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LETTERS TO THE EDITOR

NATURAL FREQUENCIES, SENSITIVITY AND MODE SHAPE DETAILS OF AN EULER–BERNOULLI BEAM WITH ONE-STEP CHANGE IN CROSS-SECTION AND WITH ENDS ON CLASSICAL SUPPORTS

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1. INTRODUCTION

The system parameters which influence the natural frequencies of the title problem are: the boundary conditions, the ratio of the flexural rigidity of the two portions, the ratio of the mass per unit length and the lengths of the two portions. A search of recent literature revealed that the range of the system parameters considered in publications were narrow. For instance, in most publications, the only example considered was a cantilever with the step at the mid-point. Natural mode shapes, position of nodes, etc., are important aspects of vibrating systems but are not found in the publications. The purpose of this paper is to present comprehensive vibratory details for all classical boundary conditions and of wide ranges of system parameters.

The title problem has been studied by several researchers. Numerical methods like Rayleigh–Ritz, finite element, finite difference, etc., have been used. In analytical approach, the frequency equations were expressed as fourth order determinant equated to zero. Brief reviews of the publications on title problem follow. Taleb and Suppiger [1] derived the frequency equation of a simply supported stepped beam and compared the exact fundamental frequency with the approximate frequency obtained by the Cauchy function method. Levinson [2] derived the same frequency equation listed in reference [1] but did not present any numerical results. Levinson concluded that "one may study the vibrations of continuous systems having discontinuous properties in an "exact" manner in the simplest cases only and that normally one must resort to numerical methods of one sort or another ... ". Balasubramanian and Subramanian [3] investigated the performance of a four-degree-of-freedom per node element when used in the vibration analysis of a cantilever with the step at the mid-point. Subramanian and Balasubramanian [4] used the same example and the method developed in reference [3] to study the beneficial effect of "steps" on vibration characteristics. The publication by Jang and Bert [5] is the first to list the frequency equations for all combinations of the classical boundary conditions, as fourth order determinants equated to zero. In reference [5], the "exact" frequencies of a beam of circular cross-section with step change in diameter at the mid-point were compared with frequencies obtained by finite element method and by a commercial code. Jang and Bert [6] published results of further combinations of boundary conditions. Manual symbolic expansion of the determinants would be laborious and has been done only for the simply supported case in references [1, 2]. Balasubramanian *et al.* [7] took the example in reference [3] to investigate the use of very high order derivatives as nodal degrees of

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freedom. Laura *et al.* [8] used the finite element method to study the vibration of the example in reference [3] and also considered various locations of the step. Popplewell and Chang [9] used the Rayleigh–Ritz method to study the example in reference [3]. Krishnan *et al.* [10] discussed the difficulties of using finite difference analysis of a pinned–pinned stepped beam with the step at mid-point.

In the present paper, the emphasis is on three types of stepped beams which are used frequently in engineering applications. Type 1 beams are of constant depth but with a step change in breadth, Type 2 beams are of constant breadth but with a step change in depth and Type 3 beams are of circular cross-section with a step change in diameter or beams in which the two portions have similar cross-section. The character of the title problem enabled the frequency equations for all combinations of classical boundary conditions to be expressed as second order determinants equated to zero. This form of the frequency equation has not appeared in previous publications. A scheme is presented to calculate the elements of the determinant and another scheme to derive the roots of the frequency equation. The first three frequency parameters are tabulated for a wide range of system parameters. A formula is given to derive the frequency parameter of a "conjugate" beam. The sensitivity of the frequency parameters for a small change in the system parameters is addressed. The second order determinantal formulation enables the mode shape to be established easily and some example mode shapes, position/s of node/s are presented.

The theory developed is applicable to other types of stepped beams as well. Publications which include the effect of support resilience and beam model based on Timoshenko theory are available but is outside the scope of this paper.

2. THEORETICAL CONSIDERATIONS

Figure 1(a) shows the Euler–Bernoulli beam $O_1O_2O_3$ with a step change in cross-section at O_2. The flexural rigidity, mass per unit length and length are EI_k, m_k and R_kL, respectively, where $k = 1$ refer to the first portion O_1O_2 and $k = 2$ to the second portion O_2O_3. R_1 and R_2 are the step location parameters. Without loss of generality, one may choose

$$R_1 + R_2 = 1. \tag{1}$$

To write the equation of transverse vibrations, two co-ordinate systems are chosen with origins at O_1 and O_2. For free vibration at frequency ω, if the ordinate $y_k(x_k)$ is the amplitude of vibration at abscissa x_k $(0 \leqslant x_k \leqslant R_kL)$, then the amplitude of bending moment $M_k(x_k)$, shearing force $Q_k(x_k)$ and the mode shape differential equation of the two portions of the beam are

$$M_k(x_k) = EI_k \frac{d^2 y_k(x_k)}{dx_k^2}, \quad Q_k(x_k) = -EI_k \frac{d^3 y_k(x_k)}{dx_k^3},$$

$$EI_k \frac{d^4 y_k(x_k)}{dx_k^4} - m_k\omega^2 y_k(x_k) = 0 \quad (k = 1, 2). \tag{2}$$

To express equation (2) in dimensionless form, one defines the dimensionless abscissa X_k, amplitude $Y_k(X_k)$, the operators D_k, D_k^n, the dimensionless bending moment $M_k(X_k)$, shearing force $Q_k(X_k)$, flexural rigidity ratio I_{21}, mass per unit length ratio μ_{21} and

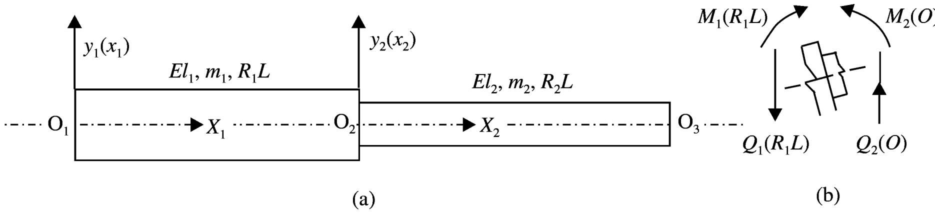

Figure 1. The stepped beam, the co-ordinate systems and the free-body diagram of element at step.

dimensionless frequency Ω_k as follows:

$$X_k = \frac{x_k}{L}, \quad Y_k(X_k) = \frac{y_k(x_k)}{L}, \quad D_k = \frac{\mathrm{d}}{\mathrm{d}X_k}, \quad D_k^n = \frac{\mathrm{d}^n}{\mathrm{d}X_k^n}, \quad M_k(X_k) = \frac{M_k(x_k)L}{EI_k},$$

$$Q_k(X_k) = \frac{Q_k(x_k)L^2}{EI_k}, \quad I_{21} = \frac{EI_2}{EI_1}, \quad \mu_{21} = \frac{m_2}{m_1}, \quad \Omega_1^2 = \frac{m_1\omega^4 L^4}{EI_1} = \alpha_1^4,$$

$$\Omega_2^2 = \alpha_2^4 = \frac{m_2\omega^2 L^4}{EI_2} = \left(\frac{\mu_{21}}{I_{21}}\right)\alpha_1^4. \tag{3}$$

In equations (3), Ω_1 is a dimensionless natural frequency, α_1 is the natural frequency parameter and the nth natural frequency parameter is denoted by $\alpha_{1,n}$.

