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LETTERS TO THE EDITOR

SOLUTIONS OF FREE NON-LINEAR OSCILLATIONS WITH POWER LAW IN ONE DIMENSION

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1. INTRODUCTION

In a conservative system the sum of a particle's kinetic and potential energies will be fixed for all time. The points at which the potential energy equals the total energy will determine the physical limits of the motion. These limits are the turning points since the velocity will be zero. If the motion is bounded by such points the region is finite. The motion of a system having one degree of freedom takes place in one dimension or along a straight line. A particle oscillates in one dimension if it moves repeatedly backward and forward between two turning points. One class of oscillations of a particle relies on a restoring force which is always directed towards an equilibrium point situated midway between the two turning points. In such a case, the motion will be symmetrical and the displacement of the particle from the midpoint, a periodic function of time which is not necessarily sinusoidal. If conservative motion is initiated either by a discrete impulse (kinetic energy) and or a static displacement (potential energy) the subsequent oscillations will be free or self-sustaining. This article will consider such restoring forces on a particle and offer a numerical solution of a non-linear equation of motion. Previous investigations of various aspects of this problem with more references [1, 2] are available.

2. THE EQUATION OF MOTION

For a particle of mass m subject to a restoring force which is proportional to any real power n of the magnitude of its displacement from an equilibrium point at the origin and oscillating symmetrically between its turning points at $x = \pm A$, with a positive constant of proportionality k, the one-dimensional Newtonian equation of motion is

$$m\ddot{x} + k\,\mathrm{sgn}(x)|x|^n = 0, \quad \infty \geqslant n \geqslant -\infty, \quad A \geqslant x \geqslant -A, \tag{1}$$

where $\mathrm{sgn}\,(x) = x/|x|$.

Now, since $\ddot{x} = \mathrm{d}\dot{x}/\mathrm{d}t = \dot{x}(\mathrm{d}\dot{x}/\mathrm{d}x)$ immediate integration of equation (1) gives

$$m\frac{(\dot{x})^2}{2} = -\frac{k|x|^{n+1}}{(n+1)} + C, \quad n \neq -1. \tag{2}$$

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When $n = -1$,

$$m\frac{(\dot{x})^2}{2} = -k\ln|x| + C. \tag{3}$$

It should be noted that dimensionally equations (2) and (3) are energy equations. The constant of integration C is decided by the value of x at either turning point when the speed $|\dot{x}| = 0$.

Equations (2) and (3) then become

$$\dot{x} = \pm\,\omega\sqrt{\frac{2(A^{n+1} - |x|^{n+1})}{n+1}}, \quad \text{where } \omega = \sqrt{\frac{k}{m}}, \quad A \geqslant |x|, \tag{4}$$

$$\dot{x} = \pm\,\omega\sqrt{2\ln\left(\frac{A}{|x|}\right)}, \quad \text{where } \omega = \sqrt{\frac{k}{m}}, \quad A \geqslant |x|. \tag{5}$$

When $x \to 0$, equations (4) and (5) will take a maximum value, v, such that

$$\lim_{x\to 0}\dot{x} = v = \begin{cases} \pm\,\omega\sqrt{\dfrac{2A^{n+1}}{n+1}}, & n > -1, \\ \to \pm\,\infty, & n \leqslant -1. \end{cases} \tag{6}$$

The singularity at the origin for $n \leqslant -1$ does not rule out oscillations. However, such inverse power restoring forces do require very large but realistically *finite* system energies to operate. Gravitational attraction with $n = -2$ is one rather interesting example with C negative and the particle's potential energy always negative with its magnitude exceeding that of its kinetic energy.

