - \mu(0)M(0)]\bar{S}_4(x). \tag{B5}$$

Using equations (12), (B2) and (B3) leads to

$$\begin{aligned} &[\bar{S}_3''(L) - \mu(0)\bar{S}_4''(L)]M(0) + \bar{S}_4''(L)Q(0) + \sum_{i=1}^{n} C_i X_i''(x_i)\bar{S}_2''(L - x_i) = 0, \\ &[\bar{S}_3'''(L) - \mu(0)\bar{S}_4'''(L)]M(0) + \bar{S}_4'''(L)Q(0) + \sum_{i=1}^{n} C_i X_i''(x_i)\bar{S}_2'''(L - x_i) = 0. \end{aligned} \tag{B6}$$

Because all $X_i(x)$ $(i = 1, 2, \ldots, n)$ have the same two initial parameters, as $X_1(x)$ has, i.e., equation (B6) has only two unknown parameters, $M(0)$ and $Q(0)$, the frequency equation is obtained by setting the second order determinant consisting of the coefficients of $M(0)$ and $Q(0)$ in equation (B6) equal to zero.

B.2. FIXED–FIXED BEAM WITH n CRACKS AT INTERMEDIATE POINTS

The mode shape function of the first segment has the same form as that of equation (B5). Using equations (B1) and (B5) and the following boundary conditions

$$X(L) = X'(L) = 0 \tag{B7}$$

one obtains

$$\begin{aligned} &[\bar{S}_3(L) - \mu(0)\bar{S}_4(L)]M(0) + \bar{S}_4(L)Q(0) + \sum_{i=1}^{n} C_i X_i''(x_i)\bar{S}_2(L - x_i) = 0, \\ &[\bar{S}'_3(L) - \mu(0)\bar{S}'_4(L)]M(0) + \bar{S}'_4(L)Q(0) + \sum_{i=1}^{n} C_i X_i''(x_i)\bar{S}'_2(L - x_i) = 0. \end{aligned} \tag{B8}$$

The frequency equation is obtained by setting the determinant consisting of the coefficients of $M(0)$ and $Q(0)$ in equation (B8) equal to zero.

B.3. FREE-FREE BEAM WITH n CRACKS AT INTERMEDIATE POINTS

The boundary conditions for this case can be written as

$$M(0) = 0, \quad \text{i.e.,} \quad X_1''(0) = 0, \qquad Q(0) = 0, \quad \text{i.e.,} \quad X_1'''(0) = 0, \tag{B9, B10}$$

$$M(L) = 0, \quad \text{i.e.,} \quad X_{n+1}''(L) = 0, \qquad Q(L) = 0, \quad \text{i.e.,} \quad X_{n+1}'''(L) = 0. \tag{B11, B12}$$

Using equations (6), (B9) and (B10) yields

$$X_1(x) = X(0)\bar{S}_1(x) + X'(0)\bar{S}_2(x). \tag{B13}$$

Using equations (12), (B11) and (B12) leads to

$$\begin{aligned} &\bar{S}_1''(L)X(0) + \bar{S}_2''(L)X'(0) + \sum_{i=1}^{n} C_i X_i''(x_i)\bar{S}_2''(L - x_i) = 0, \\ &\bar{S}_1'''(L)X(0) + \bar{S}_2'''(L)X'(0) + \sum_{i=1}^{n} C_i X_i''(x_i)\bar{S}_2'''(L - x_i) = 0. \end{aligned} \tag{B14}$$

The frequency equation can be obtained by setting the determinant consisting of the coefficients of $X(0)$ and $X'(0)$ in Equation (B14) equal to zero.

B.4. FIXED–HINGED BEAM WITH n CRACKS AT THE INTERMEDIATE POINTS

If the left end, $x = 0$, is fixed, then the mode shape function of the first segment has the same form as that of equation (B5).

Using equations (12) and (B5) and the following boundary conditions:

$$X_{n+1}(L) = X''_{n+1}(L) = 0 \tag{B15}$$

one obtains

$$\begin{aligned}&[\bar{S}_3(L) - \mu(0)\bar{S}_4(L)]M(0) + \bar{S}_4(L)Q(0) + \sum_{i=1}^{n} C_i X''_i(x_i)\bar{S}_2(L - x_i) = 0,\\ &[\bar{S}''_3(L) - \mu(0)\bar{S}''_4(L)]M(0) + \bar{S}''_4(L)Q(0) + \sum_{i=1}^{n} C_i X''_i(x_i)\bar{S}''_2(L - x_i) = 0.\end{aligned} \tag{B16}$$

Setting the determinant consisting of the coefficients of $M(0)$ and $Q(0)$ in equation (B16) equal to zero one obtains the frequency equation.