The "active" dimension of Types 1–3 beams are the width, depth and the diameter respectively. The "active" dimension factor d_{21} is defined as the ratio of the "active" dimension of the second portion to the "active" dimension of the first portion. Hence,

$$\text{for Type 1 beam:} \quad \mu_{21} = d_{21} \quad \text{and} \quad I_{21} = d_{21},$$

$$\text{for Type 2 beam:} \quad \mu_{21} = d_{21} \quad \text{and} \quad I_{21} = (d_{21})^3,$$

$$\text{for Type 3 beam:} \quad \mu_{21} = (d_{21})^2 \quad \text{and} \quad I_{21} = (d_{21})^4. \tag{4}$$

For beams which do not fall into the above categories, μ_{21} and I_{21} must be used as given. Equations (2) in dimensionless form are

$$M_k(X_k) = D_k^2[Y_k(X_k)], \quad Q_k(X_k) = -D_k^3[Y_k(X_k)], \quad D_k^4[Y_k(X_k)] - \alpha_k^4 Y_k(X_k) = 0. \tag{5}$$

2.1. MODE SHAPE OF FIRST PORTION

The solution of the mode shape differential equation of the first portion of the beam is

$$Y_1(X_1) = C_{11}\sin\alpha_1 X_1 + C_{12}\cos\alpha_1 X_1 + C_{13}\sinh\alpha_1 X_1 + C_{14}\cosh\alpha_1 X_1, \tag{6}$$

where C_{11}–C_{14} are the four constants of integration. Two of the constants of integration in equation (6) may be eliminated using the boundary conditions at O_1. The mode shape of the first portion is expressed as

$$Y_1(X_1) = B_{11}U_1(X_1) + B_{12}V_1(X_1), \tag{7}$$

where B_{11} and B_{12} are constants and the functions $U_1(X_1)$ and $V_1(X_1)$ are:

$$\text{if } O_1 \text{ is clamped:} \quad U_1(X_1) = \sin\alpha_1 X_1 - \sinh\alpha_1 X_1, \qquad V_1(X_1) = \cos\alpha_1 X_1 - \cosh\alpha_1 X_1 \tag{8}$$

or

$$\text{if } O_1 \text{ is pinned:} \quad U_1(X_1) = \sin\alpha_1 X_1, \qquad V_1(X_1) = \sinh\alpha_1 X_1 \tag{9}$$

or

$$\text{if } O_1 \text{ is sliding:} \quad U_1(X_1) = \cos\alpha_1 X_1, \qquad V_1(X_1) = \cosh\alpha_1 X_1 \tag{10}$$

or

$$\text{if } O_1 \text{ is free:} \quad U_1(X_1) = \sin\alpha_1 X_1 + \sinh\alpha_1 X_1, \qquad V_1(X_1) = \cos\alpha_1 X_1 + \cosh\alpha_1 X_1. \tag{11}$$

The derivatives of $U_1(X_1)$ and $V_1(X_1)$ are obtained easily by straightforward differentiation.

2.2. MODE SHAPE OF SECOND PORTION

To satisfy the conditions of continuity of deflection and continuity of slope at O_2 one has

$$y_1(R_1L) = y_2(0), \quad \frac{dy_1(R_1L)}{dx_1} = \frac{dy_2(0)}{dx_2}, \tag{12}$$

From the free body diagram of the element at the step shown in Figure 1(b), one has

$$EI_1\frac{d^2y_1(R_1L)}{dx_1^2} = EI_2\frac{d^2y_2(0)}{dx_2^2}, \qquad EI_1\frac{d^3y_1(R_1L)}{dx_1^3} = EI_2\frac{d^3y_2(0)}{dx_2^3}. \tag{13}$$

Equations (12) and (13) in dimensionless form are

$$Y_1(R_1) = Y_2(0), \qquad D_1[Y_1(R_1)] = D_2[Y_2(0)],$$

$$D_1^2[Y_1(R_1)] = I_{21}D_2^2[Y_2(0)], \qquad D_1^3[Y_1(R_1)] = I_{21}D_2^3[Y_2(0)]. \tag{14}$$

The mode shape of the second portion of the beam is

$$Y_2(X_2) = C_{21}\sin\alpha_2X_2 + C_{22}\cos\alpha_2X_2 + C_{23}\sinh\alpha_2X_2 + C_{24}\cosh\alpha_2X_2 \tag{15}$$

in which C_{21}–C_{24} are constants. From equations (14) and (15) one gets

$$C_{22} + C_{24} = B_{11}U_1(R_1) + B_{12}V_1(R_1), \qquad \alpha_2(C_{21} + C_{23}) = B_{11}D_1[U_1(R_1)] + B_{12}D_1[V_1(R_1)],$$

$$I_{21}\alpha_2^2(-C_{22} + C_{24}) = B_{11}\ D_1^2[U_1(R_1)] + B_{12}D_1^2[V_1(R_1)],$$

$$I_{21}\alpha_2^3(-C_{21} + C_{23}) = B_{11}\ D_1^3[U_1(R_1)] + B_{12}D_1^3[V_1(R_1)], \tag{16}$$

C_{21}–C_{24} may now be eliminated. The mode shape of the second portion is expressed as

$$Y_2(X_2) = B_{11}U_2(X_2) + B_{12}V_2(X_2), \tag{17}$$

where the two functions $U_2(X_2)$ and $V_2(X_2)$ are

$$U_2(X_2) = U_{21}\sin\alpha_2 X_2 + U_{22}\cos\alpha_2 X_2 + U_{23}\sinh\alpha_2 X_2 + U_{24}\cosh\alpha_2 X_2,$$

$$V_2(X_2) = V_{21}\sin\alpha_2 X_2 + V_{22}\cos\alpha_2 X_2 + V_{23}\sinh\alpha_2 X_2 + V_{24}\cosh\alpha_2 X_2, \tag{18}$$

in which the coefficients U_{21}–V_{24} are:

$$U_{21} = \frac{D_1[U_1(R_1)]}{2\alpha_2} - \frac{D_1^3[U_1(R_1)]}{2\alpha_2^3 I_{21}}, \qquad V_{21} = \frac{D_1[V_1(R_1)]}{2\alpha_2} - \frac{D_1^3[V_1(R_1)]}{2\alpha_2^3 I_{21}},$$

$$U_{22} = \frac{U_1(R_1)}{2} - \frac{D_1^2[U_1(R_1)]}{2\alpha_2^2 I_{21}}, \qquad V_{22} = \frac{V_1(R_1)}{2} - \frac{D_1^2[V_1(R_1)]}{2\alpha_2^2 I_{21}},$$

$$U_{23} = \frac{D_1[U_1(R_1)]}{2\alpha_2} + \frac{D_1^3[U_1(R_1)]}{2\alpha_2^3 I_{21}}, \qquad V_{23} = \frac{D_1[V_1(R_1)]}{2\alpha_2} + \frac{D_1^3[V_1(R_1)]}{2\alpha_2^3 I_{21}},$$

$$U_{24} = \frac{U_1(R_1)}{2} + \frac{D_1^2[U_1(R_1)]}{2\alpha_2^2 I_{21}}, \qquad V_{24} = \frac{V_1(R_1)}{2} + \frac{D_1^2[V_1(R_1)]}{2\alpha_2^2 I_{21}}, \tag{19}$$

$U_2(X_2)$ and $V_2(X_2)$ are transcendental functions and their derivatives are easily obtained

3. THE FREQUENCY EQUATION

The frequency equation is obtained from the boundary conditions at O_3. For example if the end O_3 is clamped, from equation (17) one has

$$Y_2(R_2) = 0 = B_{11}\ U_2(R_2) + B_{12}\quad V_2(R_2),$$

$$D_2[Y_2(R_2)] = 0 = B_{11}\ D_2[U_2(R_2)] + B_{12}\quad D_2[V_2(R_2)]. \tag{20}$$

The coefficient matrix of equations (20) must be singular. Equations of the same form as equation (20) obtain for other classical boundary conditions. The frequency equations are:

if beam end O_3 is clamped (cl), $\quad U_2(R_2)\ D_2[V_2(R_2)] - D_2[U_2(R_2)]\ V_2(R_2) = 0,$ (21)

if beam end O_3 is clamped (pn), $\quad U_2(R_2)\ D_2^2[V_2(R_2)] - D_2^2[U_2(R_2)]\ V_2(R_2) = 0,$ (22)

if beam end O_3 is sliding (sl), $\quad D_2[U_2(R_2)]\ D_2^3[V_2(R_2)] - D_2^3[U_2(R_2)]\ D_2[V_2(R_2)] = 0,$ (23)

if beam end O_3 is free (fr), $\quad D_2^2[U_2(R_2)]\ D_2^3[V_2(R_2)] - D_2^3[U_2(R_2)]\ D_2^2[V_2(R_2)] = 0.$ (24)

Equations (21)–(24) are transcendental in nature and will have an infinite number of roots.