Separating the variables in equations (4) and (5) and integrating between the limits for x at t gives

$$\int_x^A \sqrt{\frac{n+1}{2(A^{n+1} - |x|^{n+1})}}\,\mathrm{d}x = \omega\int_0^t \mathrm{d}t, \quad A \geqslant x \geqslant -A, \quad n \neq -1, \tag{7}$$

$$\int_x^A \frac{\mathrm{d}x}{\sqrt{2\ln(A/|x|)}} = \omega\int_0^t \mathrm{d}t, \quad A \geqslant x \geqslant -A, \quad n = -1. \tag{8}$$

The exact solutions of the integral equations (7) and (8) would give the position of the particle as a periodic function of time $x = f(t)$ for any value of n. From the symmetry of the motion the time taken to travel from A to 0 will thus be a quarter of the period T. Integrating equation (7) between these limits and multiplying by four therefore gives

$$T = \frac{4}{\omega}\int_0^A \sqrt{\frac{n+1}{2(A^{n+1} - x^{n+1})}}\,\mathrm{d}x. \tag{9}$$

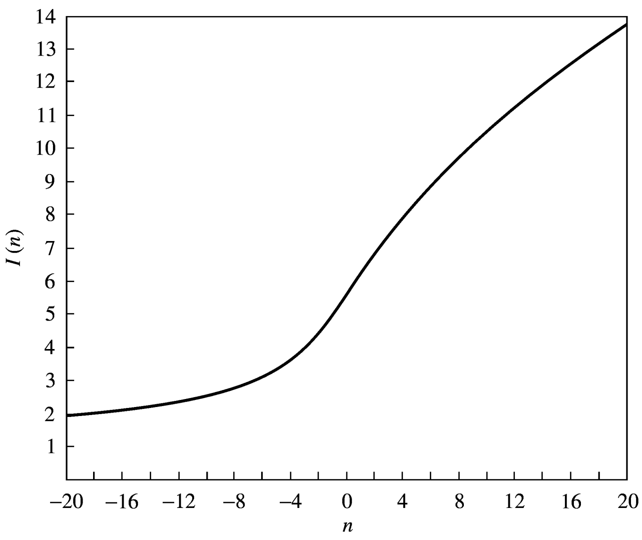


Figure 1. Computation of I_n, equation (10), over the range $-20 \leqslant n \leqslant 20$, $n \neq -1$. It should be noted that I_n is a continuous function of n with no special distinction between integers, algebraic signs and even or odd values of n. The special case of $n = -1$ where $I_n = \sqrt{8\pi}$ is also included in this curve.

Making the substitution $z = x/A$ for $0 \leqslant x \leqslant A$ then gives

$$T = \frac{A^{(1-n)/2} I_n}{\omega}, \quad \text{where } I_n = \int_0^1 \sqrt{\frac{8(n+1)}{1 - z^{n+1}}}\,\mathrm{d}z, \quad n \neq -1. \tag{10}$$

For $n = -1$ it can be shown that $T = \sqrt{8\pi}\ A/\omega$. (11)

In the particular case $n = 1$, or linear motion, the period can be seen to be uniquely independent of A with $T = 2\pi/\omega$. The integral I_n can be computed for any $\pm$ve value of n and a range of values is shown in Figure 1.

The general dependence of T on the initial displacement A suggests a practical means for the determination of n and ω. If by experiment, it was found that $T = T_1$ when $A = A_1$ and $T = T_2$ when $A = A_2$, it then follows by using equation (10)

$$T_1 = \frac{A_1^{(1-n)/2}}{\omega} I_n \quad \text{and} \quad T_2 = \frac{A_2^{(1-n)/2}}{\omega} I_n. \tag{12}$$

Hence dividing the parts of equation (12) leads to

$$n = 1 - 2\left(\frac{\ln T_1 - \ln T_2}{\ln A_1 - \ln A_2}\right). \tag{13}$$

Adding the parts of equation (12) and substituting the value of n from equation (13) then gives

$$\omega = \frac{I_n}{(T_1 + T_2)} (A_1^{(1-n)/2} + A_2^{(1-n)/2}). \tag{14}$$

3. NUMERICAL SOLUTIONS

Equations (7) and (8) do not appear to have a generalized form $x = f(t)$ but can be solved numerically. It is clear that $t = g(x)$ and since it is periodic between A and $-A$, the oscillation in this range can be plotted, rotated through 90° and finally subjected to a repeating period of T. Such a process is described for $n = 0$ in Figure A1 in Appendix A.

