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ERRORS IN PARAMETER ESTIMATES FROM THE FORCE STATE MAPPING TECHNIQUE FOR FREE RESPONSE DUE TO PHASE DISTORTION

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1. INTRODUCTION

The force state mapping (FSM) technique, as proposed by Masri *et al.* [1, 2], for a single-degree-of-freedom system uses the response measurements to construct the restoring force surface (which may be non-linear) over the displacement–velocity state plane. A surface which can be described analytically is then fitted to this empirical surface to obtain quantified system parameters. This approach to system identification has been applied to a variety of applications including non-linear and multi-degree-of-freedom systems, undergoing free or forced response [3–7]. The technique requires estimates for the system displacement, velocity and acceleration. In practice, whether these estimates are obtained by three separate instruments [7] or by careful numerical differentiation and/or integration of a single measured signal [4], the resulting signals may suffer phase distortion. Furthermore, for forced response, there is a possibility that the force signal may experience phase distortion relative to the response signals. A practical example of how these problems may manifest themselves was given in the experimental study by Worden and Tomlinson [8], in which all the measured signals were phase shifted by a multiplexed analogue–digital converter.

Wright and Al-Hadid [9] have derived expressions for the errors in the parameter estimates obtained by FSM in the presence of phase distortion of the measured signals for a linear system subject only to a sinusoidal excitation force. They have shown that for lightly damped systems, which are often of most interest, the damping parameter is the most susceptible to error particularly with phase distortion in the excitation force. It is possible to use free response data as input to the FSM technique [7] in order to avoid the phase lag between the excitation and the response and so remove the associated error in the parameter estimates. In this case, there will also be a potential for error in the parameter estimates due to relative phase error in the response signals. However, although Wright and Al-Hadid [9] reach the general conclusion that "very accurate data appear to be a basic requirement for FSM approach for lightly damped systems", they also note that different types of excitation are likely to yield different parameter estimates and hence error sensitivity. Thus, it is necessary to evaluate the sensitivity to phase errors of the force state mapping technique using free response data as input.

In this letter, the errors in the parameter estimates obtained with FSM due to phase errors for free response are assessed analytically. A novel pictorial approach is validated in the first instance with forced response and is then extended to the case of free response

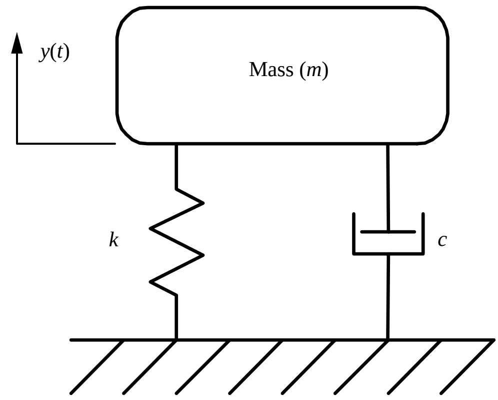

Figure 1. Schematic of a linear single-degree-of-freedom system.

which is of interest here. As in Wright and Al-Hadid's study, the system considered is linear, however, it should be noted that for systems with polynomial non-linearities, the non-linear terms can be decomposed into a dominant linear stiffness and/or damping, plus smaller contributions at higher harmonics. Thus the sensitivity of a linear system to phase errors may be indicative of the sensitivity of the non-linear system.

2. FORCED RESPONSE

The equation of motion for the system in Figure 1 under forced response is

$$m\ddot{y} + c\dot{y} + ky = F. \tag{1}$$

Under sinusoidal excitation this becomes

$$(-m\omega^2 + \mathrm{j}c\omega + k)\,\mathrm{e}^{\mathrm{j}\omega t} = f\mathrm{e}^{\mathrm{j}\theta}\mathrm{e}^{\mathrm{j}\omega t}, \tag{2}$$

where $\tan\theta = 2\zeta r/(1 - r^2)$ and $r = \omega/\omega_n$ (i.e., the ratio between the excitation and natural frequencies).

This can be represented graphically as a vector diagram, with the basis vectors aligned along the real and imaginary axes as in Figure 2(a), in which the three instantaneous responses (inertia, damping and stiffness) and the instantaneous excitation force are given by the real components of the vectors. For the purpose of force state mapping, the magnitude of both the excitation force and mass must be known in order to construct the restoring force.

