Journal of Sound and Vibration (2002) **252**(3), 577–583
doi:10.1006/jsvi.2001.3947, available online at http://www.idealibrary.com on IDEAL®

THE EFFECTS OF DISTRIBUTED MASS LOADING ON PLATE VIBRATION BEHAVIOR

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(*Received* 10 *May* 2001, *and in final form* 20 *August* 2001)

1. INTRODUCTION

Vibration problems of rectangular plates with mass loading are very common in engineering applications [1, 2]. While there are several reports on plate vibrations with added point masses [3–6], very few reports on plate vibrations with distributed mass loading can be found in the literature. It has been proved that distributed mass loading can induce significant changes of modal frequencies and shapes in beam vibrations [7, 8]. In this paper, the free bending vibration of a simply supported rectangular plate carrying distributed mass loading is analyzed by the Rayleigh–Ritz method. The effects of size and location of the mass loading on the changes of modal frequencies and shapes as demonstrated by the analysis of the numerical solution of the eigenvalue problem are investigated.

2. EIGENVALUE PROBLEM IN RAYLEIGH–RITZ METHOD

By neglecting the effects of shear deformation and rotatory inertia effects, the dynamic equation for free vibration of a uniform isotropic rectangular plate is given by [9]

$$\nabla^4\left[\frac{Eh^3 w(x,y,t)}{12(1-v^2)}\right]+\frac{\partial^2(\rho h w(x,y,t)}{\partial t^2}=0, \tag{1}$$

where h is the thickness of the plate, ρ, E and v are the density, Young's modulus and the Poisson's ratio of the plate material respectively.

Consider the plate to be loaded with a uniformly distributed mass on area $x_c \times y_c$ as shown in Figure 1 and assume that the mass does not prevent any bending of the plate segment on which it is. The dynamic equation of the loaded plate may be written as

$$\nabla^4\left[\frac{Eh^3 w(x,y,t)}{12(1-v^2)}\right]+\frac{\partial^2(\rho h w(x,y,t))}{\partial t^2}+\frac{\partial^2(MA' w(x,y,t))}{\partial t^2}\;=\;0, \tag{2}$$

where M is the distributed mass loading per area and A' is the area of the plate with the added distributed mass loading.

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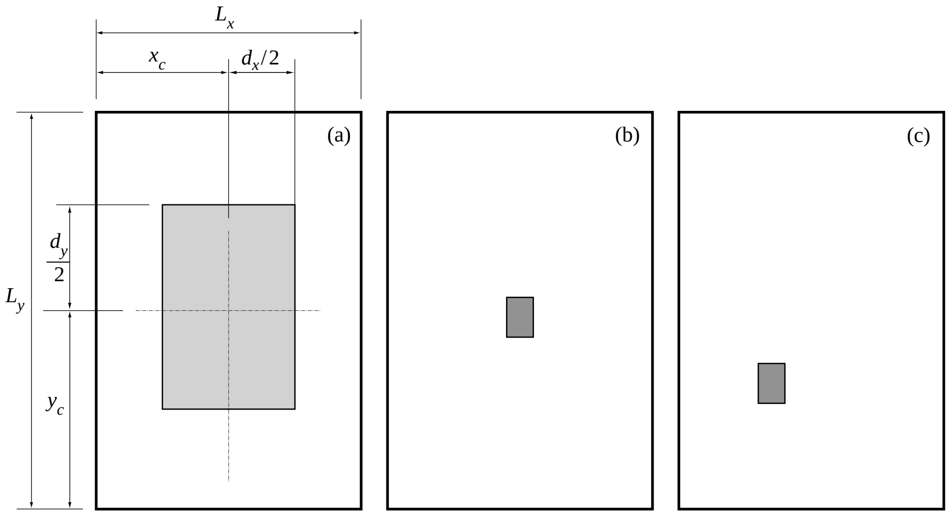

Figure 1. Rectangular plates with distributed mass loadings of different size and location. The total mass loadings in all three cases are the same: ▩—region with uniformly distributed mass loading. (a) Loading case 1: $L_x = 1, L_y = 1{\cdot}5, x_c = 0{\cdot}5, y_c = 0{\cdot}75, d_x = 0{\cdot}5, d_y = 0{\cdot}75, M/\rho h = 0{\cdot}4$. (b) Loading case 2: $L_x = 1, L_y = 1{\cdot}5, x_c = 0{\cdot}5, y_c = 0{\cdot}75, d_x = 0{\cdot}1, d_y = 0{\cdot}15, M/\rho h = 10$. (c) Loading case 3: $L_x = 1, L_y = 1{\cdot}5, x_c = 0{\cdot}3, y_c = 0{\cdot}45, d_x = 0{\cdot}1, d_y = 0{\cdot}15, M/\rho h = 10$.