3.1. NATURAL FREQUENCY CALCULATIONS

The roots of a frequency equation were determined by a "search" to bracket an approximate range within which a root is present followed by an iterative procedure based on linear interpolation. The procedure is as follows. For the selected boundary conditions at O_1, one chose $U_1(X_1)$ and $V_1(X_1)$ from the applicable equations (8)–(11). A trial frequency was assumed ($\Omega_1 = 0{\cdot}1$ say) and with $X_1 = R_1$ the values of $U_1(R_1)$, $V_1(R_1)$, $D_1[U_1(R_1)]$, $D_1[V_1(R_1)]$, etc., were calculated. One proceeded to calculate the coefficients U_{21}–V_{24} from equation (19). From the boundary conditions at O_3, one chose and calculated the left-hand side of the applicable frequency equations (21)–(24). The value of Ω_1 was increased in steps of 0·1 and the calculations described were repeated till a sign change in the left-hand side occurred. The sign change indicated the presence of a root within this range. A "search" was made within this range but with change of 0·01 in Ω_1 to narrow the range within which the root lies. At this stage an iterative procedure based on linear interpolation was invoked to calculate the root to the pre-set accuracy. The "search" procedure was continued (from the value of the first root) to locate the second root and so on. The first three frequency parameters $\alpha_{1,1}$, $\alpha_{1,2}$ and $\alpha_{1,3}$ in Tables 1–4 are for the three types of beams with combinations of the "active" dimension factors $d_{21} = 0{\cdot}5$ and 0·8, step location parameters $R_1 = 0{\cdot}250$, 0·375, 0·625 and 0·750 and for a total of 16 combinations of *cl*, *pn*, *sl* or *fr* boundary conditions. In addition, the frequency parameters for combinations of $d_{21} = 0{\cdot}5$ and 0·8 and $R_1 = 0{\cdot}500$ and are included in Tables 5 and 6.

3.1.1. *The "conjugate" beam*

To reflect the dependence of α_1 on the system parameters and the boundary conditions, one may use the notation $\alpha_1[(i,j),\ \mu_{21},\ I_{21},\ R_1]$ in which (i,j) indicates the boundary conditions with i or $j = 1, 2, 3$ or 4 denote *cl*, *pn*, *sl* or *fr* supports. Clearly,

$$\alpha_1\{(i,j), \mu, I, R\} = \left(\frac{\mu}{I}\right)^{1/4} \alpha_1 \left\{(j,i), \frac{1}{\mu}, \frac{1}{I}, (1-R)\right\}. \tag{25}$$

Such pairs of beams are referred to as "conjugates". Thus frequency parameters of active dimension ratio $d_{21} = 2{\cdot}0$ and 1·25 may be read from Tables 1–3 or 4.

3.2. SENSITIVITY OF THE FREQUENCY PARAMETER

Sensitivity S_R and S_d is the rate at which the frequency parameter changes with respect to small change in R_I and d_{21}, respectively, around nominal values and are defined as

$$S_R = \frac{\partial\{\alpha_1[(i,j), \mu, I, R]\}}{\partial R}, \qquad S_d = \frac{\partial\{\alpha_1[(i,j), \mu, I, R]\}}{\partial d_{21}}. \tag{26}$$

Substantial labour will be required to obtain the symbolic expressions of S_R and S_d. One may calculate S_R and S_d numerically based on the following formulae:

$$S_R = \frac{\alpha_1(R_1 + \varepsilon_R) - \alpha_1(R_1)}{\varepsilon_R}, \qquad S_d = \frac{\alpha_1(d_{21} + \varepsilon_d) - \alpha_1(d_{21})}{\varepsilon_d}, \tag{27}$$

The values of S_R (tabulated in italics) in Table 5 were obtained with $\varepsilon_R = 10^{-3}$ in the immediate vicinity of $R_1 = 0{\cdot}5$ and $d_{21} = 0{\cdot}8$. The values of S_d (tabulated in italics) in

TABLE 1

Frequency parameters of the three types of stepped cl–cl, cl–pn, cl–sl and cl–fr beams for the stated system parameters

System parameters		First three frequency parameters								
		Type 1 beam			Type 2 beam			Type 3 beam		
(i, j)	(d_{21}, R_1)	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$
cl–cl	(0·5, 0·250)	4·9579	7·9043	10·9708	3·9906	6·1618	8·3054	4·1766	6·3824	8·2155
	(0·5, 0·375)	4·8166	7·8428	11·0407	3·9788	6·1444	8·8724	4·2022	6·0081	8·8763
	(0·5, 0.625)	4·5956	7·8192	11·0598	3·9082	6·8726	9·4997	3·6900	7·0322	9·3598
	(0·5, 0.750)	4.5506	7·7164	10·9395	4·1370	6·8312	10·0182	3·8970	6·6511	10·1551
	(0·8, 0·250)	4·7972	7·8781	10·9966	4·4531	7·2243	10·1150	4·5326	7·2473	10·0981
	(0·8, 0·375)	4·7620	7·8547	10·9982	4·4284	7·3450	10·2660	4·4653	7·3302	10·2910
	(0·8, 0·625)	4·6931	7·8469	11·0045	4·5058	7·4659	10·5907	4·4599	7·4668	10·6084
	(0·8, 0·750)	4·6681	7·8194	10·9859	4·5360	7·5845	10·6208	4·4859	7·5330	10·6132
cl–pn	(0·5, 0·250)	4·1692	7·1650	10·1857	3·3603	5·6457	7·7404	3·5036	5·9023	7·7143
	(0·5, 0·375)	4·0905	7·0426	10·2586	3·4450	5·5281	8·1829	3·7005	5·4418	8·1261
	(0·5, 0·625)	3·9058	7·0929	10·2219	3·3051	6·3105	8·6510	3·2260	6·4922	8·4520
	(0·5, 0·750)	3·8996	7·0107	10·2324	3·4891	6·2480	9·4669	3·2859	6·2509	9·6590
	(0·8, 0·250)	3·9990	7·1036	10·2141	3·7327	6·5206	9·3836	3·8121	6·5611	9·3703
	(0·8, 0·375)	3·9778	7·0708	10·2132	3·7177	6·5934	9·5553	3·7793	6·5769	9·5730
	(0·8, 0·625)	3·9330	7·0695	10·2114	3·7647	6·7532	9·8186	3·7500	6·7748	9·8073
	(0·8, 0·750)	3·9265	7·0618	10·2126	3·8513	6·8105	9·9003	3·8327	6·7881	9·9225
cl–sl	(0·5, 0.250)	2·5846	5·6881	8·6524	2·0777	4·5565	6·6761	2·1464	4·7836	6·8258
	(0·5, 0·375)	2·6133	5·5274	8·6482	2·2637	4·4543	6·8008	2·4234	4·5882	6·6623
	(0·5, 0·625)	2·5198	5·5260	8·5794	2·3226	4·8226	7·3653	2·5308	4·7357	7·4054
	(0·5, 0.750)	2·4447	5·5443	8·6643	2·2417	5·1986	7·7632	2·3410	5·3861	7·5708
	(0·8, 0·250)	2·4355	5·5547	8·6551	2·3063	5·1287	7·9370	2·3677	5·1995	7·9440
	(0·8, 0·375)	2·4401	5·5176	8·6397	2·3380	5·1227	8·0917	2·4176	5·1350	8·0845
	(0·8, 0·625)	2·4153	5·4992	8·6325	2·3214	5·3199	8·2203	2·3775	5·3269	8·1973
	(0·8, 0·750)	2·3964	5·5007	8·6420	2·3119	5·3502	8·4461	2·3399	5·3769	8·4463
cl–fr	(0·5, 0·250)	2·0849	4·9153	7·9045	1·6687	3·9528	6·1635	1·7151	4·1395	6·3841
	(0·5, 0·375)	2·1526	4·7950	7·8343	1·8696	3·9387	6·1399	1·9777	4·1638	6·0002
	(0·5, 0·625)	2·1526	4·7950	7·8343	2·1193	4·0976	6·8479	2·3651	3·9885	7·0045
	(0·5, 0·750)	2·0849	4·9153	7·9045	2·0813	4·6760	6·9380	2·2454	4·8251	6·7812
	(0·8, 0·250)	1·9464	4·7592	7·8791	1·8594	4·4137	7·2254	1·9123	4·4913	7·2482
	(0·8, 0·375)	1·9644	4·7318	7·8531	1·9205	4·3974	7·3419	1·9995	4·4354	7·3249
	(0·8, 0·625)	1·9644	4·7318	7·8531	1·9600	4·5939	7·4659	2·0488	4·6096	7·4660
	(0·8, 0·750)	1·9464	4·7592	7·8791	1·9458	4·7263	7·6760	2·0138	4·7920	7·6708