In the absence of noise, the FSM process as applied to a system governed by equation (2) is simply a matter of determining the magnitude of stiffness and damping which will close the vector diagram. Expressions for the errors in the stiffness and damping estimates can be obtained by geometric considerations. It should be noted that the response signals $(y, \dot{y}, \ddot{y})$ have a relative phase angle of 90°.

The error in the parameter estimates due to a phase error in the excitation signal can be assessed by examining the situation when the measured force is delayed relative to the actual force by a time Δt. The excitation vector will be rotated clockwise by an angle $\tau = \omega\Delta t$. (Note that a positive angle is a time lag). The force diagram for this scenario is shown in Figure 2(b).

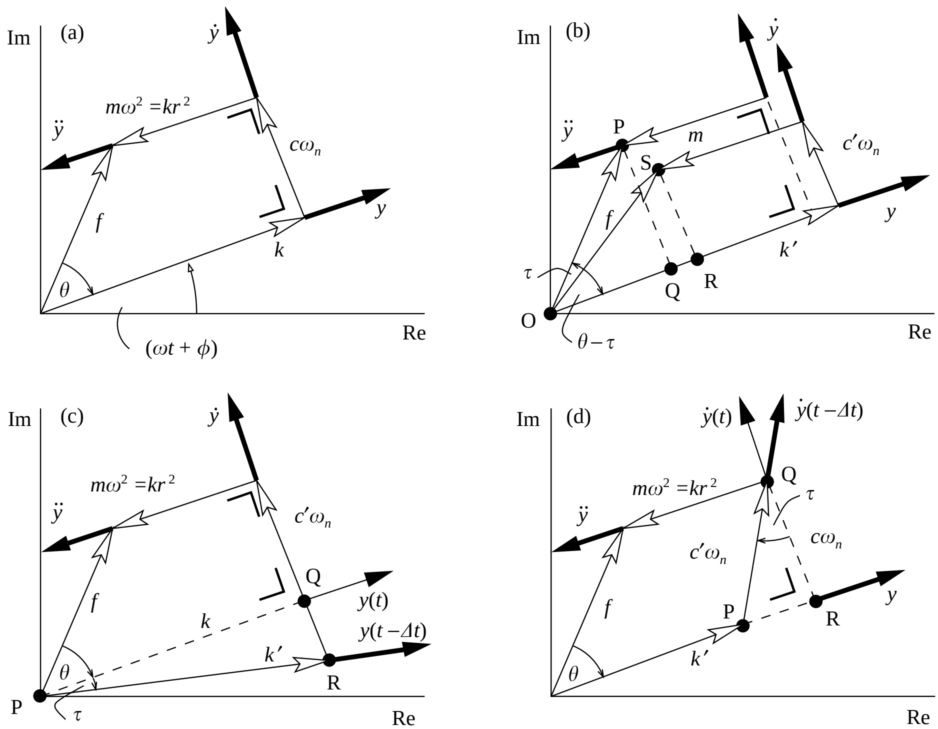

Figure 2. Force diagrams for forced response: (a) no phase error; (b) phase lag in excitation force; (c) phase lag in displacement and (d) phase lag in velocity.

Define the signed normalized error in the estimated stiffness k' as

$$\varepsilon_k = (k' - k)/k. \tag{3}$$

Comparing the triangles OPQ and ORS in Figure 2(b), the difference between the actual stiffness k and the estimate k' is

$$k' - k = f\cos(\theta - \tau) - f\cos\tau. \tag{4}$$

Some manipulation yields

$$\Rightarrow \varepsilon_k = (1 - \cos\tau)(1 - r^2) - 2\zeta r \sin\tau. \tag{5}$$

Similarly, the signed normalized error in the estimated damping c' can be defined as

$$\varepsilon_c = (c' - c)/c. \tag{6}$$

By considering triangle ORS in Figure 2(b), this quantity can be expressed as

$$\varepsilon_c = (\cos\tau - 1) - \sin\tau\,((1 - r^2)/2\zeta). \tag{7}$$

Figure 2(c) and 2(d) depicts the case for phase lag in displacement and velocity respectively. Using the same approach as for phase lag in the excitation force, error estimates can be obtained for phase lag in displacement and velocity. These expressions are summarized in Table 1.