With the single exception of $n = 1$, equation (10) shows that T depends crucially on the magnitude of A and by implication allows an infinite range of frequencies. However, for any value of A it is still possible to create a Fourier sine series representation of the periodic wave form $x = f(t)$. The "fundamental" $2\pi/T$ will have the usual integer multiples.

$$x(t) = \sum_{s=1}^{\infty} B_s \sin\left(\frac{2s\pi t}{T}\right), \quad \text{where } B_s = \frac{4}{T}\int_0^{T/2} f(t) \sin\left(\frac{2s\pi t}{T}\right) \mathrm{d}t. \tag{15}$$

Unfortunately, the above strategy has a fatal flaw: $x = f(t)$ is the very solution being sought. However, two particular cases, $n = 1$ and 0, do have exact solutions which can be tested against the numerical method. For convenience let $x(0) = 0$ giving $x(T/4) = A$, then

$$n = 1, \quad m\ddot{x} + kx = 0 \Rightarrow x = A \sin\left(\frac{2\pi t}{T}\right), \quad \text{where } T = 2\pi\sqrt{\frac{m}{k}} \quad \text{from equation (10)} \tag{16}$$

$$n = 0, \quad m\ddot{x} + k\,\mathrm{sgn}\,(x) = 0 \Rightarrow x = \frac{32A}{\pi^3}\sum_{s=1}^{\infty}\frac{1}{(2s-1)^3}\sin\left[\frac{2(2s-1)\pi t}{T}\right],$$

$$\text{where } T = \sqrt{\frac{32mA}{k}} \quad \text{from equation (10).} \tag{17}$$

Equation (17) has the particle subjected to a constant restoring force directed towards the origin. The solution is obtained by integrating twice to derive a quadratic expression (a parabola above and below the time axis) or $x = \pm(8At/T)(1 - 2t/T)$ for $0 \leqslant t \leqslant T/2$. This exact and intrinsically periodic solution will clearly repeat from the beginning of every cycle of T and can be shown to be the above Fourier sine series. Equation (17) can be compared exactly with equation (19) in reference [2]. The above solutions, equations (16) and (17), provide an excellent fit and confirmation with the graphical trace. Figure 2 shows the superimposed wave forms

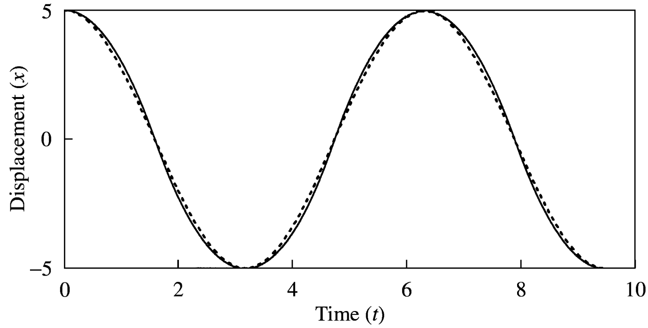
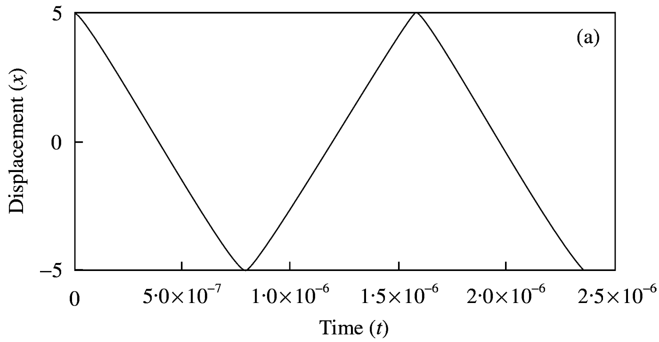


Figure 2. Superimposed $n = 1$ and 0 cases. The period T is chosen to be the same. The dotted line is for $n = 1$ and the full line $n = 0$. The parabola for $n = 0$ shows the initial trend towards a square wave as $n \to -\infty$.