TABLE 2

Frequency parameters of the three types of stepped pn–cl, pn–pn, pn–sl and pn–fr beams for the stated system parameters

System parameters		First three non-zero frequency parameters								
		Type 1 beam			Type 2 beam			Type 3 beam		
(i, j)	(d_{21}, R_1)	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$
pn–cl	(0·5, 0·250)	3·8738	7·0168	10·2318	2·7832	5·1907	7·7462	2·6688	5·0203	7·6366
	(0·5, 0·375)	3·8220	7·1086	10·2217	2·8428	5·6323	8·3061	2·6337	5·5445	8·4646
	(0·5, 0·625)	3·7639	6·9978	10·2767	3·2019	6·0506	8·9770	2·9185	6·1179	8·9625
	(0·5, 0·750)	3·7372	6·9218	10·1282	3·4058	6·1239	9·1973	3·2144	5·8705	9·2708
	(0·8, 0·250)	3·9186	7·0637	10·2124	3·5397	6·4775	9·4284	3·5207	6·4504	9·4255
	(0·8, 0·375)	3·9070	7·0746	10·2113	3·5946	6·6412	9·4853	3·5557	6·6474	9·5052
	(0·8, 0.625)	3·8756	7·0560	10·2193	3·7239	6·7060	9·8224	3·6719	6·6880	9·8517
	(0·8, 0·750)	3·8600	7·0284	10·1949	3·7340	6·8311	9·8532	3·6831	6·7776	9·8314
pn–pn	(0·5, 0·250)	3·1160	6·2218	9·4248	2·2270	4·5706	7·0984	2·1602	4·3995	6·9661
	(0·5, 0·375)	3·0890	6·2831	9·4667	2·2637	4·9160	7·6854	2·1257	4·7788	7·7817
	(0·5, 0·625)	3·0890	6·2831	9·4667	2·5443	5·5403	8·1049	2·3111	5·7147	7·9441
	(0·5, 0·750)	3·1160	6·2218	9·4248	2·8238	5·4805	8·6897	2·6160	5·4039	8·8739
	(0·8, 0·250)	3·1390	6·2767	9·4248	2·8274	5·7369	8·6953	2·8174	5·7089	8·6833
	(0·8, 0·375)	3·1361	6·2832	9·4292	2·8664	5·8917	8·7843	2·8443	5·8811	8·8121
	(0·8, 0·625)	3·1361	6·2832	9·4292	3·0163	5·9757	9·0820	2·9873	5·9867	9·0890
	(0·8, 0·750)	3·1390	6·2767	9·4248	3·0929	6·0646	9·1163	3·0770	6·0358	9·1294
pn–sl	(0·5, 0·250)	1·5664	4·6601	7·8129	1·1104	3·3739	5·8169	1·0986	3·2389	5·6516
	(0·5, 0·375)	1·5581	4·6575	7·9080	1·1116	3·5350	6·3320	1·0787	3·3516	6·2958
	(0·5, 0·625)	1·5361	4·7692	7·7995	1·1353	4·2643	6·5929	1·0450	4·3136	6·5098
	(0·5, 0·750)	1·5312	4·7545	7·8984	1·1760	4·4463	7·1796	1·0588	4·6397	7·0791
	(0·8, 0·250)	1·5704	4·7070	7·8496	1·4072	4·2705	7·2171	1·4055	4·2491	7·1921
	(0·8, 0·375)	1·5695	4·7066	7·8597	1·4125	4·3731	7·3669	1·4075	4·3447	7·3851
	(0·8, 0·625)	1·5671	4·7184	7·8483	1·4409	4·5632	7·4961	1·4257	4·5866	7·4670
	(0·8, 0·750)	1·5666	4·7169	7·8586	1·4686	4·5693	7·6874	1·4495	4·5949	7·7003
pn–fr	(0·5, 0·250)	3·8996	7·0107	10·2324	2·8062	5·1858	7·7466	2·7143	5·0110	7·6373
	(0·5, 0·375)	3·9058	7·0929	10·2219	2·9206	5·6184	8·3072	2·7774	5·5228	8·4661
	(0·5, 0·625)	4·0905	7·0426	10·2586	3·6083	6·0359	8·9552	3·5956	6·1025	8·9281
	(0·5, 0·750)	4·1692	7·1650	10·1857	4·0546	6·3328	9·1810	4·2831	6·1678	9·2551
	(0·8, 0·250)	3·9265	7·0618	10·2126	3·5513	6·4748	9·4287	3·5387	6·4463	9·4259
	(0·8, 0·375)	3·9330	7·0695	10·2114	3·6342	6·6332	9·4857	3·6165	6·6354	9·5058
	(0·8, 0·625)	3·9778	7·0708	10·2132	3·8985	6·7197	9·8113	3·9385	6·7081	9·8351
	(0·8, 0·750)	3·9990	7·1036	10·2141	3·9832	6·9541	9·8712	4·0589	6·9656	9·8575

TABLE 3

Frequency parameters of the three types of stepped sl–cl, sl–pn, sl–sl and sl–fr beams for the stated system parameters

System parameters		First three non-zero frequency parameters								
		Type 1 beam			Type 2 beam			Type 3 beam		
(i, j)	(d_{21}, R_1)	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$
sl–cl	(0·5, 0·250)	2·2288	5·5629	8·6638	1·6841	4·3640	6·7589	1·5072	4·3981	6·9422
	(0·5, 0·375)	2·2029	5·5391	8·5844	1·7503	4·5526	6·6859	1·5624	4·7460	6·6179
	(0·5, 0·625)	2·1659	5·3675	8·6531	1·8750	4·4913	7·6854	1·7477	4·3026	7·8801
	(0·5, 0·750)	2·1631	5·3312	8·5175	1·8839	4·7811	7·5938	1·7946	4·4790	7·5066
	(0·8, 0·250)	2·3277	5·5069	8·6419	2·1499	5·1047	7·9152	2·1060	5·1293	7·9366
	(0·8, 0·375)	2·3140	5·5036	8·6341	2·1664	5·1110	8·0126	2·1175	5·1374	7·9865
	(0·8, 0·625)	2·2952	5·4674	8·6414	2·1667	5·2196	8·2510	2·1136	5·1712	8·2717
	(0·8, 0·750)	2·2965	5·4435	8·6122	2·1685	5·2998	8·3389	2·1115	5·2481	8·2942
sl–pn	(0·5, 0·250)	1·5312	4·7545	7·8984	1·1627	3·7186	6·1853	1·0510	3·7129	6·3368
	(0·5, 0·375)	1·5361	4·7692	7·7995	1·2305	3·9488	6·1312	1·1040	4·0929	6·1658
	(0·5, 0·625)	1·5581	4·6575	7·9080	1·4243	3·9045	7·0060	1·3214	3·8706	7·1002
	(0·5, 0·750)	1·5664	4·6601	7·8129	1·5151	4·0706	7·0665	1·4615	3·8475	7·1487
	(0·8, 0·250)	1·5666	4·7169	7·8586	1·4609	4·3838	7·2138	1·4422	4·3974	7·2413
	(0·8, 0·375)	1·5671	4·7184	7·8483	1·4939	4·4049	7·2626	1·4757	4·4345	7·2439
	(0·8, 0·625)	1·5695	4·7066	7·8597	1·5487	4·4622	7·5456	1·5409	4·4401	7·5741
	(0·8, 0·750)	1·5704	4·7070	7·8496	1·5637	4·5838	7·5709	1·5608	4·5541	7·5605
sl–sl	(0·5, 0·250)	3·1426	6·3444	9·4259	2·4358	4·9800	7·3143	2·3699	5·0506	7·5128
	(0·5, 0·375)	3·1863	6·2843	9·3836	2·6421	5·0851	7·2844	2·6532	5·2747	7·1431
	(0·5, 0·625)	3·1863	6·2843	9·3836	2·8811	5·3332	8·2082	3·0779	5·1494	8·3583
	(0·5, 0·750)	3·1426	6·3444	9·4259	2·8312	5·9247	8·3701	2·9341	6·0575	8·1532
	(0·8, 0·250)	3·1417	6·2897	9·4249	2·9324	5·8084	8·6201	2·9273	5·8362	8·6325
	(0·8, 0·375)	3·1464	6·2833	9·4204	2·9815	5·8109	8·7705	2·9974	5·8219	8·7437
	(0·8, 0·625)	3·1464	6·2833	9·4204	2·9994	6·0590	8·9688	3·0256	6·0485	8·9625
	(0·8, 0·750)	3·1417	6·2897	9·4249	3·0038	6·1405	9·1948	3·0106	6·1691	9·1822
sl–fr	(0·5, 0·250)	2·4447	5·5443	8·6643	1·9022	4·3446	6·7601	1·8637	4·3724	6·9437
	(0·5, 0·375)	2·5198	5·5260	8·5794	2·1066	4·5245	6·6849	2·1179	4·7114	6·6164
	(0·5, 0·625)	2·6133	5·5274	8·6482	2·5317	4·5973	7·6576	2·7581	4·4513	7·8445
	(0·5, 0·750)	2·5846	5·6881	8·6524	2·5739	5·2535	7·6305	2·7724	5·2764	7·5452
	(0·8, 0·250)	2·3964	5·5007	8·6420	2·2595	5·0941	7·9157	2·2718	5·1137	7·9372
	(0·8, 0·375)	2·4153	5·4992	8·6325	2·3356	5·1003	8·0108	2·3727	5·1217	7·9837
	(0·8, 0·625)	2·4401	5·5176	8·6397	2·4294	5·2963	8·2420	2·5072	5·2873	8·2587
	(0·8, 0·750)	2·4355	5·5547	8·6551	2·4339	5·4923	8·3994	2·5049	5·5432	8·3845