If the vibration is assumed to be a simple harmonic motion, the solution of equation (2) may be written as

$$w(x, y, t) = W(x, y)\sin \omega t. \tag{3}$$

Maximum strain energy of the plate is

$$U_{max} = \iint\limits_A \frac{D}{2}\left[\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + \left(\frac{\partial^2 W}{\partial y^2}\right)^2 + 2\nu\,\frac{\partial^2 W}{\partial x^2}\,\frac{\partial^2 W}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 W}{\partial x \partial y}\right)^2\right] \mathrm{dA}. \tag{4}$$

Maximum kinetic energy of the plate is

$$T_{max} = \frac{\rho h \omega^2}{2} \iint\limits_A W^2\,\mathrm{d}A + \frac{M\omega^2}{2} \iint\limits_{A'} W^2\,\mathrm{d}A', \tag{5}$$

where $D = Eh^3/[12(1-\nu^2)]$ is the flexural rigidity of the plate.

To apply the Ritz [9, 10] method in solving equation (2), the following series is used to represent the deflection $W(x, y)$:

$$W(x, y) = \sum_m \sum_n A_{mn}\phi_m(x)\psi_n(y), \tag{6}$$

where $\phi_m(x)$ and $\psi_n(y)$ are appropriate functions which individually satisfies at least the geometric boundary conditions in the x and y directions respectively.

Substitution of the deflection function $W(x, y)$ in equation (6) into the kinetic and strain energy expression and minimization of the Rayleigh quotient with respect to the coefficients

A_{ij} leads to the eigenvalue equation [10]:

$$\sum_m \sum_n \left\{ C_{mnij} - \lambda \left[E_{mi}^{(0,0)} F_{mi}^{(0,0)} + (M/\rho h) \hat{E}_{mi}^{(0,0)} \hat{F}_{mi}^{(0,0)} \right] \right\} A_{mn} = 0, \tag{7}$$

$$C_{mnij} = E_{mi}^{(2,2)} F_{nj}^{(0,0)} + \alpha^4 E_{mi}^{(0,0)} F_{nj}^{(2,2)} + v\alpha^2 (E_{mi}^{(0,2)} F_{nj}^{(2,0)}$$

$$+ E_{mi}^{(2,0)} F_{nj}^{(0,2)}) + 2(1-v)\alpha^2 E_{mi}^{(1,1)} F_{nj}^{(1,1)},$$

$$E_{mi}^{(r,s)} = \int_0^1 \left(\frac{\mathrm{d}^r \phi_m}{\mathrm{d}x^r} \right) \left(\frac{\mathrm{d}^s \phi_n}{\mathrm{d}x^r} \right) \mathrm{d}x, \quad F_{nj}^{(r,s)} = \int_0^1 \left(\frac{\mathrm{d}^r \psi_n}{\mathrm{d}y^r} \right) \left(\frac{\mathrm{d}^s \psi_j}{\mathrm{d}y^s} \right) \mathrm{d}y,$$

$$\hat{E}_{mi}^{(r,s)} = \int_{x_c - d_x/2}^{x_c + d_x/2} \left(\frac{\mathrm{d}^r \phi_m}{\mathrm{d}x^r} \right) \left(\frac{\mathrm{d}^s \phi_n}{\mathrm{d}x^r} \right) \mathrm{d}x, \quad \hat{F}_{nj}^{(r,s)} = \int_{y_c - d_y/2}^{y_c + d_y/2} \left(\frac{\mathrm{d}^r \psi_m}{\mathrm{d}y^r} \right) \left(\frac{\mathrm{d}^s \psi_n}{\mathrm{d}y^r} \right) \mathrm{d}y,$$

$$\alpha = L_x/L_y, \quad \lambda = \rho h \omega^2 L_x^4/D, \quad m, n, i, j = 1, 2, 3, \ldots, N, \; r, s = 0, 1, 2,$$

where $(x_c - d_x/2, \ y_c - d_y/2)$, $(x_c - d_x/2, \ y_c + d_y/2)$, $(x_c + d_x/2, \ y_c + d_y/2)$ and $(x_c + d_x/2, \ y_c - d_y/2)$ are the co-ordinates of the four corners of the loaded area A' as shown in Figure 1(a).

