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OSCILLATIONS OF NON-LINEAR SYSTEM WITH RESTORING FORCE CLOSE TO SIGN(*X*)

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1. INTRODUCTION

Consider the equation

$$\ddot{x} + x^{(1/2n+1)} = 0 \tag{1}$$

with the initial conditions

$$x(0) = 0, \qquad \dot{x}(0) = A \tag{2}$$

for $n \to \infty$.

A similar model has very often been used in the theory of vibro-impact system [1]. A solution to equation (1) can be found using the special Ateb functions proposed by Rosenberg [2], being an inversion of the incomplete Beta functions. However, in this work a construction of an asymptotical solution using a small δ-method [3–5] is proposed.

2. ANALYSIS

In the limit as $n \to \infty$ one gets the following equation:

$$\ddot{x}_0 + \operatorname{sign}(x_0) = 0, \tag{3}$$

$$\operatorname{sign}(x) = \begin{cases} +1 & \text{for } x > 0, \\ -1 & \text{for } x < 0, \end{cases}$$

which has been analyzed using various methods in references [6–8]. An analytical solution to equation (3) can be presented in the form of a Fourier series [7], piecewise continuous functions [6] or by "saw-tooth" functions [8].

In order to construct a solution to equation (1), the so-called small δ-method can be applied [3–5]. By taking $\delta = (2n+1)^{-1}$ and supposing $\delta \ll 1$ we use the following

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approximation (considering only the values of $x > 0$, since a solution for $x < 0$ can be obtained using a symmetric mapping):

$$x^{\delta} = 1 + \delta \ln|x| + \cdots. \tag{4}$$

For equation (1) with initial conditions (2) the following solution is sought:

$$x = x_0 + \delta x_1 + \cdots. \tag{5}$$

Substituting equation (5) into equations (1) and (2) yields (in the first approximation) the following equation:

$$\ddot{x}_1 = -\ln|x_0|. \tag{6}$$

In order to avoid singularities consider a solution to equation (6) on a quarter part of its period. A solution to the boundary value problem can be presented in the form [6]

$$x(t) = \begin{cases} -\dfrac{t}{2}(t - 2A), & 0 \leqslant t \leqslant 2A, \\ \dfrac{t^2}{2} - 3At + 4A^2, & 2A \leqslant t \leqslant 4A, \end{cases}$$
$$x(t + nT) = x(t), \qquad T = 4A.$$

Therefore, the first approximation on the interval $0 \leqslant t \leqslant A$ yields

$$\ddot{x}_1 = -\ln\left(tA - \frac{t^2}{2}\right), \qquad x_1(0) = 0, \quad \dot{x}(0) = 0. \tag{7, 8}$$

Integrating equation (7) twice and taking into account equation (8) gives (in this case "Mathematica" computations have been used)

$$x_1(t) = 2At - 3t^2 - t^2 \ln 2 - 4A(A - t)\ln|-2A| + t^2 \ln|2A - t|$$
$$+ t^2 \ln t + 4A^2 \ln|-2A + t| - 4At \ln|-2A + t|. \tag{9}$$

Although the solution can be extended up to the terms of $\delta^2, \delta^3, \ldots,$ only zero and first order approximations are used:

$$x \approx x_0 + \delta x_1. \tag{10}$$

The Padé approximants can be used to extend the application area of solution (1) [3, 5]. For this case one gets

$$x \approx \frac{x_0^2}{(x_0 - \delta x_1)}. \tag{11}$$

In addition, the exponential approximation can also be applied [9]:

$$x \approx x_0 \exp(\delta x_1 / x_0). \tag{12}$$

3. RESULTS AND CONCLUSIONS

Some of the numerical results for $A = 1$ are presented in Figure 1.

In addition, the numerically estimated periods together with the corresponding relative errors with respect to the results obtained using the fourth order Runge–Kutta method are included in Table 1. For some values of n the periods have not been given. The reason can be simply explained by tracing the corresponding drawings in Figure 1: in some cases, the curves corresponding to approximations (11) and (13) do not have a minimum. It should be noted that the periods obtained using equations (10)–(12) are generally larger than the "exact period", calculated from the fourth order Runge–Kutta method, for small values of n (see the results in Table 1).

The analysis leads to the following conclusions. Only the small δ method allows for the estimation of all of the periods in the considered interval of changes of n values. The examination of the figures shows that only equation (10) gives the correct qualitative