TABLE 4

Frequency parameters of the three types of stepped fr–cl, fr–pn, fr–sl and fr–fr beams for the stated system parameters

System parameters		First three non-zero frequency parameters								
		Type 1 beam			Type 2 beam			Type 3 beam		
(i, j)	(d_{21}, R_1)	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$
fr–cl	(0·5, 0·250)	1·6432	4·5178	7·7201	1·1622	3·2121	5·5895	1·0001	3·0870	5·3921
	(0·5, 0·375)	1·6081	4·5318	7·8311	1·1387	3·2915	6·0102	0·9670	3·0870	5·8236
	(0·5, 0·625)	1·6081	4·5318	7·8311	1·1533	3·7289	6·9026	0·9732	3·3849	7·0719
	(0·5, 0·750)	1·6432	4·5178	7·7201	1·2036	4·0787	6·8458	1·0187	3·7974	6·6746
	(0·8, 0·250)	1·8015	4·6339	7·8216	1·6116	4·1587	7·0938	1·5445	4·1065	7·0462
	(0·8, 0·375)	1·7870	4·6500	7·8517	1·6000	4·2257	7·3254	1·5225	4·1723	7·3010
	(0·8, 0·625)	1·7870	4·6500	7·8517	1·6150	4·4502	7·4755	1·5342	4·3901	7·4804
	(0·8, 0·750)	1·8015	4·6339	7·8216	1·6509	4·5044	7·5878	1·5758	4·4534	7·5375
fr–pn	(0·5, 0·250)	3·7372	6·9218	10·1282	2·6509	4·9845	7·4568	2·5400	4·8010	7·2539
	(0·5, 0·375)	3·7639	6·9978	10·2767	2·7053	5·3002	8·1678	2·5525	5·0862	8·1110
	(0·5, 0·625)	3·8220	7·1086	10·2217	3·0204	6·3368	8·6581	2·7435	6·5251	8·4634
	(0·5, 0·750)	3·8738	7·0168	10·2318	3·3815	6·2702	9·4657	3·0932	6·2824	9·6575
	(0·8, 0·250)	3·8600	7·0284	10·1949	3·4593	6·3549	9·3295	3·4038	6·3063	9·2896
	(0·8, 0·375)	3·8756	7·0560	10·2193	3·5016	6·5503	9·5641	3·4507	6·5126	9·5858
	(0·8, 0·625)	3·9070	7·0746	10·2113	3·7029	6·7642	9·8189	3·6555	6·7911	9·8078
	(0·8, 0·750)	3·9186	7·0637	10·2124	3·8315	6·8154	9·8998	3·8022	6·7954	9·9218
fr–sl	(0·5, 0·250)	2·1631	5·3312	8·5175	1·5304	3·8039	6·2015	1·4167	3·6621	5·9946
	(0·5, 0·375)	2·1659	5·3675	8·6531	1·5362	3·9534	6·7241	1·4261	3·7400	6·5699
	(0·5, 0·625)	2·2029	5·5391	8·5844	1·6007	4·7991	7·3896	1·4545	4·7069	7·4409
	(0·5, 0·750)	2·2288	5·5629	8·6638	1·6745	5·2254	7·7664	1·4952	5·4204	7·5758
	(0·8, 0·250)	2·2965	5·4435	8·6122	2·0548	4·8958	7·8352	1·9961	4·8451	7·7887
	(0·8, 0·375)	2·2952	5·4674	8·6414	2·0571	5·0074	8·0856	1·9988	4·9587	8·0756
	(0·8, 0·625)	2·3140	5·5036	8·6341	2·1064	5·3241	8·2248	2·0541	5·3331	8·2041
	(0·8, 0·750)	2·3277	5·5069	8·6419	2·1589	5·3623	8·4461	2·1092	5·3947	8·4463
fr–fr	(0·5, 0·250)	4·5506	7·7164	10·9395	3·2363	5·5867	8·0971	3·1150	5·3869	7·9049
	(0·5, 0·375)	4·5956	7·8192	11·0598	3·3427	6·0011	8·8741	3·1705	5·8080	8·8783
	(0·5, 0·625)	4·8166	7·8428	11·0407	4·0393	6·8760	9·4863	3·9057	7·0430	9·3372
	(0·5, 0·750)	4·9579	7·9043	10·9708	4·7025	6·9471	9·9933	4·8512	6·7968	10·1295
	(0·8, 0·250)	4·6681	7·8194	10·9859	4·1903	7·0916	10·0783	4·1369	7·0435	10·0443
	(0·8, 0·375)	4·6931	7·8469	11·0045	4·2700	7·3194	10·2778	4·2234	7·2925	10·3082
	(0·8, 0·625)	4·7620	7·8547	10·9982	4·6101	7·4723	10·5808	4·6191	7·4765	10·5935
	(0·8, 0·750)	4·7972	7·8781	10·9966	4·7621	7·6760	10·6256	4·8296	7·6715	10·6200

TABLE 5

The sensitivity S_R (shown in italics and calculated with $\varepsilon_R = 10^{-3}$) in the vicinity of the parameters $(R_1, d_{21}) = (0{\cdot}5, 0{\cdot}50)$