According to the Ritz method, the assumed displacement function W approaches the exact solution as N approaches infinity if the system of chosen functions $\phi_i(x)$ and $\psi_i(y)$ satisfies the following conditions [11, 12]: (1) $\phi_i(x)$ and $\psi_i(y)$ are linearly independent; (2) $\phi_i(x)$ and $\psi_i(y)$ each form a complete system of functions, and (3) $\phi_i(x)$ and $\psi_i(y)$ satisfy the geometric boundary conditions of the plate in the x and y directions respectively. A similar method for solving plate vibration excited by a uniformly distributed force acting over a rectangular portion of the plate can be found in reference [13].

3. NUMERICAL RESULTS AND DISCUSSION

The transverse vibration of an isotropic rectangular plate with simple supports along all its edges is studied using the proposed method. $\phi_i(x)$ and $\psi_i(y)$ are taken as $\sin(i\pi x/L_x)$ and $\sin(j\pi y/L_y)$ respectively. A computer program is written for equation (6) and solved by the Matlab software. The proposed modelling method is verified by comparing the results of a test case reported in references [3, 5] with the result obtained with the Matlab program. The 50 kg point mass loading in the test case is represented by a mass loading distributed in a very small region so that it can be solved by the present method. The parameters used in the simulation are $M = 1{\cdot}25 \times 10^7$ kg/m^2, $E = 2{\cdot}051 \times 10^{11}$ N/m^2, $L_x = L_y = 2$ m, $x_c = y_c = 0{\cdot}5$, $\rho = 7850$ kg/m^3, $h = 0{\cdot}005$ m, $v = 0{\cdot}3$, $d_x = d_y = 0{\cdot}002$ and $N = 5$. The first five natural frequencies (in rad/s) are shown in Table 1. Numerical results for the test case

TABLE 1

The first five natural frequencies for a uniform square simply supported plate carrying a concentrated mass of 50 *kg located at* $x_c/L_x = 0{\cdot}25$ and $y_c/L_y = 0{\cdot}25$

	ω_1	ω_2	ω_3	ω_4	ω_5
The present method	31·8536	63.5505	95·4149	128·0735	180·8910
Cha [3]	31·8140	63·2319	95·4148	127·6160	180·5930
Reference [1]	31·8248	63·3182	95·4150	127·7414	180·6767

TABLE 2

Change of the non-dimensional eigenfrequencies, $\sqrt{\lambda} = (\sqrt{\rho h / D L_x^2})\omega$, of a simply supported rectangular plate of aspect ratio $L_y/L_x = 1{\cdot}5$ loaded with distributed masses: effect of size and location of the mass loading

Non-dimensional eigenfrequencies	Unloaded plate	Loading case 1 (refer to Figure 1)	% decrease of natural frequency	Loading case 2 (refer to Figure 1)	% decrease of natural frequency	Loading case 3 (refer to Figure 1)	% decrease of natural frequency
$(\sqrt{\rho h/DL_x^2})\omega_{11}$	14·2561	12·65588	11·2	12·0753	15·3	13·0495	8
$(\sqrt{\rho h/DL_x^2})\omega_{12}$	27·4156	25·38666	7·4	26·8918	1·9	24·7544	10
$(\sqrt{\rho h/DL_x^2})\omega_{21}$	43·8649	40·61045	7·4	41·9978	4·3	39·9536	9
$(\sqrt{\rho h/DL_x^2})\omega_{13}$	49·348	46·57419	5·6	45·3518	8·1	48·5313	2
$(\sqrt{\rho h/DL_x^2})\omega_{22}$	57·0244	54·28112	4·8	56·8956	0·2	53·1536	7
$(\sqrt{\rho h/DL_x^2})\omega_{23}$	78·9568	74·64819	5·5	73·7955	6·5	77·6171	2
$(\sqrt{\rho h/DL_x^2})\omega_{14}$	80·0535	76·07834	5·0	78·4533	2·0	77·3189	3
$(\sqrt{\rho h/DL_x^2})\omega_{31}$	93·2129	87·95304	5·6	86·4268	7·3	91·6198	2
$(\sqrt{\rho h/DL_x^2})\omega_{32}$	106·3724	102·5845	3·6	105·2100	1·1	102·5937	4
$(\sqrt{\rho h/DL_x^2})\omega_{24}$	109·6623	104·7244	4·5	108·9475	0·7	106·1183	3

reported in references [3, 5] are also listed in the table for comparison. The comparison of the results indicates that the proposed method is appropriate for the vibration analysis of plates carrying mass loading.