B.5. HINGED-HINGED BEAM WITH n CRACKS AT INTERMEDIATE POINTS

The boundary conditions for this case can be written as

$$X(0) = M(0) = 0, \qquad X_{n+1}(L) = X''_{n+1}(L) = 0. \tag{B17, B18}$$

Using equation (6) and equation (B17) obtains

$$X_1(x) = X'(0)\bar{S}_2(x) - \frac{Q(0)}{K(0)}\bar{S}_4(x). \tag{B19}$$

Using equation (12), (B18) and (B19) leads to

$$\begin{aligned}&\bar{S}_2(L)X'(0) - \frac{1}{K(0)}\bar{S}_4(L)Q(0) + \sum_{i=1}^{n} C_i X''_i(x_i)\bar{S}_2(L - x_i) = 0,\\ &\bar{S}''_2(L)X'(0) - \frac{1}{K(0)}\bar{S}''_4(L)Q(0) + \sum_{i=1}^{n} C_i X''_i(x_i)\bar{S}''_2(L - x_i) = 0.\end{aligned} \tag{B20}$$

The frequency equation can be obtained by setting the determinant consisting of the coefficients of $X'(0)$ and $Q(0)$ in equation (B20) equal to zero.

B.6. FIXED-SPRING BEAM WITH n CRACKS AT INTERMEDIATE POINTS AND WITH A CONCENTRATED MASS AT THE SPRING END (Figure A1)

The boundary conditions for this case are

$$\begin{aligned}&x = 0, \qquad X(0) = X'(0) = 0,\\ &x = L, \qquad X''_{n+1}(L) = -\frac{K_{\varphi L} X'_{n+1}(L)}{K(L)},\\ &\qquad\qquad X'''_{n+1}(L) = -\frac{1}{K(L)}[(m_L\omega^2 - K_L)X_{n+1}(L) - \mu(L)K_{\varphi L}X'_{n+1}(L)],\end{aligned} \tag{B21}$$

where $K_{\varphi L}$ and K_L are the rotational spring and translational spring constants at the right end of the beam respectively.

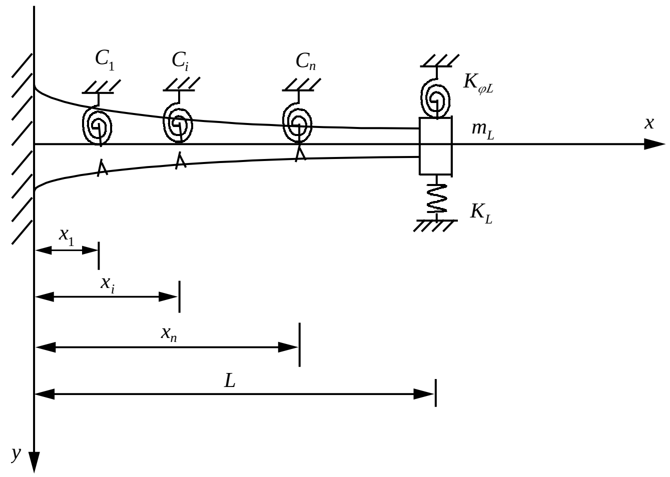

Figure A1. A fixed-spring beam with n cracks and a concentrated mass at the spring end.

The mode shape function of the first segment is given in equation (B5). Using the boundary conditions at $x = L$ leads to

$$[\bar{S}''_3(L) - \mu(0)\bar{S}''_4(L) + \mu_{\varphi L}\bar{S}'_3(L) - \mu_{\varphi L}\mu(0)\bar{S}'_4(L)]M(0) + [\bar{S}''_4(L) + \mu_{\varphi L}\bar{S}'_4(L)]Q(0) + \sum_{i=1}^{n} C_i X''_i(x_i)[\bar{S}''_2(L - x_i) + \mu_{\varphi L}\bar{S}'_2(L - x_i)] = 0, \tag{B22a}$$

$$[\bar{S}'''_3(L) - \mu(0)\bar{S}'''_4(L) + K_{mL}\bar{S}_3(L) - \mu(0)\bar{S}_4(L) - \mu(L)\mu_{\varphi L}\bar{S}'_3(L) - \mu(0)\mu(L)\mu_{\varphi L}\bar{S}'_4(L)]M(0) + [\bar{S}'''_4(L) + K_{mL}\bar{S}_4(L) - \mu(L)\mu_{\varphi L}\bar{S}'_4(L)]Q(0) + \sum_{i=1}^{n} C_i X''_i(x_i)[\bar{S}'''_2(L - x_i) + K_{mL}\bar{S}_2(L - x_i) - \mu(L)\mu_{\varphi L}\bar{S}_2(L - x_i)] = 0, \tag{B22b}$$

in which

$$\mu_{\varphi L} = \frac{K_{\varphi L}}{K(L)}, \qquad K_{mL} = \frac{m_L\omega^2 - K_L}{K(L)}. \tag{B23}$$

The frequency equation can be obtained by setting the determinant consisting of the coefficients of $M(0)$ and $Q(0)$ in equations (B22a) and (B22b) equal to zero.