(i, j)	(d_{21}, R_1)	Type 1 $\alpha_{1,1}$	Type 1 $\alpha_{1,2}$	Type 1 $\alpha_{1,3}$	Type 2 $\alpha_{1,1}$	Type 2 $\alpha_{1,2}$	Type 2 $\alpha_{1,3}$	Type 3 $\alpha_{1,1}$	Type 3 $\alpha_{1,2}$	Type 3 $\alpha_{1,3}$
cl–cl	(0·5, 0·50)	4·6745	7·9120	10·9368	3·8519	6·6601	9·0420	3·8301	6·6690	9·1232
		– 0·8803	*– 0·0164*	*0·1063*	*– 0·4623*	*4·4233*	*– 1·2793*	*– 2·4399*	*6·9918*	*– 5·1233*
cl–pn	(0·5, 0·50)	3·9748	7·1027	10·1646	3·3427	5·8964	8·5297	3·4637	5·8042	8·7006
		– 0·8031	*0·5129*	*– 0·7801*	*– 0·8683*	*4·3792*	*– 1·0863*	*– 2·5294*	*6·0306*	*– 2·8989*
cl–sl	(0·5, 0·50)	2·5849	5·4733	8·6725	2·3533	4·4513	7·3568	2·6025	4·3224	7·4669
		– 0·4107	*0·1446*	*– 0·6905*	*0·1812*	*1·5306*	*2·9858*	*0·4542*	*0·4101*	*5·6710*
cl–fr	(0·5, 0·50)	2·1773	4·7271	7·8949	2·0448	3·8559	6·6404	2·2517	3·8407	6·6393
		– 0·0016	*0·0048*	*– 0·0062*	*1·0880*	*0·3052*	*4·2913*	*1·8788*	*– 1·4977*	*6·8309*
pn–cl	(0·5, 0·50)	3·7872	7·1174	10·1726	2·9837	6·0799	8·2611	2·7110	6·2330	8·1716
		– 0·2082	*– 0·6705*	*0·9131*	*1·4788*	*1·8116*	*1·4020*	*1·1357*	*3·6576*	*– 1·9587*
pn–pn	(0·5, 0·50)	3·0771	6·3418	9·3659	2·3603	5·3848	7·6999	2·1626	5·4334	7·7370
		0·0008	*– 0·0047*	*0·0103*	*1·0913*	*3·2264*	*– 0·7449*	*0·6906*	*5·3865*	*– 4·3127*
pn–sl	(0·5, 0·50)	1·5468	4·7134	7·8539	1·1178	3·8542	6·6603	1·0569	3·7153	6·8529
		– 0·0938	*0·5636*	*– 0·9227*	*0·0844*	*3·1497*	*– 0·0832*	*– 0·1507*	*3·9877*	*– 0·1274*
pn–fr	(0·5, 0·50)	3·9748	7·1027	10·1646	3·1771	6·0537	8·2596	3·0463	6·1977	8·1696
		0·8065	*– 0·5197*	*0·7971*	*2·7506*	*1·7453*	*1·3235*	*3·1849*	*3·6168*	*– 2·0762*
sl–cl	(0·5, 0·50)	2·1825	5·4401	8·6894	1·8246	4·4796	7·1368	1·6538	4·5432	7·0136
		– 0·1527	*– 0·7797*	*0·7535*	*0·5427*	*– 0·7600*	*5·8568*	*0·8013*	*– 2·8255*	*7·6460*
sl–pn	(0·5, 0·50)	1·5468	4·7134	7·8539	1·3210	3·9627	6·3620	1·1950	4·1278	6·1837
		0·0940	*– 0·5638*	*0·9233*	*0·7970*	*– 0·6235*	*4·4292*	*0·8808*	*– 1·7377*	*4·8334*
sl–sl	(0·5, 0·50)	3·2054	6·2246	9·4836	2·8238	5·0196	7·9075	2·9630	4·9531	7·8685
		– 0·0013	*0·0046*	*– 0·0106*	*1·0496*	*0·5169*	*6·0225*	*2·0894*	*– 1·6869*	*8·8528*
sl–fr	(0·5, 0·50)	2·5849	5·4733	8·6725	2·3430	4·4660	7·1230	2·4515	4·5276	6·9914
		0·4083	*– 0·1369*	*0·6787*	*1·8585*	*– 0·4071*	*5·7137*	*2·8578*	*– 2·4287*	*7·4454*
fr–cl	(0·5, 0·50)	1·5976	4·5314	7·9319	1·1355	3·4505	6·6620	0·9591	3·1592	6·6708
		0·0007	*0·0001*	*– 0·0085*	*0·0522*	*1·7205*	*4·9228*	*0·0227*	*1·1082*	*7·5091*
fr–pn	(0·5, 0·50)	3·7872	7·1174	10·1726	2·8131	5·8687	8·5593	2·5954	5·7731	8·7412
		0·2090	*0·6605*	*– 0·8955*	*1·1892*	*5·1537*	*– 1·1045*	*0·6596*	*6·9528*	*– 2·8004*
fr–sl	(0·5, 0·50)	2·1825	5·4401	8·6894	1·5596	4·2727	7·3743	1·4386	4·0481	7·4857
		0·1538	*0·7821*	*– 0·7659*	*0·2489*	*3·4313*	*3·3291*	*0·0987*	*3·7677*	*6·0994*
fr–fr	(0·5, 0·50)	4·6745	7·9120	10·9368	3·5790	6·6401	9·0694	3·3802	6·6392	9·1640
		0·8849	*0·0015*	*– 0·0781*	*2·6635*	*4·8027*	*– 1·4841*	*2·7166*	*7·3690*	*– 5·3261*

TABLE 6

The sensitivity S_d (shown in italics and calculated with $\varepsilon_d = 10^{-3}$) in the vicinity of the parameters $(R_1, d_{21}) = (0{\cdot}5, 0{\cdot}80)$

		Type 1			Type 2			Type 3		
(i, j)	(d_{21}, R_1)	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$
cl–cl	(0·8, 0·50)	4·7242	7·8594	10·9894	4·4466	7·4586	10·3328	4·4287	7·4840	10·3095
		0·0139	*− 0·0088*	*0·0063*	*0·3754*	*0·2703*	*0·3764*	*0·4141*	*0·2384*	*0·3949*
cl–pn	(0·8, 0·50)	3·9511	7·0705	10·2045	3·7142	6·7285	9·5825	3·7321	6·7367	9·5760
		− 0·0283	*− 0·0050*	*0·0055*	*0·3243*	*0·2780*	*0·3625*	*0·2987*	*0·2728*	*0·3592*
cl–sl	(0·8, 0·50)	2·4315	5·4989	8·6410	2·3362	5·2106	8·1590	2·4128	5·1929	8·1840
		− 0·1579	*0·0033*	*− 0·0039*	*0·0282*	*0·3435*	*0·2954*	*− 0·1760*	*0·3896*	*0·2576*
cl–fr	(0·8, 0·50)	1·9703	4·7181	7·8548	1·9535	4·4535	7·4491	2·0466	4·4543	7·4691
		− 0·2739	*− 0·0207*	*− 0·0045*	*− 0·2064*	*0·3440*	*0·2774*	*− 0·4414*	*0·3707*	*0·2489*
pn–cl	(0·8, 0·50)	3·8925	7·0754	10·2070	3·6681	6·6822	9·6310	3·6168	6·7117	9·6038
		0·0562	*− 0·0088*	*0·0041*	*0·4314*	*0·2863*	*0·3672*	*0·5543*	*0·2373*	*0·4003*
pn–pn	(0·8, 0·50)	3·1348	6·2894	9·4186	2·9325	5·9725	8·8538	2·9018	5·9987	8·8298
		0·0241	*− 0·0110*	*0·0073*	*0·4563*	*0·2603*	*0·3809*	*0·5711*	*0·2174*	*0·4051*
pn–sl	(0·8, 0·50)	1·5683	4·7125	7·8540	1·4230	4·4935	7·3795	1·4132	4·4849	7·3930
		0·0176	*− 0·0003*	*0·0000*	*0·5896*	*0·2973*	*0·3346*	*0·6620*	*0·3289*	*0·3055*
pn–fr	(0·8, 0·50)	3·9511	7·0705	10·2045	3·7624	6·6709	9·6280	3·7607	6·6953	9·5992
		− 0·0283	*− 0·0050*	*0·0055*	*0·3095*	*0·2962*	*0·3681*	*0·3564*	*0·2509*	*0·4017*
sl–cl	(0·8, 0·50)	2·3026	5·4881	8·6465	2·1711	5·1307	8·2181	2·1206	5·1168	8·2271
		0·1471	*0·0143*	*− 0·0074*	*0·4512*	*0·4005*	*0·2887*	*0·5741*	*0·4148*	*0·2886*
sl–pn	(0·8, 0·50)	1·5683	4·7125	7·8540	1·5246	4·4067	7·4528	1·5108	4·4162	7·4393
		0·0177	*− 0·0003*	*0·0000*	*0·2260*	*0·3615*	*0·3253*	*0·3349*	*0·3270*	*0·3541*
sl–sl	(0·8, 0·50)	3·1484	6·2770	9·4310	3·0004	5·8938	8·9455	3·0306	5·8677	8·9694
		− 0·0241	*0·0110*	*− 0·0073*	*0·2071*	*0·3993*	*0·2787*	*0·1040*	*0·4437*	*0·2556*
sl–fr	(0·8, 0·50)	2·4315	5·4989	8·6410	2·3958	5·1405	8·2096	2·4591	5·1314	8·2144
		− 0·1579	*0·0033*	*− 0·0039*	*− 0·0439*	*0·4001*	*0·2932*	*− 0·1767*	*0·4148*	*0·2954*
fr–cl	(0·8, 0·50)	1·7825	4·6565	7·8671	1·6003	4·3347	7·4705	1·5189	4·2737	7·5007
		0·2863	*0·0535*	*− 0·0131*	*0·8985*	*0·5050*	*0·2645*	*1·2007*	*0·6265*	*0·2307*
fr–pn	(0·8, 0·50)	3·8925	7·0754	10·2070	3·5833	6·7333	9·5895	3·5323	6·7437	9·5866
		0·0562	*− 0·0088*	*0·0041*	*0·5652*	*0·2789*	*0·3569*	*0·6806*	*0·2745*	*0·3507*
fr–sl	(0·8, 0·50)	2·3026	5·4881	8·6465	2·0740	5·1780	8·1705	2·0190	5·1441	8·2010
		0·1471	*0·0143*	*− 0·0074*	*0·7468*	*0·3926*	*0·2873*	*0·8932*	*0·4637*	*0·2464*
fr–fr	(0·8, 0·50)	4·7242	7·8594	10·9894	4·4172	7·4583	10·3359	4·3840	7·4832	10·3143
		0·0139	*− 0·0087*	*0·0063*	*0·4506*	*0·2714*	*0·3729*	*0·5318*	*0·2407*	*0·3896*