TABLE 1

Errors in parameter estimates for forced response

		Phase lag in	
Error	Displacement (Figure 2(b))	Velocity (Figure 2(c))	Force (Figure 2(d))
ε_k	$\operatorname{cosec}\tau - 1$	$2\zeta r\tan\tau$	$(1-\cos\tau)(1-r^2) - 2\zeta r\sin\tau$
ε_c	$(\tan\tau)2\zeta r$	$\operatorname{cosec}\tau - 1$	$(\cos\tau - 1) - (\sin\tau)(1-r^2)/2\zeta r$

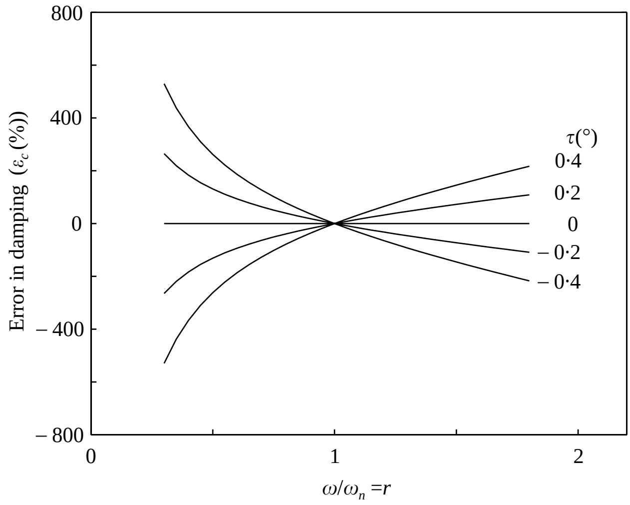

Figure 3. Error in damping estimate caused by time lag in excitation force ($\zeta = 0{\cdot}2\%$).

For systems with small damping, an order of magnitude analysis reveals that the damping estimate is the most susceptible to error, in particular when the excitation force experiences a phase shift, as previously predicted. Wright and Al-Hadid illustrated the variation of damping estimate error for a range of frequency ratio and phase shift with a specific example (Figure 5 of their paper [9]). Figure 3 shows variation of ε_c obtained with equation (6) for the same range of frequency ratio, phase shift and the same damping ratio ($\zeta = 0{\cdot}2\%$) . It should be noted that here a phase lag is defined as a positive angle whereas Wright and Al-Hadid defined a phase lead as positive. Excellent agreement can be seen between the two graphs indicating that the general pictorial approach to error estimation is valid.

The next most significant error in the damping estimate occurs when a phase shift is present in the displacement. For small phase lags this error can be written as $\varepsilon_c = \tau/2\zeta r$. Using the definitions of τ and r

$$\varepsilon_c = \omega_n \Delta t/2\zeta. \tag{8}$$

This indicates that for a given time lag (rather than phase lag), the error in the damping estimate is constant and does not depend on the excitation frequency.

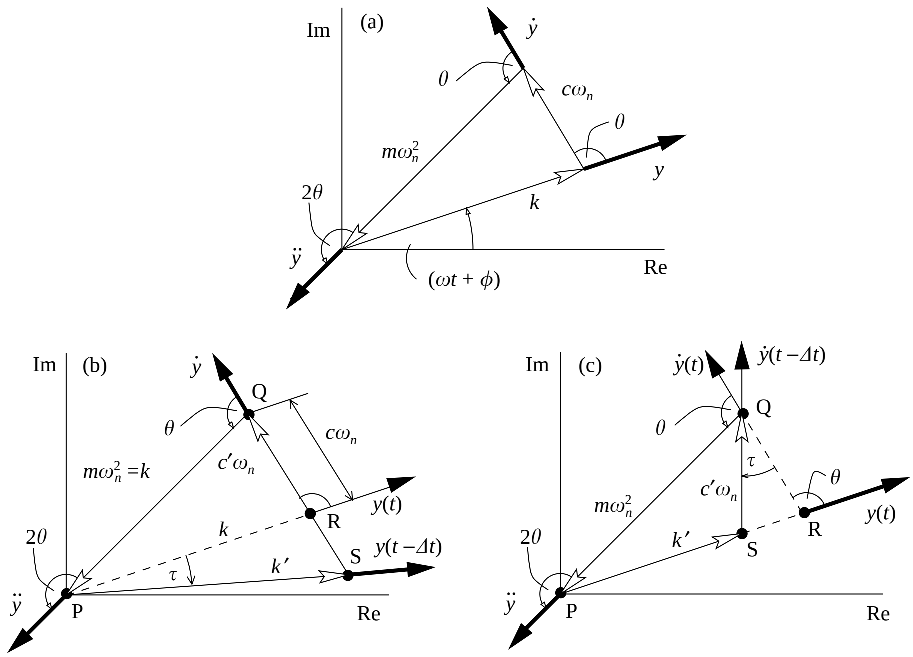

Figure 4. Force diagrams for free response: (a) no phase error; (b) phase lag in displacement and (c) phase lag in velocity.