B.7. SPRING-SPRING BEAM WITH n CRACKS AND $(n + 2)$ CONCENTRATED MASSES (Figure A2)

It is assumed that $(n + 2)$ concentrated masses are attached to a spring-spring beam shown in Figure A2. We assume that cracks occur at the sections where the concentrated masses are attached. The boundary conditions for this case are

$$x = 0, \qquad X''_1(0) = \mu_{\varphi 0}X'_1(0), \qquad X'''_1(0) = K_{m0}X_1(0) - \mu_{\varphi 0}K_{\varphi 0}X'_1(0), \tag{B24}$$

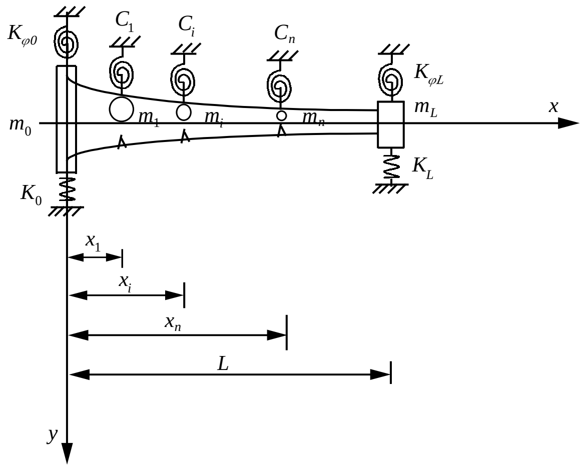

Figure A2. A spring-spring beam with n cracks and $(n+2)$ concentrated masses.

$$x = L, \qquad X''_{n+1}(L) = -\mu_{\varphi L} X'_{n+1}(L), \qquad \text{(B25)}$$
$$X'''_{n+1}(L) = K_{mL} X_{n+1}(L) - \mu_{\varphi L} K_{\varphi L} X'_{n+1}(L)$$

where

$$K_{m0} = \frac{m_0 \omega^2 - K_0}{K(0)}, \qquad \mu_{\varphi 0} = \frac{K_{\varphi 0}}{K(0)}. \qquad \text{(B26)}$$

K_{mL} and $\mu_{\varphi L}$ are defined in equation (B23). $K_{\varphi 0}$ and K_0 are the rotational spring and translational spring constants attached to the left end of the beam, respectively, and $K(0)$ is the flexural stiffness at the left end of the beam. Using equation (B24) leads to

$$X_1(x) = X(0)\bar{S}_{11}(x) + X'(0)\bar{S}_{22}(x) \qquad \text{(B27)}$$

where

$$\bar{S}_{11}(x) = \bar{S}_1(x) + K_{m0}\bar{S}_4(x), \qquad \text{(B28)}$$
$$\bar{S}_{22}(x) = \bar{S}_2(x) + \mu_{\varphi 0}\bar{S}_3(x) - \mu_{\varphi 0}\mu_0\bar{S}_4(x).$$

The mode shape function can be written as

$$X_{n+1}(x) = X_1(x) + \sum_{i=1}^{n} C_i X''_i(x_i)\bar{S}_2(x - x_i)H(x - x_i) + \sum_{i=1}^{n} K_{mi} X_i(x_i)\bar{S}_4(x - x_i)H(x - x_i) \qquad \text{(B29)}$$

where

$$K_{mi} = \frac{m_i \omega^2}{K(x_i)} \quad i = 1, 2, \ldots, n. \qquad \text{(B30)}$$

It is evident that there are only two unknown parameters, $X(0)$ and $X'(0)$, in the expressions of $X_1(x)$, $X_i(x)$ and $X_{n+1}(x)$. Using equation (B25) leads to

$$[\bar{S}''_{11}(L) - \mu_{\varphi L}\bar{S}'_{11}(L)]X(0) + [\bar{S}''_{22}(L) - \mu_{\varphi L}\bar{S}'_{22}(L)]X'(0)$$
$$+ \sum_{i=1}^{n} C_i X''_i(x_i)[\bar{S}''_2(L - x_i) + \mu_{\varphi L}\bar{S}'_2(L - x_i)]$$

$$+\sum_{i=1}^{n} K_{mi} X_i(x_i)[\bar{S}''_4(L-x_i)+\mu_{\varphi L}\bar{S}'_4(L-x_i)]=0,$$

$$[\bar{S}'''_{11}(L)-K_{mL}\bar{S}'''_{11}(L)+\mu_{\varphi L}K_{\varphi L}\bar{S}'_{11}(L)]X(0)$$

$$+[\bar{S}'''_{22}(L)-K_{mL}\bar{S}_{22}(L)+\mu_{\varphi L}K_{\varphi L}\bar{S}'_{22}(L)]X'(0)$$

$$+\sum_{i=1}^{n} C_i X''_i(x_i)[\bar{S}'''_2(L-x_i)-K_{mL}\bar{S}_2(L-x_i)+\mu_{\varphi L}K_{\varphi L}\bar{S}'_2(L-x_i)]$$

$$+\sum_{i=1}^{n} K_{mi} X_i(x_i)[\bar{S}'''_4(L-x_i)-K_{mL}\bar{S}_2(L-x_i)+\mu_{\varphi L}K_{\varphi L}\bar{S}'_4(L-x_i)]=0. \quad \text{(B31)}$$

The frequency equation can be obtained by setting the determinant consisting of the coefficients of $X(0)$ and $X'(0)$ in equation (B31) equal to zero.