Table 6 were obtained with $\varepsilon_d = 10^{-3}$ in the immediate vicinity of $d_{21} = 0{\cdot}5$ and $R_1 = 0{\cdot}5$. The tables indicate that Type 3 beam is more sensitive to R_1 and d_{21} compared to Type 1 beam. Since S_d and S_R were obtained numerically, they are not useful in quantitative calculations. This is the reason why the frequency parameters of $R_1 = 0{\cdot}5$ interpolated from Tables 1–3 or 4 did not compare well with the exact value in Tables 5 and 6. If the absolute value of sensitivity is large, interpolated values are of limited accuracy.

3.3. MODE SHAPES

The beam mode shape is the juxtaposition of the mode shapes of the two portions which were expressed as

$$Y_k(X_k) = B_{11}U_k(X_k) + B_{12}V_k(X_k) \quad (k = 1, 2). \tag{28}$$

One may choose the deflection at the step to be A and, without loss of generality, one may choose $A = 1{\cdot}0$. Hence,

$$A = 1 = Y_1(R_1) = B_{11}U_1(R_1) + B_{12}V_1(R_1). \tag{29}$$

For clamped or pinned support at O_3, one has

$$Y_2(R_2) = B_{11}U_2(R_2) + B_{12}V_2(R_2) = 0. \tag{30}$$

Hence from equations (29) and (30), for clamped or pinned supports at O_3:

$$B_{11} = \frac{AV_2(R_2)}{U_1(R_1)V_2(R_2) - U_2(R_2)V_1(R_1)}, \qquad B_{12} = -\frac{AU_2(R_2)}{U_1(R_1)V_2(R_2) - U_2(R_2)V_1(R_1)}. \tag{31}$$

Similarly for sliding or free supports at O_3:

$$B_{11} = \frac{AD_2^3[V_2(R_2)]}{U_1(R_1)D_2^3[V_2(R_2)] - D_2^3[U_2(R_2)]V_1(R_1)},$$

$$B_{12} = -\frac{AD_2^3[U_2(R_2)]}{U_1(R_1)D_2^3[V_2(R_2)] - D_2^3[U_2(R_2)]V_1(R_1)}. \tag{32}$$

To establish the mode shape, the natural frequency Ω_1 was calculated for the selected combination of boundary conditions and system parameters μ_{21}, I_{21} and R_1. Depending on the boundary conditions at O_3, one chose the relevant equation (31) or (32) and calculated B_{11} and B_{12}. For the mode shape of the first portion, X_1 was increased from 0 to R_1 in suitable steps and $Y_1(X_1)$ calculated from equation (7). For the mode shape of the second portion, X_2 was increased from 0 to R_2 in suitable steps and $Y_2(X_2)$ calculated from equation (17). The examples of mode shapes shown in Figure 2 are of the first three natural modes of the three types of beams clamped at O_1 and pinned, sliding or free at O_3. The mode shapes are normalized at $X_1 = 0{\cdot}3$ and some are truncated to highlight others. Approximate locations of node/s may be read from Figure 2.

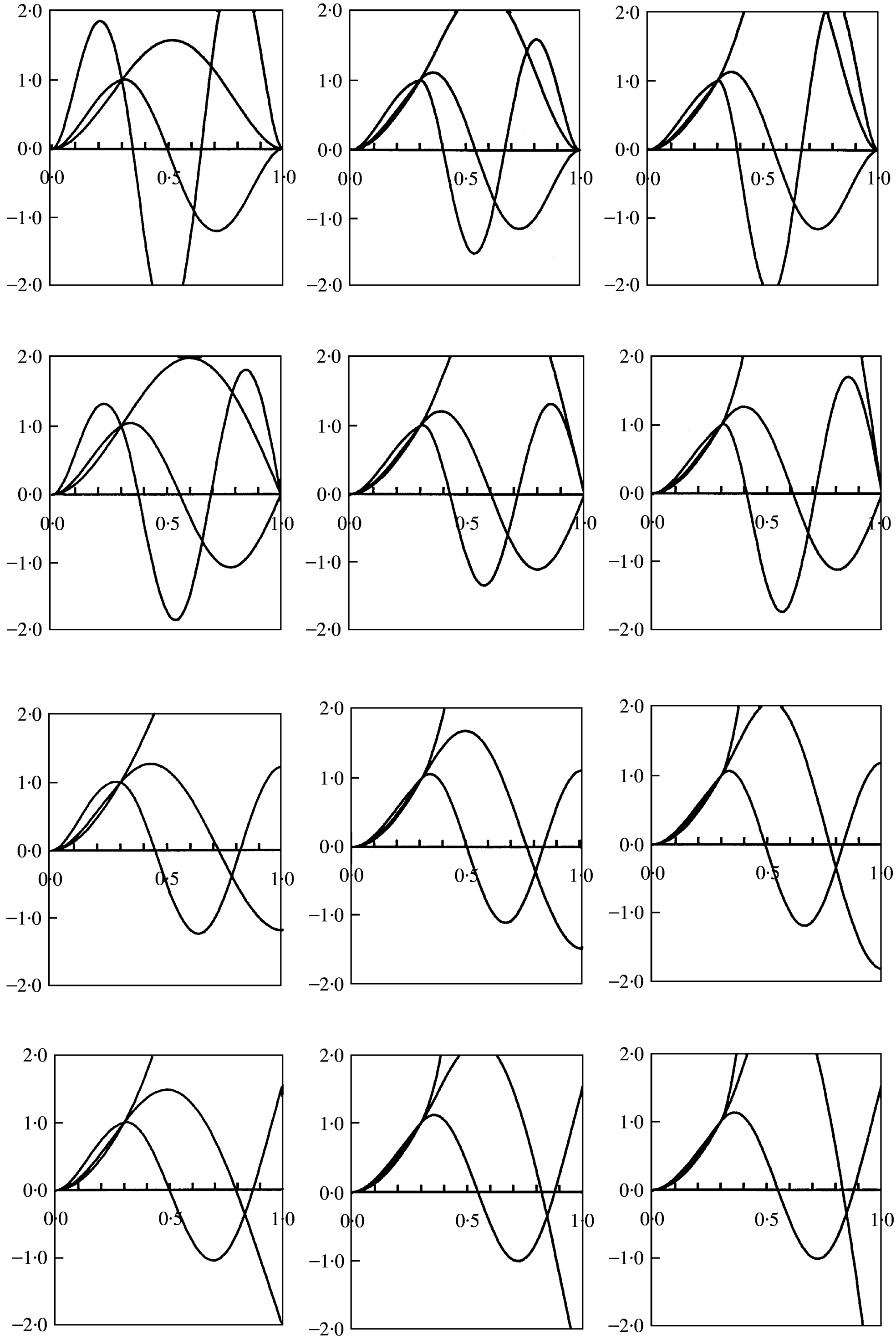

Figure 2. The first three normalized mode shapes of the three types of beams. Columns 1, 2 and 3 are of Types 1, 2 and 3 beams respectively. Rows 1, 2, 3 and 4 are *cl–cl*, *cl–pn*, *cl–sl* and *cl–fr* boundary conditions. System parameters $(R_1, d_{21}) = (0{\cdot}3, 0{\cdot}5)$.