3. FREE RESPONSE

For a linear system undergoing free response the displacement, velocity and acceleration is

$$y(t) = A\mathrm{e}^{-\zeta\omega_n t}\mathrm{e}^{\,\mathrm{j}\,(\omega t+\phi)}; \qquad \dot{y}(t) = A\mathrm{e}^{-\zeta\omega_n t}(\omega_n \mathrm{e}^{\mathrm{j}\theta})\,\mathrm{e}^{\,\mathrm{j}\,(\omega t+\phi)}, \tag{9a, b}$$

$$\ddot{y}(t) = A\mathrm{e}^{-\zeta\omega_n t}(\omega_n^2 \mathrm{e}^{\mathrm{j}2\theta})\,\mathrm{e}^{\,\mathrm{j}\,(\omega t+\phi)}, \tag{9c}$$

where $\zeta = c/2\sqrt{km}$, $\sin\theta = \sqrt{1-\zeta^2}$, $\cos\theta = -\,\zeta$, $\omega_n = \sqrt{k/m}$, $\omega = \omega_n\sqrt{1-\zeta^2}$ and ϕ, A are constants depending on $y(0)$ and $\dot{y}(0)$.

Substituting the responses into equation (1) and setting the right-hand-side to zero yields

$$A\mathrm{e}^{-\zeta\omega_n t}(m\omega_n^2\mathrm{e}^{\mathrm{j}2\theta} + \mathrm{j}c\omega_n\mathrm{e}^{\mathrm{j}\,\theta} + k)\,\mathrm{e}^{\mathrm{j}\omega_n t} = 0 \quad \Rightarrow \quad m\omega_n^2\mathrm{e}^{\mathrm{j}2\theta} + \mathrm{j}c\omega_n\mathrm{e}^{\mathrm{j}\theta} + k = 0. \tag{10, 11}$$

In force state mapping from free response data, the restoring force is simply the inertial force $m\ddot{y}$, thus the mass must be known. As with the forced response, the parameter estimation process is simply a matter of closing the vector triangle shown in Figure 4(a).

The force diagram for a phase lag applied to the displacement signal is shown in Figure 4(b). To evaluate the error in the estimated stiffness, consider the triangle PQS:

$$k'/\sin(\theta) = m\omega_n^2/\sin(\theta+\tau) = k/\sin(\theta+\tau) \quad \Rightarrow \quad k'/k = \sin(\theta)/\sin(\theta+\tau). \tag{12, 13}$$

Some manipulation yields an expression for the normalized error as

$$\varepsilon_k = (\tan\theta\,(1-\cos\tau) - \sin\tau)/(\tan\theta\cos\tau + \sin\tau). \tag{14}$$

TABLE 2

Errors in parameter estimates for free response

Error	Phase lag in	
	Displacement	Velocity
ε_k	$\dfrac{\tan\theta(1-\cos\tau)-\sin\tau}{\tan\theta\cos\tau+\sin\tau}$	$\dfrac{2\tan\tau}{\tan\theta-\tan\tau}$
ε_c	$-\dfrac{\sin\tau\sin\theta\tan\theta+\sin\tau\cos\theta}{2(\sin\theta\cos\tau+\sin\tau\cos\theta)}$	$\dfrac{\tan\theta(1-\cos\tau)+\sin\tau}{\tan\theta\cos\tau-\sin\tau}$

TABLE 3

Approximate errors in parameter estimates for free response

Error	Phase lag in	
	Displacement	Velocity
ε_k	$-\zeta\tau$	$2\zeta\tau$
ε_c	$\tau/2\zeta$	$\zeta\tau$