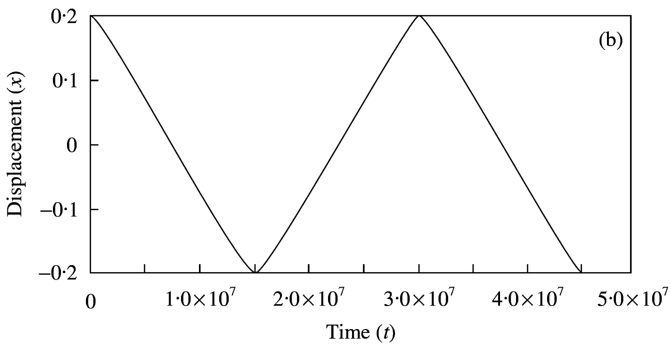


Figure 3. Case $n = 20$, $\omega = 2$, (a) for $A = 5$; (b) for $A = A/5$. This confirms equation (19) and the effect of the turning point when $n \to \infty$. The period when $A = 1$ is 6·89885 s.

Two further solutions, which can be regarded as limiting cases, are worth considering.

The first case is for large positive value of n

$$n = N \to \infty \Rightarrow m\ddot{x} + k\,\mathrm{sgn}(x)|x|^N = 0. \tag{18}$$

The example for $n = 20$ in Figure 3 gives a clear indication that the motion is approaching a triangular shape in the time domain. The interpretation is that the particle is travelling at a constant speed $|v|$ with almost instantaneous reversal at either $x = A$ or $-A$. One could imagine the bob of a very long pendulum which is caught between two vertical plates and undergoing perfectly elastic rebounds. A very good approximation on limiting solution for large values of n will therefore be a Fourier sine series for a triangular shape which can be obtained in the following form:

$$x \simeq \frac{8A}{\pi^2} \sum_{s=1}^{\infty} \frac{(-1)^{s+1}}{(2s-1)^2} \sin\left[\frac{2(2s-1)\pi t}{T}\right], \quad \text{where } T = \frac{4A}{v} \to \begin{cases} 0, & A > 1, \\ \infty, & A < 1. \end{cases} \tag{19}$$

In Figure 3 the effect of the magnitude of the separation of the turning points is also shown for $n = 20$, for cases $A > 1$ and $A < 1$.

Differentiating equation (19) to obtain the velocity at any time produces a step function with jumps at $x = A$ and $-A$. The appearance of the Gibbs' phenomenon in the vicinity of these points is peculiar to the Fourier sine series approximation close to a discontinuity and is not exhibited if the exact form, equation (4), is used.