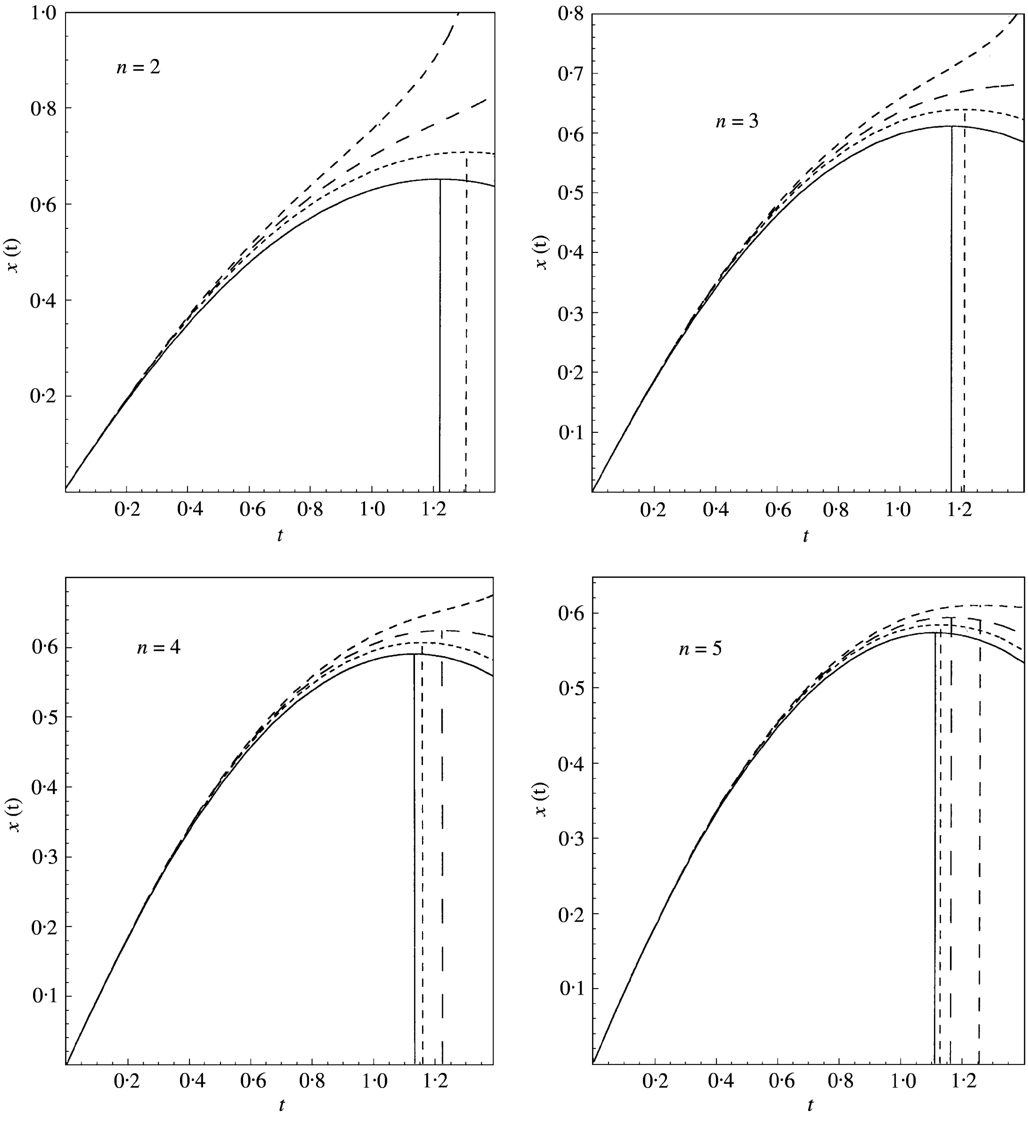

Figure 1. Solutions to the Cauchy problem (1), (2) for $A = 1$ using the fourth order Runge–Kutta method (——), and approximations: (10) (- - - - - -); (11) (- - -); (12) (– – –) for different n values.

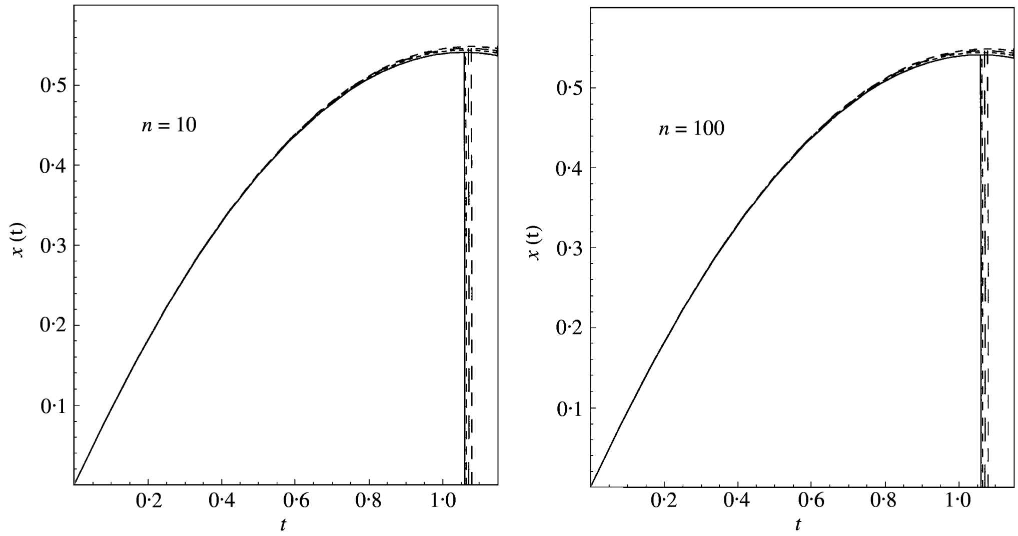

Figure 1. *Continued*

TABLE 1

Numerical estimations of $\frac{1}{4}$ T_i, $i = 1, 2, 3$, *where* T_1, T_2 T_3 *are the periods related to approximations* (10), (11) *and* (12) *respectively*

	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 10$	$n = 100$
T_{num}	1·21981	1·16507	1·13200	1·10991	1·05976	1·00684
T_1	1·30571	1·20769	1·15748	1·12686	1·06436	1·00652
Δ_1 (%)	7·04	3·66	2·25	1·53	0·434	0·00473
T_2	—	—	—	1·25367	1·07934	1·00664
Δ_2 (%)	—	—	—	12·95	1·85	0·0160
T_3	—	—	1·22309	1·16116	1·07097	1·00658
Δ_3 (%)	—	—	8·05	4·62	1·06	0·0103

behaviour for small values of *n*. In addition, it gives the best approximation to the real (numerical) values of the periods sought. The relative errors decrease quickly with an increase of *n*. Although the latter behaviour can also be observed in a case of the Padé and exponential approximations, it seems that the exponential approximations is more suitable than the Padé one. However, with an increase of *n* all of them are suitable to approximate the "real" periods.

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