Table 7

The location/s of node/s of Type 1 *beam of the first three non-zero frequency natural modes*, 16 *boundary conditions and system parameter* $(R_1, d_{21}) = (0{\cdot}3,\ 0{\cdot}5)$

(i, j)	(d_{21}, R_1)	Type 1 beam		Type 2 beam		Type 3 beam	
		$\alpha_{1,1}$	Node/s at	$\alpha_{1,2}$	Node/s at	$\alpha_{1,3}$	Nodes at
cl–cl	(0·5, 0·3)	4·9113	None	7·8429	0·4946	11·0094	0·3466 & 0·6425
cl–pn	(0·5, 0·3)	4·1495	None	7·0880	0·5559	10·2036	0·3748 & 0·6925
cl–sl	(0·5, 0·3)	2·6035	None	5·6220	0·7229	8·6140	0·4480 & 0·8179
cl–fr	(0·5, 0·3)	2·1163	None	4·8727	0·7920	7·8405	0·4982 & 0·8677
pn–cl	(0·5, 0·3)	3·8520	None	7·0472	0·4273	10·2676	0·2920 & 0·6169
pn–pn	(0·5, 0·3)	3·1047	None	6·2357	0·4853	9·4716	0·3175 & 0·6690
pn–sl	(0·5, 0·3)	1·5636	None	4·6514	0·6555	7·8540	0·3845 & 0·8006
pn–fr	(0·5, 0·3)	3·8960	0·7268	7·0375	0·4311 & 0·8533	10·2684	0·2919, 0·6196 & 0·8988
sl–cl	(0·5, 0·3)	2·2172	None	5·5680	0·2533	8·6131	0·1730 & 0·5416
sl–pn	(0·5, 0·3)	1·5321	None	4·7701	0·2981	7·8551	0·1869 & 0·6001
sl–sl	(0·5, 0·3)	3·1612	0·4552	6·3336	0·2259 & 0·7531	9·3768	0·1611, 0·4955 & 0·8326
sl–fr	(0·5, 0·3)	2·4743	0·5065	5·5489	0·2577 & 0·8150	8·6124	0·1729, 0·5449 & 0·8795
fr–cl	(0·5, 0·3)	1·6256	None	4·5264	0·1856	7·7525	0·1271 & 0·4841
fr–pn	(0·5, 0·3)	3·7483	0·2204	6·9432	0·1379 & 0·5415	10·1989	0·1015, 0·3714 & 0·6924
fr–sl	(0·5, 0·3)	2·1613	0·3740	5·3447	0·1671 & 0·7019	8·5617	0·1175, 0·4406 & 0·8169
fr–fr	(0·5, 0·3)	4·5673	0·1921 & 0·7674	7·7463	0·1269, 0·4877 & 0·8664	11·0173	0·0947, 0·3440, 0·6454 & 0·9057

TABLE 8

The locations of nodes of first three non-zero frequency natural modes of three types of fr–fr beams with $d_{21} = 0{\cdot}5$ and various values of R_1

Beam type	(d_{21}, R_1)	First mode		Second mode		Third mode	
		$\alpha_{1,1}$	Node/s at	$\alpha_{1,2}$	Node/s at	$\alpha_{1,3}$	Node/s at
1	(0·5, 0·30)	4·5673	0·1921 & 0·7674	7·7463	0·1269, 0·4877 & 0·8664	11·0173	0·0947, 0·3440, 0·6454 & 0·9057
	(0·5, 0·40)	4·6072	0·2053 & 0·7650	7·8469	0·1322, 0·4842 & 0·8687	11·0389	0·0949, 0·3436, 0·6461 & 0·9059
	(0·5, 0·50)	4·6745	0·2152 & 0·7599	7·9120	0·1326, 0·4780 & 0·8703	10·9368	0·0952, 0·3546, 0·6389 & 0·9053
	(0·5, 0·60)	4·7842	0·2188 & 0·7516	7·8594	0·1329, 0·4871 & 0·8680	11·0220	0·0942, 0·3554, 0·6306 & 0·9066
	(0·5, 0·70)	4·9113	0·2166 & 0·7387	7·8429	0·1326, 0·4963 & 0·8602	11·0094	0·0943, 0·3558, 0·6330 & 0·9061
2	(0·5, 0·30)	3·2685	0·1989 & 0·7700	5·7159	0·1405, 0·5095 & 0·8720	8·3870	0·1143, 0·3932, 0·6705 & 0·9124
	(0·5, 0·40)	3·3768	0·2181 & 0·7730	6·1192	0·1547, 0·5343 & 0·8808	9·0074	0·1165, 0·4159, 0·6936 & 0·9184
	(0·5, 0·50)	3·5790	0·2350 & 0·7779	6·6401	0·1563, 0·5554 & 0·8909	9·0694	0·1162, 0·4117, 0·6958 & 0·9190
	(0·5, 0·60)	3·9252	0·2432 & 0·7851	6·8901	0·1532, 0·5393 & 0·8959	9·2912	0·1122, 0·4148, 0·6953 & 0·9214
	(0·5, 0·70)	4·4377	0·2347 & 0·7879	6·8265	0·1538, 0·5521 & 0·8917	9·9897	0·1039, 0·3927, 0·6885 & 0·9279
3	(0·5, 0·30)	3·1284	0·1722 & 0·7572	5·5001	0·1329, 0·4838 & 0·8673	8·2430	0·1128, 0·3748, 0·6650 & 0·9109
	(0·5, 0·40)	3·1952	0·1999 & 0·7525	5·9468	0·1531, 0·5108 & 0·8779	9·0863	0·1160, 0·4083, 0·6965 & 0·9191
	(0·5, 0·50)	3·3802	0·2250 & 0·7515	6·6392	0·1563, 0·5427 & 0·8913	9·1640	0·1154, 0·4033, 0·6993 & 0·9198
	(0·5, 0·60)	3·7665	0·2409 & 0·7587	7·0828	0·1497, 0·5180 & 0·8993	9·0982	0·1149, 0·4201, 0·6840 & 0·9200
	(0·5, 0·70)	4·4411	0·2337 & 0·7685	6·8070	0·1549, 0·5473 & 0·8911	10·1601	0·1021, 0·3865, 0·6709 & 0·9294

3.3. THE "EXACT" POSITION OF NODE/S

The "exact" location of the node/s was calculated while the mode shape was being established. Once the approximate location of a node is known, the iterative procedure based on linear interpolation was invoked to find the "exact" location of the node. Table 7 is the tabulation of the location of the node/s of Type 1 beam with system parameters $(d_{21}, R_1) = (0{\cdot}5, 0{\cdot}8)$, for 16 boundary conditions. The rigid-body rotation of *pn–fr*, *fr–pn*, and *fr–fr* and the rigid-body translation of *sl–sl*, *sl–fr*, *fr–sl* and *fr–fr* beams are not included. Table 8 is the tabulation of the position of node/s of the three types of *fr–fr* beams with $d_{21} = 0{\cdot}5$ and for various values of R_1. Rigid-body rotation and translation modes are not included.

4. CONCLUSIONS

The character of the title problem enabled the frequency equations to be expressed as second order determinants equated to zero. A scheme was presented to calculate the elements of the determinant and a scheme to evaluate the roots of the frequency equation.

The first three frequency parameters are tabulated for three types of beams, all the combinations of classical boundary conditions, several locations of the step. Type 3 beams are more sensitive to the system parameters compared to Type 1 beams. The first three mode shapes, position/s node/s, etc., are presented for selected system parameters. The theory developed is applicable to other types of stepped beams as well.

The results may be used as benchmarks to compare solutions obtained by approximate methods.

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