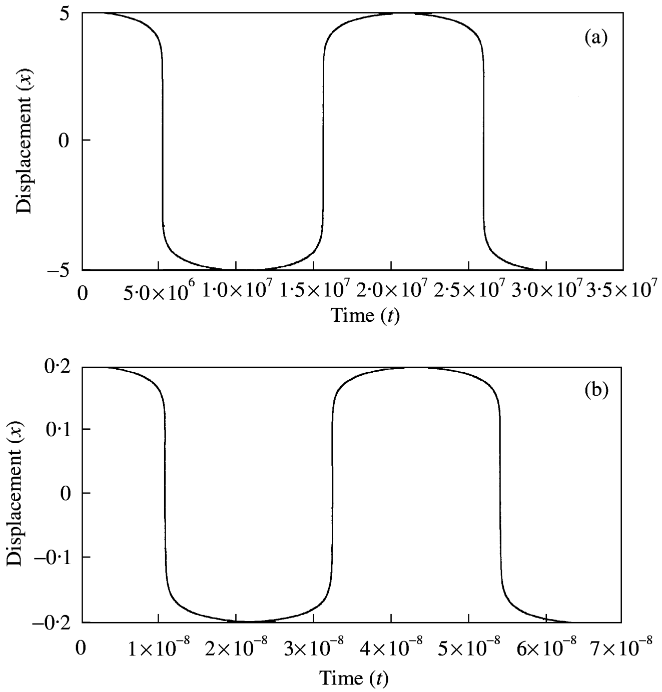


Figure 4. Case $n = -20$, $\omega = 2$, (a) for $A = 5$; (b) for $A = A/5$. This confirms equation (21) and the effect of the turning point when $n \to -\infty$. The period when $A = 1$ is 0·95157 s.

The second case is for large negative value of n

$$n = -N \to -\infty \Rightarrow m\ddot{x} + \frac{k\,\mathrm{sgn}(x)}{|x|^N} = 0. \tag{20}$$

It will be seen in Figure 4 for $n = -20$ for cases of $A > 1$ and $A < 1$ that a vibration with quasi-square wave characteristics is indicated. In the limit, $n \to -\infty$, therefore a particle is predicted to jump periodically and instantaneously between its turning points.

The corresponding Fourier series would be

$$x \simeq \frac{4A}{\pi^2} \sum_{s=1}^{\infty} \frac{1}{(2s-1)} \sin\left[\frac{2(2s-1)\pi t}{T}\right], \quad \text{where } T \to \begin{cases} \infty, & A > 1, \\ 0, & A < 1. \end{cases} \tag{21}$$

Equation (21) could be the description of a relaxation phenomenon with the particle notionally at rest at almost all times, i.e., pure potential energy. It would be the exact antithesis of $n \to \infty$ where the particle travels at constant speed between its turning points, i.e., pure kinetic energy. Newton's first law that a particle stays at rest or continues to move in a straight line with constant speed unless acted on by an external force could therefore be satisfied by $n \to -\infty$ and $n \to \infty$ respectively.

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APPENDIX A: NUMERICAL SOLUTION OF THE NON-LINEAR EQUATION OF MOTION (1) USING MATHCAD

No special requirements are necessary and the worksheet below should be self-explanatory.

The technique is to construct the oscillation piecemeal using the period T as the repeating factor. Time (t) is plotted along the horizontal axis with the displacement (x) on the vertical

$$A := 5 \quad n := 0 \quad \omega := 2 \quad x := -A, \left(-A + \frac{A}{399}\right) \cdot\cdot\ A$$

$$T(n) := \frac{4}{\omega}\int_0^A \sqrt{\frac{n+1}{2[A^{n+1} - (|x|)^{n+1}]}}\,\mathrm{d}x \qquad T(n) = 6{\cdot}325$$

$$t_1(x, n) := \frac{1}{\omega}\int_x^A \sqrt{\frac{n+1}{2[A^{n+1} - (|x|)^{n+1}]}}\,\mathrm{d}x \qquad t_2(x, n) := -t_1(x, n)$$

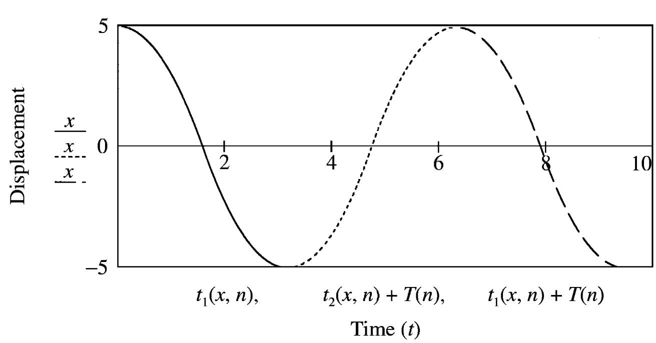

Figure A1. Plot of numerical solution of equation (7) for $A = 5$, $\omega = 2$ and $n = 0$.