The effect of size and location of distributed mass loading on the transverse vibration of a simply supported rectangular plate is investigated by studying three different cases of mass loading as shown in Figure 1. The additional mass loading in all three cases is 10% of the mass of the unloaded plate. The loaded mass in both cases 1 and 2 is distributed around the center of the plate while that in case 3 is closer to one of the corners of the plate. The loaded areas in cases 1, 2 and 3 are 25, 1 and 1% of the total plate surface area respectively. The natural frequencies of the three loaded plates are calculated by setting $N = 20$ and they are compared with those of the unloaded plate. The changes of natural frequency in these three loading cases are shown in Table 2. As depicted by equation (4), the maximum kinetic energy of the vibrating plate is affected most if the mass loading is added on an antinode of the plate in which case a large change of natural frequency of the corresponding vibration mode will be effected. This prediction is substantiated by observing the change of frequency in loading case 2 as shown in Table 2. Vibration modes with mass loading on an antinode such as φ_{11}, φ_{13} and φ_{31} have relatively larger changes of natural frequency than the modes with mass loading about a node such as φ_{22} and φ_{24}. Since the mass loading in case 2 is more concentrated than that in case 1, the effects of the loading on the maximum kinetic energy of the plate for some vibration modes are more dramatic in case 2 than that in case 1. Therefore, the variation of frequency changes in case 1 is generally not as great as that in case 2. In loading case 3, the added mass is moved closer to a corner of the plate. The change of natural frequencies is more significant for modes with the mass added on an antinode such as φ_{12}, φ_{21} and φ_{22}.

The mode shapes of the simply supported rectangular plate with mass loading case 2 ($L_x = 1$, $L_y = 1{\cdot}5$, $x_c = 0{\cdot}5$, $y_c = 0{\cdot}75$, $d_x = 0{\cdot}1$, $d_y = 0{\cdot}15$, $M/\rho h = 10$) have also been

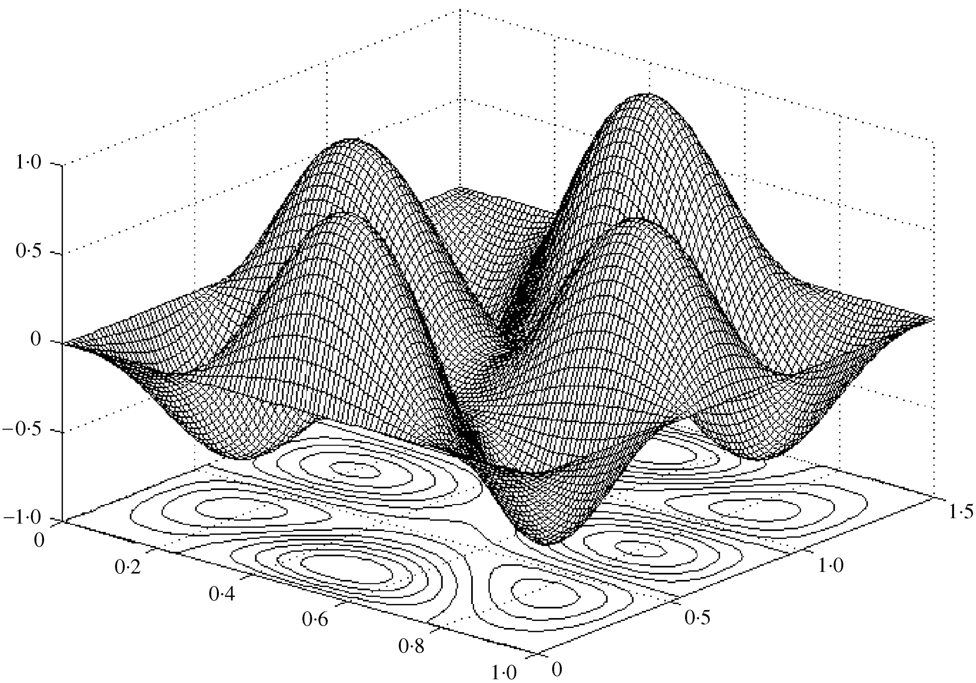

Figure 2. Normalized mode shape, φ_{33} of a rectangular plate carrying a distributed mass with its center (loading case 2).