The error in the damping estimate due to a time lag in the displacement can be evaluated by first working on triangle PQR in Figure 4(b), namely

$$c\omega_n = -\, m\omega_n^2 \sin(2\theta)/\sin(\theta) \tag{15}$$

and then on triangle PQS:

$$c'\omega_n = -\, m\omega_n^2 \sin(2\theta+\tau)/\sin(\theta+\tau). \tag{16}$$

Equations (15) and (16) yield

$$\varepsilon_c = -\,(\sin\tau\sin\theta\tan\theta + \sin\tau\cos\theta)/2(\sin\theta\cos\tau + \sin\tau\cos\theta). \tag{17}$$

Figure 4(c) shows a situation similar to Figure 4(b), but now the time lag is applied to the velocity. The same geometric approach can be applied to obtain errors in the estimated stiffness and damping as

$$\varepsilon_k = 2\tan\tau/(\tan\theta - \tan\tau), \tag{18}$$

$$\varepsilon_c = (\tan\theta(1-\cos\tau) + \sin\tau)/(\tan\theta\cos\tau - \sin\tau). \tag{19}$$

The error estimates of equations (14), (17)–(19) are summarized in Table 2.

In many applications the damping is low [7, 5, 9]. Thus, $\zeta \ll 1 \Rightarrow \tan\theta \approx 1/\zeta$, $\sin\theta \approx 1$. Furthermore, for most experimental set-ups the time lag Δt will be small and so small angle approximations can be applied. If both these conditions are valid, then the errors can be well approximated as shown in Table 3.

A cursory inspection of these quantities will reveal that only ε_c for a phase error in the displacement will be significant as it is a quotient of two small numbers, while the other errors are products of two small numbers.

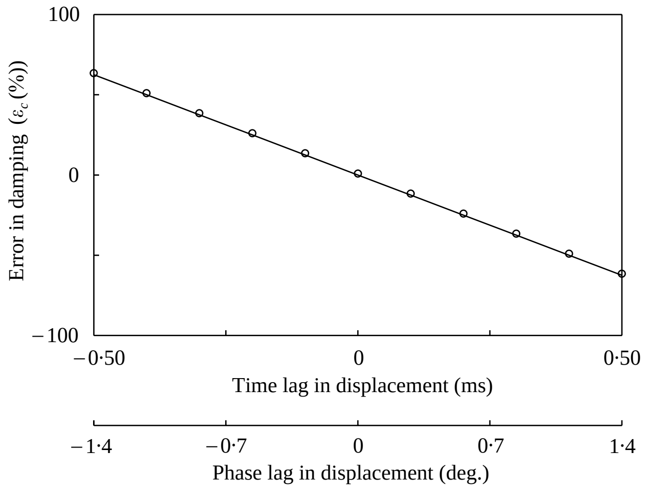

Figure 5. Error in damping estimate caused by time lag in displacement: ○, FSM identification; ——, expression in Table 3.

Using the definitions of τ and noting that for free response $\omega \approx \omega_n$, the error in the damping estimate for a given time lag is

$$\varepsilon_c = \omega_n \Delta t / 2\zeta. \tag{20}$$

This is exactly the same as equation (8) indicating that damping estimates from free response data are equally prone to error caused by displacement phase error as those from forced response data.

To demonstrate the validity of the expression for ε_c, consider a specific example of the system in Figure 1 with system parameters, $m = 1\,\mathrm{kg}$, $c = 2{\cdot}0\,\mathrm{N\,s/m}$, $k = 2500\,\mathrm{N/m}$ (i.e., $\omega_n = 50\,\mathrm{rad/s}$, $\zeta = 2\%$). The response was obtained with equations (9). Various time lags were applied to the displacement and the system parameters were estimated using force state mapping (FSM). As predicted, the error in the stiffness estimate was always very small ($< 1\%$), however, as can be seen in Figure 5, the errors in the damping are significant and follow the predicted trend.

4. CONCLUSIONS

A novel method of assessing the sensitivity of the parameter estimates obtained from force state mapping to a phase lag in the response signals has been described for a linear system undergoing either free or forced response.

It has been shown that in systems with low damping, the sensitivity of the damping estimate due to phase distortion in displacement is comparable for both free and forced response. However, the free response data has the advantage over forced response that it is not prone to the most significant source of error in the damping estimates (a phase distortion of the excitation force).

Although this cannot be extended directly to non-linear systems, for certain classes of non-linearities the error trends obtained here will be indicative of the error associated with the non-linear parameters.

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