calculated by substituting the coefficients A_{mn} into equation (5) after solving equation (6) using the Matlab program. The mode shape φ_{33} is shown in Figure 2. It is observed that the expected antinode at the center becomes unobservable and all vibration antinodes around the added mass appear to be shifted towards the loaded region. This phenomenon has also been observed in beam vibration with distributed mass loading [7]. Contour maps of four other mode shapes are shown in Figure 3. The shifting of vibration antinodes towards the loaded region appears in modes φ_{13}, φ_{25} and φ_{35}. Furthermore, in respect of modes φ_{13} and φ_{35}, the amplitude of the antinodes at the loaded region, i.e., the center, is smaller than those of the other antinodes. No significant change of mode shape for mode φ_{24} is observed because the center of the mass loading region is a node of the vibration mode. This minimal effect of mass loading on mode φ_{24} is confirmed by the small frequency change (0·7%) when compared with the change of frequencies of the same vibration mode in the cases of loadings 1 and 2 as shown in Table 2.

4. CONCLUSION

The solution to the eigenvalue problem of the bending vibration of plates with distributed mass loading is formulated using the Rayleigh–Ritz method. The natural frequencies and

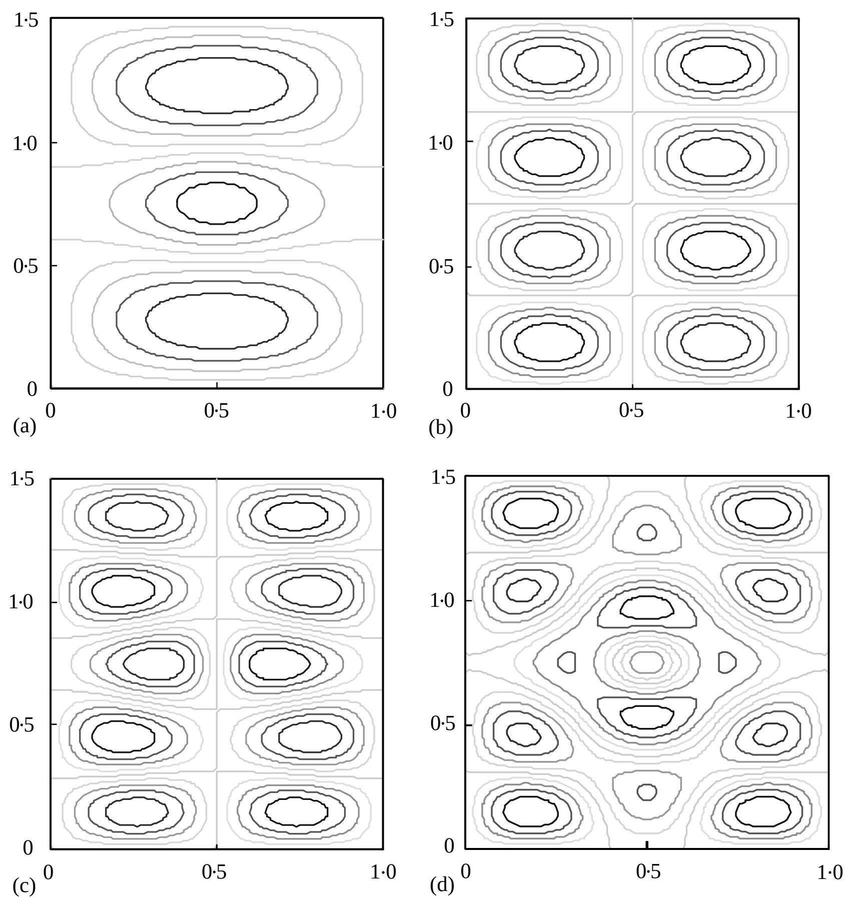

Figure 3. Contours of vibration modes of a simply supported rectangular plate carrying a distributed mass with its center at (loading case 2): (a) φ_{13}; (b) φ_{24}; (c) φ_{25}; (d) φ_{35}.

mode shapes of a simply supported rectangular plate are calculated by numerically solving the eigenvalue problem. The effects of the size and location of a distributed mass loading on the plate are investigated. It is found that both the natural frequency and mode shape of a certain vibration mode will have relatively larger changes if the mass loading is placed on an antinode of the vibration mode. In the three particular cases being studied, it is observed that the added mass would reduce the amplitude of an antinode close to it and all vibration antinodes around the added mass appear to be shifted towards the loaded region.

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